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ON THE MAZUR-ULAM THEOREM IN FUZZY ANTI-2-NORMED SPACES

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Abstract: In this article, we study the notions of 2-isometries in fuzzy anti-2-normed spaces and prove a Mazur-Ulam type theorem in the 2-strictly convex fuzzy anti-2-normed spaces.

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1. Introduction

The theory of fuzzy sets was introduced by L. Zadeh [14] in 1965 and thereafter several authors applied it different branches of pure and applied mathematics. Many mathematicians considered the fuzzy normed spaces in several angels (see [3], [10], [13]). In [8] Iqbal H. Jebril and Samanta introduced fuzzy anti-norm on a linear space depending on the idea of fuzzy anti-norm was introduced by Bag and Samanta [1] and investigated their important properties.

Let X and Y be metric spaces with metrics d_X and d_Y, respectively. A map $f : X \to Y$ is called an isometry if $d_Y(f(x), f(y)) = d_X(x, y)$ for every $x, y \in X$. In 1932, the theory of isometric mappings was originated in the classical paper

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[11] by Mazur and Ulam. They have proved the following theorem.

Mazur-Ulam Theorem. *Let f be an isometric transformation from a real normed vector space X onto a real normed vector space Y with $f(0) = 0$. Then f is linear.*

The theorem is not true for normed complex vector spaces. It was a natural ask if the theorem holds without the onto assumption. In fact, the hypothesis of surjectivity is essential. Without this assumption, Baker [2] proved that every isometry from a normed real space into a strictly convex normed real space is linear up to translation. Chu et al. [4] have defined the notion of a 2-isometry which is suitable to represent the concept of an area preserving mapping in linear 2-normed spaces. In [5], Chu proved that the Mazur-Ulam theorem holds in linear 2-normed spaces under the condition that a 2-isometry preserves collinearity. Chu et al. [6] discussed characteristics of 2-isometries. Recently, some authors have studied the Mazur-Ulam theorem on non-Archimedean normed linear spaces [9], [12]. In this paper, we prove that the Mazur-Ulam theorem holds under some conditions in the fuzzy anti-2-normed spaces. We establish a Mazur-Ulam type theorem in the framework of strictly convex normed spaces by using some ideas of [7]. Now we recall some notations and definitions used in this paper.

2. Preliminaries

Definition 2.1. Let X be a linear space over a real field F. A fuzzy subset N of $X \times X \times \mathbb{R}$ is called a fuzzy anti-2-norm on X if the following conditions are satisfied for all $x, y \in X$:

$(a - N_1)$ For all $t \in \mathbb{R}$ with $t \leq 0, N(x, y; t) = 1$,

$(a - N_2)$ For all $t \in \mathbb{R}$ with $t > 0, N(x, y; t) = 0$ if and only if x and y are linearly dependent,

$(a - N_3)$ For all $t \in \mathbb{R}$ with $t > 0$ and $x, y \in X$, $N(x, y; t) = N(y, x; t)$,

$(a - N_4)$ For all $t \in \mathbb{R}$ with $t > 0, N(\alpha x, y; t) = N(x, y; t/|\alpha|)$, for all $\alpha \neq 0, \alpha \in F$,

$(a - N_5)$ For all $s, t \in \mathbb{R}, N(x + y, z; t + s) \leq \max\{N(x, z, t), N(y, z, s)\}$,

$(a - N_6)$ $N(x, y, t)$ is a non-increasing function of $t \in \mathbb{R}$ and $\lim_{t\to\infty} N(x, y, t) = 0$.

Then the pair (X, N) is called a fuzzy anti-2-normed linear space.

Example 2.2. Let $(X, \|.,.\|)$ be a 2-normed space. If we define

$$N(x,y;t) = \begin{cases} \frac{\|x,y\|}{t+\|x,y\|} & \text{if} \quad t > 0 \\ 1 & \text{if} \quad t \leq 0. \end{cases}$$

then (X, N) is called the standard fuzzy anti-2-norm induced by the 2-norm $\|.,.\|$.

Definition 2.3. A fuzzy anti-2-normed space X is called 2-strictly convex if $N(x+y, z, s+t) = \max\{N(x,z,s), N(y,z,t)\}$ and $N(x,z,s) = N(y,z,t)$ implies that $x = y$ and $s = t$.

Definition 2.4. Let (X, N) and (Y, N) be two fuzzy anti-2-normed spaces. We call that $f : (X, N) \to (Y, N)$ is a fuzzy 2-isometry if $N(x-z, y-z, t) = N(f(x)-f(z), f(y)-f(z), t)$ for all $x, y, z \in X$ and $t > 0$.

Definition 2.5. Let X be a real linear space and x, y, z mutually disjoint elements of X. Then x, y and z are said to be 2-collinear if $y - z = r(x - z)$ for some real number r.

3. Main Results

In this section we will prove that the Mazur-Ulam theorem under some conditions in the fuzzy real anti-2-normed 2-strictly convex spaces. First, we prove some lemmas that is require for the main theorem of our paper.

Lemma 3.1. *Let (X, N) be a fuzzy anti-2-normed space. Then*

$$N(x,y,t) = N(x, y+rx, t) \quad for\ all\ r \in \mathbb{R}.$$

Proof. Let $x, y \in X$ and let $r \in \mathbb{R}$. Without loss of generality, we may assume $s, t > 0$. Then

$$N(x, y+rx, t) \leq \max\{N(x,y,t), N(x, rx, t)\} = N(x,y,t).$$

Conversely,

$$N(x,y,t) = N(x, y+rx-rx, t)$$

$$\begin{aligned}
&\leq N(x, y+rx-rx, t+s)\\
&\leq \max\{N(x, y+rx, t), N(x, -rx, s)\}\\
&= N(x, y+tx, t).
\end{aligned}$$

Thus $N(x, y, t) = N(x, y+rx, t)$ for all $r \in \mathbb{R}$.

Lemma 3.2. *Let (X, N) be a fuzzy anti-2-normed space which is strictly convex and let $x, y, z \in X$ and $t > 0$. Then $u = \frac{x+y}{2}$ is unique element of X such that*

$$N(x-z, x-u, t) = N(x-z, y-z, 2t)$$

and

$$N(y-u, y-z, t) = N(x-z, y-z, 2t),$$

and x, y, u are 2-collinear.

Proof. First, we prove that x, y, u are 2-collinear. Since $u = \frac{x+y}{2}$ we have

$$\begin{aligned}
x-u = x - \frac{x+y}{2} &= \frac{x}{2} - \frac{y}{2} = \frac{x+y-y}{2} - \frac{y}{2}\\
&= -\big(y - \frac{x+y}{2}\big) = -(y-u)
\end{aligned}$$

Thus x, y and u are 2colinear. By using lemma (3.1), we have

$$\begin{aligned}
N(x-z, x-u, t) &= N\big(x-z, x - \frac{x+y}{2}, t\big)\\
&= N(x-z, x-y, 2t)\\
&= N(x-z, y-z, 2t),
\end{aligned}$$

and similarly $N(y-u, y-z, t) = N(x-z, y-z, 2t)$. To show the uniqueness, assume that $v \in X$, satisfies the above properties. Since x, y and v are 2-colinear, there exists a real number s such that $v := sx + (1-s)y$. In view of lemma (3.1) and definition (2.3), we obtain

$$\begin{aligned}
N(x-z, y-z, 2t) &= N(x-z, x-v, t)\\
&= N(x-z, x-(sx+(1-s)y), t)\\
&= N(x-z, x-y, \frac{t}{|1-s|})\\
&= N(x-z, y-z, \frac{t}{|1-s|}).
\end{aligned}$$

Thus $2t = \frac{t}{|1-s|}$. Since $t > 0$, $|1-s| = \frac{1}{2}$. Also,

$$\begin{aligned} N(x-z, y-z, 2t) &= N(y-v, y-z, t) \\ &= N(y-(sx+(1-s)y), y-z, t) \\ &= N(x-y, y-z, \frac{t}{|s|}) \\ &= N(x-z, y-z, \frac{t}{|s|}). \end{aligned}$$

So, $2t = \frac{t}{|s|}$. Hence $\frac{1}{2} = |s| = |1-s|$ and so $s = \frac{1}{2}$. Therefore we obtain that $u = v$ and this complete the proof.

Lemma 3.3. *Let $f : (X, N) \to (Y, N)$ is a fuzzy 2-isometry;*
(i) For every $x, y, z, u \in X$, if x, y and z are 2-colinear, then $f(x), f(y)$ and $f(z)$ are 2-colinear.
(ii) If $f(0) = 0$, then for every $x, y \in X$ and $t > 0$

$$N(x, y, t) = N(f(x), f(y), t).$$

Proof. (i) Since x, y and z are 2-colinear, there exists a real number s such that $y - x = s(z - x)$. So we have

$$\begin{aligned} N(f(y)-f(x), f(u)-f(x), t) &= N(y-x, u-x, t) \\ &= N(s(z-x), u-x, t) \\ &= N(z-x, u-x, \frac{t}{|s|}) \\ &= N(f(z)-f(x), f(u)-f(x), \frac{t}{|s|}) \\ &= N(s(f(z)-f(x)), f(u)-f(x), t) \end{aligned}$$

and by definition (2.3), we have that $f(y) - f(x) = s(f(z) - f(x))$.
To prove (ii), we have

$$\begin{aligned} N(x, y, t) &= N(x-0, y-0, z-0, t) \\ &= N(f(x)-f(0), f(y)-f(0), t) \\ &= N(f(x), f(y), t). \end{aligned}$$

Theorem 3.4. *Every fuzzy 2-isometry $f : (X, N) \to (Y, N)$ is affine.*

Proof. Let $g(x) := f(x) - f(0)$. Then g is fuzzy 2-isometry and $g(0) = 0$. Hence, it is enough to show that g is linear. Let $x, y \in X$. By (i) of Lemma (3.3), we obtain that $g(\frac{x+y}{2}), g(x)$ and $g(y)$ are 2colinear. Now, from Lemma (3.2), for all $x, y \in X$, we have

$$g(\frac{x+y}{2}) = g(\frac{x}{2}) + g(\frac{y}{2}).$$

It follows that g is $\mathbb{Q}$-linear. We have to show that g is $\mathbb{R}-$linear.

Let $r \in \mathbb{R}^+$ with $r \neq 1$ and $x \in X$. By (i) of Lemma (3.3), since $0, x$ and rx are 2-collinear $g(0), g(x)$ and $g(rx)$ are also 2-collinear. Since $g(0) = 0$, there exists $r' \in \mathbb{R}$ such that $g(rx) = r'g(x)$. Now, we will proved that $r = r'$. From (ii) of Lemma (3.3), for each $y \in X$ and $t > 0$ we can write

$$\begin{aligned} N(x, y, \frac{t}{r}) &= N(rx, y, t) \\ &= N(g(rx), g(y), t) \\ &= N(r'g(x), g(y), t) \\ &= N(g(x), g(y), \frac{t}{|r'|}) \\ &= N(x, y, \frac{t}{|r'|}). \end{aligned}$$

So, $r = \pm r'$. We assume that $r' = -r$, that is, $g(rx) = -rg(x)$. Then there exists $q_1, q_2 \in \mathbb{Q}$ such that $0 < q_1 < r < q_2$. For each $y \in X$, we have

$$\begin{aligned} N(g(x), g(y) - g(q_2x), \frac{t}{q_2 + r}) &= N(q_2g(x) - (-rg(x)), g(y) - g(q_2x), t) \\ &= N(g(rx) - g(q_2x), g(y) - g(q_2x), t) \\ &= N(rx - q_2x, y - q_2x, t) \\ &= N(x, y - q_2x, \frac{t}{q_2 - r}) \\ &\geq N(x, y - q_2x, \frac{t}{q_2 - q_1}) \\ &= N(q_1x - q_2x, y - q_2x, t) \\ &= N(g(q_1x) - g(q_2x), g(y - q_2x), t) \\ &= N(g(x), g(y - q_2x), \frac{t}{q_2 - q_1}) \end{aligned}$$

By $(a - N_6)$, we have $q_2 + r \leq q_2 - q_1$ which is a contradiction. Hence $r = r'$, that is, $g(rx) = rg(x)$ for all positive real numbers r. Hence for every real number $r, g(rx) = rg(x)$. Therefore $f = g - g(0)$ is $\mathbb{R}$-linear and the proof is complete.

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