International Journal of Pure and Applied Mathematics
Volume 96 No. 2 2014, 289-298
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v96i2.10

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A NOTE ON SYMMETRIC k-TRIDIAGONAL MATRIX FAMILY AND THE FIBONACCI NUMBERS

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Abstract: In this note, we provide arbitrary integer powers for another type of k-tridiagonal matrix family whose integer powers are specified to the famous Fibonacci numbers.

AMS Subject Classification: 15A18, 15A15
Key Words: k-tridiagonal matrix, Chebyshev polynomial, determinant, Fibonacci numbers

1. Introduction

The k-tridiagonal matrix [8] has received much attenson in recent years in the field of special matrices. $T_n^{(k)}$ is defined as follows:

Received: July 14, 2014

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$$T_n^{(k)} = \begin{pmatrix} d_1 & 0 & \cdots & 0 & a_1 & 0 & \cdots & 0 \\ 0 & d_2 & & & & a_2 & & \vdots \\ \vdots & & \ddots & & & & \ddots & 0 \\ 0 & & & d_{n-k} & & & & a_{n-k} \\ b_{k+1} & & & & \ddots & & & 0 \\ 0 & b_{k+2} & & & & \ddots & & \vdots \\ \vdots & & \ddots & & & & d_{n-1} & 0 \\ 0 & \cdots & 0 & b_n & 0 & \cdots & 0 & d_n \end{pmatrix}$$

It has recently been found that [9] the k-tridiagonal matrix plays important role in describing generalized k-Fibonacci numbers. Moreover, the authors ([3]-[6]) computed integer powers of some special types of these matrices by exploiting some properties of Chebyshev polynomials. The authors ([8]-[10]) investigated $k-$tridiagonal matrices. The paper ([13]) shows the importance of powers of a matrix for a generalized Fibonacci numbers, where elements of the matrix powers generates the generalized Fibonacci numbers. This also motivates us to find such matrices for ordinary Fibonacci numbers.

Among numerical sequences, the Fibonacci numbers which is defined by the recurrence $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$, has achieved a kind of celebrity status [2]. Binet defined an explicit formula for Fibonacci numbers [2] using the roots of quadratic equation $x^2 - x - 1 = 0$ as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1}$$

where α is positive root and β is negative root of the equation. It is well known that the determinant of a special tridiagonal matrix yields Fibonacci numbers, see, e.g., [1] for the recent overview.

In this note, we consider a specialized form of symmetric k-tridiagonal matrix, i.e.,

$$A_{n,k} = \begin{cases} 1, & a_{i,i+k} = a_{i+k,i}, \text{ for } i = 1, 2, \ldots, 3k, \\ 0, & \text{otherwise}. \end{cases} \tag{2}$$

where $n = 4k$ ($k = 1, 2, \ldots$). The matrix (2) is not only theoretical interest, but arises in scientific computing, e.g. a finite difference discretization of Laplace operator over thin rectangular domain.

The aim of this note is to compute integer powers of these type of matrices and show that all integer powers of A are specified to the famous Fibonacci numbers with positive and negative powers.

2. Eigenvalues of $A_{n,k}$

Lemma 1 ([4]). *Let $\triangle_n(\alpha)$ be of the form*

$$\triangle_n(\alpha) = \det \begin{pmatrix} \alpha & 1 & & & \\ 1 & \alpha & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \alpha & 1 \\ & & & 1 & \alpha \end{pmatrix}.$$

Then it holds

$$\triangle_n(\alpha) = \alpha\triangle_{n-1}(\alpha) - \triangle_{n-2}(\alpha), \tag{3}$$

with $\triangle_0(\alpha) = 1, \triangle_1(\alpha) = \alpha, \triangle_2(\alpha) = \alpha^2 - 1$. Moreover, the solution of difference equation (3) is $\triangle_n(\alpha) = U_n(\alpha/2)$. Here $U_n(\alpha/2)$ is the nth Chebyshev polynomial of the second kind:

$$U_n(x) = \frac{\sin(n+1)\arccos x}{\sin\arccos x}, \quad -1 \leq x \leq 1.$$

All roots of $U_n(x)$ are included in the interval $[-1, 1]$ and can be found using the relation

$$x = \cos\frac{k\pi}{n+1}, \quad k = 1, 2, \ldots, n.$$

Proposition 2. *All eigenvalues of $A_{n,k}$ are of the form*

$$\lambda_k = -2\cos\frac{k\pi}{5}, \quad k = 1, 2, 3, 4.$$

Proof. Let $P_n(x) = |A_{n,k} - xI|$ be the characteristic polynomial of the matrix $A_{n,k}$. Then, $P_n(x) = [\triangle_4(-x)]^{\frac{n}{4}}$. Since $n = 4k$, it follows from Lemma 1 that

$$P_n(x) = \left[U_4\left(-\frac{x}{2}\right)\right]^k. \tag{4}$$

Thus, from Lemma 1, zeros of $P_n(x)$ are given by $\lambda_k = -2\cos(k\pi/5)$, $k = 1, 2, 3, 4$. This completes the proof. $\square$

Proposition 2 may be interesting since all eigenvalues of $A_{n,k}$ does not depend on matrix size n. Moreover, we can write

$$\begin{aligned} P_n(x) &= (x^4 - 3x^2 + 1)^{\frac{n}{4}} \\ &= [(x^2 - 1 + x)(x^2 - 1 - x)]^{\frac{n}{4}} \end{aligned}$$

$$= [(x^2 - \alpha^2)(x^2 - \beta^2)]^{\frac{n}{4}},$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. All eigenvalues of the matrix are multiple and ($l_k = k$, $k = 1, 2, \ldots, n/4$) is the multiplicity of the eigenvalue of λ_k. Then, we write Jordan's form of the matrix $A_{n,k}$ as:

$$J = \operatorname{diag}(\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \lambda_3, \ldots, \lambda_3, \lambda_4, \ldots, \lambda_4). \tag{5}$$

From (5) and Proposition 2, we easily have the following result:

Corollary 3. *Let A be n-square matrix as in (2). Then*

$$\det(A_{n,k}) = 1.$$

Proof. From Proposition 2, the product of all eigenvalues is

$$\prod_{i=1}^{4} \left(-2\cos\frac{i\pi}{5}\right)^k = \left[\prod_{i=1}^{4} \left(-2\cos\frac{i\pi}{5}\right)\right]^k = 1^k = 1.$$

This concludes the proof. □

3. Eigenvectors of $A_{n,k}$ and Transforming Matrix

Proposition 4. *Let T be a matrix such that $J = T^{-1}A_{n,k}T$, where matrix J corresponds to (5). Let $m_i = (4 - \lambda_i^2)/(2n + 2)$. Let T_i's and t_i's are submatrices of T and T^{-1} such that $T = (T_1, T_2, T_3, T_4)$ and $T^{-1} = (t_1, t_2, t_3, t_4)$ respectively. Then, T_i's and t_i's are given by*

$$T_j = \left(\begin{array}{cccc}
U_0(\frac{\lambda_j}{2}) & 0 & \cdots & 0 \\
0 & U_0(\frac{\lambda_j}{2}) & & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & \cdots & U_0(\frac{\lambda_j}{2}) \\
U_1(\frac{\lambda_j}{2}) & 0 & \cdots & 0 \\
0 & U_1(\frac{\lambda_j}{2}) & & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & \cdots & U_1(\frac{\lambda_j}{2}) \\
U_2(\frac{\lambda_j}{2}) & 0 & \cdots & 0 \\
0 & U_2(\frac{\lambda_j}{2}) & & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & \cdots & U_2(\frac{\lambda_j}{2}) \\
U_3(\frac{\lambda_j}{2}) & 0 & \cdots & 0 \\
0 & U_3(\frac{\lambda_j}{2}) & & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & \cdots & U_3(\frac{\lambda_j}{2})
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} k \ times \\
\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} k \ times \\
\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} k \ times \\
\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} k \ times
\end{array}
\quad j = 1, 2, 3, 4,$$

$$t_j = \left(\begin{array}{cccc}
m_1U_{j-1}(\frac{\lambda_1}{2}) & 0 & \cdots & 0 \\
0 & m_1U_{j-1}(\frac{\lambda_1}{2}) & & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & \cdots & m_1U_{j-1}(\frac{\lambda_1}{2}) \\
m_2U_{j-1}(\frac{\lambda_2}{2}) & 0 & \cdots & 0 \\
0 & m_2U_{j-1}(\frac{\lambda_2}{2}) & & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & \cdots & m_2U_{j-1}(\frac{\lambda_2}{2}) \\
m_3U_{j-1}(\frac{\lambda_3}{2}) & 0 & \cdots & 0 \\
0 & m_3U_{j-1}(\frac{\lambda_3}{2}) & & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & \cdots & m_3U_{j-1}(\frac{\lambda_3}{2}) \\
m_4U_{j-1}(\frac{\lambda_4}{2}) & 0 & \cdots & 0 \\
0 & m_4U_{j-1}(\frac{\lambda_4}{2}) & & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & \cdots & m_4U_{j-1}(\frac{\lambda_4}{2})
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} k \ times \\
\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} k \ times \\
\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} k \ times \\
\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} k \ times
\end{array}$$

where $j = 1, 2, 3, 4.$

We now give the main result.

Theorem 5. *Let $A_{n,k}$ be a matrix as in (2). Then*

$$A^r_{n,k} = \begin{cases} a_{i,i} = a_{3k+i,3k+i} \\ \qquad = \sum_{t=1}^{4} \lambda_t^r m_t U_0^2(\frac{\lambda_t}{2}), & \text{for } 1 \le i \le k \\ a_{k+i,k+i} \\ \qquad = \sum_{t=1}^{4} \lambda_t^r m_t U_1^2(\frac{\lambda_t}{2}), & \text{for } 1 \le i \le 2k \\ a_{i,2k+i} = a_{2k+i,i} \\ \qquad = \sum_{t=1}^{4} \lambda_t^r m_t U_0(\frac{\lambda_t}{2}) U_2(\frac{\lambda_t}{2}), & \text{for } 1 \le i \le 2k \\ a_{i,k+i} = a_{k+i,i} = a_{3k+i,2k+i} = a_{2k+i,3k+i} \\ \qquad = \sum_{t=1}^{4} \lambda_t^r m_t U_0(\frac{\lambda_t}{2}) U_1(\frac{\lambda_t}{2}), & \text{for } 1 \le i \le k \\ a_{k+i,2k+i} = a_{2k+i,k+i} \\ \qquad = \sum_{t=1}^{4} \lambda_t^r m_t U_1(\frac{\lambda_t}{2}) U_2(\frac{\lambda_t}{2}), & \text{for } 1 \le i \le k \\ a_{3k+i,i} = a_{i,3k+i} \\ \qquad = \sum_{t=1}^{4} \lambda_t^r m_t U_0(\frac{\lambda_t}{2}) U_3(\frac{\lambda_t}{2}), & \text{for } 1 \le i \le k \\ 0, & \text{otherwise} \end{cases}$$

here r is any arbitrary integer.

Proof. By matrix multiplication (5) and Proposition 4, we obtain the expression which is desired. □

Moreover it can be seen that arbitrary integer power of the matrix can be given as below:

$$A_{n,k}^{r} = \begin{cases} a_{i,i} = a_{3k+i,3k+i} \\ \quad = \dfrac{-\alpha^r\beta + \beta^r\alpha + (-\beta)^r\alpha - (-\alpha)^r\beta}{2(\alpha^2-\beta^2)}, & \text{for } 1 \le i \le k \\ a_{k+i,k+i} \\ \quad = \dfrac{\alpha^{r+1} - \beta^{r+1} - \beta(-\beta)^r + \alpha(-\alpha)^r}{2(\alpha^2-\beta^2)}, & \text{for } 1 \le i \le 2k \\ a_{i,2k+i} = a_{2k+i,i} \\ \quad = \dfrac{\alpha^r - \beta^r - (-\beta)^r + (-\alpha)^r}{2(\alpha^2-\beta^2)}, & \text{for } 1 \le i \le 2k \\ a_{i,k+i} = a_{k+i,i} = a_{3k+i,2k+i} = a_{2k+i,3k+i} \\ \quad = \dfrac{\alpha^r - \beta^r + (-\beta)^r - (-\alpha)^r}{2(\alpha^2-\beta^2)}, & \text{for } 1 \le i \le k \\ a_{k+i,2k+i} = a_{2k+i,k+i} \\ \quad = \dfrac{\alpha^{r+1} - \beta^{r+1} + \beta(-\beta)^r\alpha - \alpha(-\alpha)^r}{2(\alpha^2-\beta^2)}, & \text{for } 1 \le i \le k \\ a_{3k+i,i} = a_{i,3k+i} \\ \quad = \dfrac{-\alpha^r\beta + \beta^r\alpha - (-\beta)^r\alpha + (-\alpha)^r\beta}{2(\alpha^2-\beta^2)}, & \text{for } 1 \le i \le k \\ 0, & \text{otherwise} \end{cases}$$

here $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$.

4. Illustrative Examples

We can find arbitrary integer powers of the matrix $A_{n,k}$, taking into account derived expressions. For example, let $k = 2$, then $n = 8$, i.e.,

$$A_{8,2} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

The transforming matrix of $A_{8,2}$ is

$$T = \begin{pmatrix} U_0\left(\frac{\lambda_1}{2}\right) & 0 & U_0\left(\frac{\lambda_2}{2}\right) & 0 & U_0\left(\frac{\lambda_3}{2}\right) & 0 & U_0\left(\frac{\lambda_4}{2}\right) & 0 \\ 0 & U_0\left(\frac{\lambda_1}{2}\right) & 0 & U_0\left(\frac{\lambda_2}{2}\right) & 0 & U_0\left(\frac{\lambda_3}{2}\right) & 0 & U_0\left(\frac{\lambda_4}{2}\right) \\ U_1\left(\frac{\lambda_1}{2}\right) & 0 & U_1\left(\frac{\lambda_2}{2}\right) & 0 & U_1\left(\frac{\lambda_3}{2}\right) & 0 & U_1\left(\frac{\lambda_4}{2}\right) & 0 \\ 0 & U_1\left(\frac{\lambda_1}{2}\right) & 0 & U_1\left(\frac{\lambda_2}{2}\right) & 0 & U_1\left(\frac{\lambda_3}{2}\right) & 0 & U_1\left(\frac{\lambda_4}{2}\right) \\ U_2\left(\frac{\lambda_1}{2}\right) & 0 & U_2\left(\frac{\lambda_2}{2}\right) & 0 & U_2\left(\frac{\lambda_3}{2}\right) & 0 & U_2\left(\frac{\lambda_4}{2}\right) & 0 \\ 0 & U_2\left(\frac{\lambda_1}{2}\right) & 0 & U_2\left(\frac{\lambda_2}{2}\right) & 0 & U_2\left(\frac{\lambda_3}{2}\right) & 0 & U_2\left(\frac{\lambda_4}{2}\right) \\ U_3\left(\frac{\lambda_1}{2}\right) & 0 & U_3\left(\frac{\lambda_2}{2}\right) & 0 & U_3\left(\frac{\lambda_3}{2}\right) & 0 & U_3\left(\frac{\lambda_4}{2}\right) & 0 \\ 0 & U_3\left(\frac{\lambda_1}{2}\right) & 0 & U_3\left(\frac{\lambda_2}{2}\right) & 0 & U_3\left(\frac{\lambda_3}{2}\right) & 0 & U_3\left(\frac{\lambda_4}{2}\right) \end{pmatrix}.$$

The inverse of the matrix T is

$$T^{-1} = \operatorname{diag}(m_1, m_1, m_2, m_2, m_3, m_3, m_4, m_4) \cdot \begin{pmatrix} U_0\left(\frac{\lambda_1}{2}\right) & 0 & U_1\left(\frac{\lambda_1}{2}\right) & 0 & U_2\left(\frac{\lambda_1}{2}\right) & 0 & U_3\left(\frac{\lambda_1}{2}\right) & 0 \\ 0 & U_0\left(\frac{\lambda_1}{2}\right) & 0 & U_1\left(\frac{\lambda_1}{2}\right) & 0 & U_2\left(\frac{\lambda_1}{2}\right) & 0 & U_3\left(\frac{\lambda_1}{2}\right) \\ U_0\left(\frac{\lambda_2}{2}\right) & 0 & U_1\left(\frac{\lambda_2}{2}\right) & 0 & U_2\left(\frac{\lambda_2}{2}\right) & 0 & U_3\left(\frac{\lambda_2}{2}\right) & 0 \\ 0 & U_0\left(\frac{\lambda_2}{2}\right) & 0 & U_1\left(\frac{\lambda_2}{2}\right) & 0 & U_2\left(\frac{\lambda_2}{2}\right) & 0 & U_3\left(\frac{\lambda_2}{2}\right) \\ U_0\left(\frac{\lambda_3}{2}\right) & 0 & U_1\left(\frac{\lambda_3}{2}\right) & 0 & U_2\left(\frac{\lambda_3}{2}\right) & 0 & U_3\left(\frac{\lambda_3}{2}\right) & 0 \\ 0 & U_0\left(\frac{\lambda_3}{2}\right) & 0 & U_1\left(\frac{\lambda_3}{2}\right) & 0 & U_2\left(\frac{\lambda_3}{2}\right) & 0 & U_3\left(\frac{\lambda_3}{2}\right) \\ U_0\left(\frac{\lambda_4}{2}\right) & 0 & U_1\left(\frac{\lambda_4}{2}\right) & 0 & U_2\left(\frac{\lambda_4}{2}\right) & 0 & U_3\left(\frac{\lambda_4}{2}\right) & 0 \\ 0 & U_0\left(\frac{\lambda_4}{2}\right) & 0 & U_1\left(\frac{\lambda_4}{2}\right) & 0 & U_2\left(\frac{\lambda_4}{2}\right) & 0 & U_3\left(\frac{\lambda_4}{2}\right) \end{pmatrix},$$

and Jordan form is

$$J = \operatorname{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_3, \lambda_3, \lambda_4, \lambda_4).$$

For $r = 4, 5$,

$$A_{8,2}^4 = \begin{pmatrix} 2 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 3 \\ 3 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad A_{8,2}^5 = \begin{pmatrix} 0 & 0 & 5 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 3 \\ 5 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 & 5 \\ 3 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 5 & 0 & 0 \end{pmatrix}.$$

For $r = -4$,

$$A_{8,2}^{-4} = \begin{pmatrix} 5 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & -3 \\ -3 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

For $r = -5$,

$$A_{8,2}^{-5} = \begin{pmatrix} 0 & 0 & 5 & 0 & 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & -8 \\ 5 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 & 5 \\ -8 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & -8 & 0 & 0 & 0 & 5 & 0 & 0 \end{pmatrix}.$$

From these examples, we see that all nonzero elements in the powers of matrix $A_{n,k}$ are closely related to Fibonacci numbers.

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