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ON CERTAIN ESTIMATES FOR PARABOLIC MARCINKIEWICZ INTEGRAL AND EXTRAPOLATION

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Abstract: In this article, we establish L^p boundedness of the parametric Marcinkiewicz integral operators with rough kernels. These estimates and extrapolation arguments improve and extend some known results on parabolic Marcinkiewicz integrals.

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1. Introduction

Throughout this article, let $\mathbf{R}^n$, $n \geq 2$, be the n-dimensional Euclidean space, and let $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ which is equipped with the normalized Lebesgue surface measure $d\sigma = d\sigma(\cdot)$. Also, let p' denote to the exponent conjugate to p; that is $1/p + 1/p' = 1$.

Let $\alpha_i \geq 1$, $(i = 1, 2, \cdots, n)$, be fixed real numbers. For fixed $x \in \mathbf{R}^n$ and $\rho > 0$, let $F(x,\rho) = \sum_{i=1}^{n} \frac{x_i^2}{\rho^{2\alpha_i}}$. Then it is easy to see that for any fixed $x \in \mathbf{R}^n \setminus \{0\}$, $F(x,\rho)$ is strictly decreasing function in $\rho > 0$. The unique solution of the equation $F(x,\rho) = 1$ is denoted by $\rho(x)$.It was proved in [13]

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that $\rho(x)$ is a metric on $\mathbf{R}^n$, and $(\mathbf{R}^n, \rho)$ is the mixed homogeneity space.

For $\lambda > 0$, let $A_\lambda = \begin{bmatrix} \lambda^{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & \lambda^{\alpha_n} \end{bmatrix}$. For $\tau = a + ib$ ($a, b \in \mathbf{R}$ with $a > 0$), let $K_{\Omega,h}(u) = \Omega(u)h(\rho(u))\rho(u)^{\tau-\alpha}$, where $h : [0, \infty) \to \mathbf{C}$ is a measurable function and Ω is a real valued and measurable function on $\mathbf{R}^n$ with $\Omega \in L^1(\mathbf{S}^{n-1})$ that satisfying the conditions

$$\Omega(A_\lambda x) \quad = \quad \Omega(x), \quad \textit{and} \tag{1}$$

$$\int_{\mathbf{S}^{n-1}} \Omega(x')J(x')d\sigma(x') = 0, \tag{2}$$

where $J(x')$ is a function will be defined later. Define the parabolic Marcinkiewicz integral operator $\mathcal{M}^\tau_{\Omega,h}$ for $f \in \mathcal{S}(\mathbf{R}^n)$ by

$$\mathcal{M}^\tau_{\Omega,h} f(x) = \left(\int_0^\infty \left| \frac{1}{t^\tau} \int_{\rho(u) \le t} K_{\Omega,h}(u) f(x-u) du \right|^2 \frac{dt}{t} \right)^{1/2}. \tag{3}$$

When $\tau = 1$ and $h = 1$, we denote $\mathcal{M}^\tau_{\Omega,h}$ by μ_Ω. The parabolic Littlewood-Paley operator μ_Ω was introduced by Xue, Ding and Yabuta in [20] in which they proved that μ_Ω is bounded for $p \in (1, \infty)$ provided that $\Omega \in L^q(\mathbf{S}^{n-1})$ for $q > 1$. Subsequently, the study of the L^p boundedness of μ_Ω under various conditions on the function Ω has been studied by many authors. For example, Cheng and Ding improved the above result in [8]; they obtained the L^p boundedness of μ_Ω when Ω belongs to the Hardy space $H^1(\mathbf{S}^{n-1})$ for $1 < p < \infty$. However, the authors of [6] established that μ_Ω is bounded under the condition $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ for $1 < p < \infty$. In addition, the authors of [10] found that μ_Ω is bounded when Ω belongs to the block space $B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ for $1 < p < \infty$ and $q > 1$. Recently, Wang, Chen and Yu verified in [19] that if we replace n by $n+1$, and y by $(y, \phi(\rho(y)))$, where ϕ is a polynomial of degree m, then μ_Ω is bounded on $L^p(\mathbf{R}^{n+1})$ for $p \in (\frac{2+2\nu}{1+2\nu}, 2+2\nu)$ provided that $\Omega \in F(\nu, \mathbf{S}^{n-1})$ for some $\nu > 0$, where $F(\nu, \mathbf{S}^{n-1})$ denotes the set of all Ω which are integrable over $\mathbf{S}^{n-1}$ and satisfying

$$\sup_{\xi \in \mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} |\Omega(u)| (\ln \frac{1}{|u \cdot \xi|})^{1+\nu} du < \infty.$$

We point out that the class of the operators μ_Ω is related to the class of the parabolic singular integral operators

$$T_\Omega f(x) = p.v. \int_{\mathbf{R}^n} \frac{\Omega(u)}{\rho(u)^\alpha} f(x-u)du.$$

The class of the operators T_Ω belongs to the class of singular Radon transforms, which has considered to study by many mathematicians (we refer the readers, in particular, to [13], [15] and [16]).

If $\alpha_1 = \cdots = \alpha_n = 1$, then $\rho(x) = |x|$, $\alpha = n$ and $(\mathbf{R}^n, \rho) = (\mathbf{R}^n, |\cdot|)$. In this case, μ_Ω is just the classical Marcinkiewicz integral, which was introduced by Stein in [18]. For more information about the importance and the recent advances on the study of such operators, the readers are refereed (for instance to [2], [3], [5], [9], [11], [12], [14], and the references therein).

Our main interest in this paper is to study the L^p boundedness of the parabolic Marcinkiewicz integral under weak conditions on Ω and h, and then apply an extrapolation method to establish new improved results. In this work, we let $\Delta_\gamma(\mathbf{R}^+)$ (for $\gamma > 1$) denote the collection of all measurable functions $h : [0,\infty) \to \mathbf{C}$ satisfying

$$\|h\|_{\Delta_\gamma(\mathbf{R}^+)} = \sup_{R \in \mathbf{Z}} \left(\frac{1}{R} \int_0^R |h(\rho)|^\gamma d\rho \right)^{1/\gamma} < \infty.$$

In this article, we extend and improve some known results in the parabolic Marcinkiewicz operators (see [1], [6], [10], and [20]). Our main result is formulated as follows:

Theorem 1. *Let $\Omega \in L^q\left(\mathbf{S}^{n-1}\right)$ for some $1 < q \le 2$ and $h \in \triangle_\gamma(\boldsymbol{R}^+)$ for some $\gamma > 1$. Then for any $f \in L^p(\mathbf{R}^n)$ with p satisfying $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$, a constant C_p (independent of Ω, h, γ, and q) exists such that*

$$\left\|\mathcal{M}^\tau_{\Omega,h} f\right\|_{L^p(\mathbf{R}^n)} \le C_p A(\gamma)(q-1)^{-1/2} \|h\|_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbf{R}^n)},$$

where $A(\gamma) = \begin{cases} \gamma^{1/2} & \text{if } \gamma > 2, \\ (\gamma-1)^{-1/2} & \text{if } 1 < \gamma \le 2. \end{cases}$

The fruit of our result is earned by using its conclusion and the extrapolation method (see [4]). In particular, Theorem 1 and extrapolation lead to the following theorem.

Theorem 2. *Suppose that Ω satisfies the conditions* (1)-(2) *and $h \in \triangle_\gamma(\boldsymbol{R}^+)$ for some $\gamma > 1$.*

(i) If $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ *for some* $q > 1$, *then for* $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$,

$$\left\|\mathcal{M}^{\tau}_{\Omega,h} f\right\|_{L^p(\mathbf{R}^n)} \leq C_p A(\gamma) \|h\|_{\triangle_\gamma(\boldsymbol{R}^+)} \|f\|_{L^p(\boldsymbol{R}^n)} \left(1 + \|\Omega\|_{B_q^{(0,-1/2)}(\mathbf{S}^{n-1})}\right);$$

(ii) If $\Omega \in L(logL)^{1/2}(\mathbf{S}^{n-1})$, *then*

$$\left\|\mathcal{M}^{\tau}_{\Omega,h} f\right\|_{L^p(\mathbf{R}^n)} \leq C_p A(\gamma) \|h\|_{\triangle_\gamma(\boldsymbol{R}^+)} \|f\|_{L^p(\boldsymbol{R}^n)} \left(1 + \|\Omega\|_{L(logL)^{1/2}(\mathbf{S}^{n-1})}\right)$$

for $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$.

Throughout this paper, the letter C denotes a bounded positive constant that may vary at each occurrence but independent of the essential variables.

2. Some Lemmas

In this section, we give some auxiliary lemmas used in the sequel. Let us first recall the polar coordinates transform in the mixed homogeneity space $(\mathbf{R}^n, \rho)$. For $x = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n$, set

$$\begin{aligned} x_1 &= \rho^{\alpha_1} \cos\vartheta_1 \cdots \cos\vartheta_{n-2} \cos\vartheta_{n-1}, \\ x_2 &= \rho^{\alpha_2} \cos\vartheta_1 \cdots \cos\vartheta_{n-2} \sin\vartheta_{n-1}, \\ &\vdots \\ x_{n-1} &= \rho^{\alpha_{n-1}} \cos\vartheta_1 \sin\vartheta_2, \\ x_n &= \rho^{\alpha_n} \sin\vartheta_1. \end{aligned}$$

Thus, $dx = \rho^{\alpha-1} J(\vartheta_1, \cdots, \vartheta_{n-1}) d\rho d\sigma(x')$, where $\rho^{\alpha-1} J(\vartheta_1, \cdots, \vartheta_{n-1})$ is the Jacobian of the above transforms. It was shown in [13] that $J(\vartheta_1, \cdots, \vartheta_{n-1})$ is a $C^\infty\left((0, 2\pi)^{n-2} \times (0, \pi)\right)$ function in the variable $x' \in \mathbf{S}^{n-1}$, and that a real constant $M \geq 1$ exists so that $1 \leq J(\vartheta_1, \cdots, \vartheta_{n-1}) \leq M$. For simplicity, we let $J(x')$ denote $J(\vartheta_1, \cdots, \vartheta_{n-1})$.

In order to prove Theorem 1, we need the following lemmas.

Lemma 3. *[17]. Suppose that* $\lambda_i's$ *and* $\alpha_i's$ *are fixed real numbers, and* $\Gamma(t) = (\lambda_1 t^{\alpha_1}, \cdots, \lambda_n t^{\alpha_n})$ *is a function from* $\boldsymbol{R}^+$ *to* $\boldsymbol{R}^n$. *For suitable* f, *let* $\mathcal{M}_\Gamma$ *be the maximal operator defined on* $\boldsymbol{R}^n$ *by*

$$\mathcal{M}_\Gamma(f)(x) = \sup_{h>0} \frac{1}{h} \left| \int_0^h f(x - \Gamma(t)) dt \right|$$

for $x \in \boldsymbol{R}^n$. *Then for* $1 < p \leq \infty$, *there exists a constant* $C_p > 0$ *such that*

$$\|\mathcal{M}_\Gamma(f)\|_{L^p(\boldsymbol{R}^n)} \leq C_p \|f\|_{L^p(\boldsymbol{R}^n)}.$$

The constant C_p *is independent of* $\lambda_i's$ *and* f.

Lemma 4. *[6]. Let $\beta \in [0,1]$. Then for any $u, \xi \in \boldsymbol{R}^n$,*

$$\left|\int_1^2 e^{-iA_\lambda u\cdot\xi}\frac{d\lambda}{\lambda}\right| \leq C\,|u\cdot\xi|^{-\frac{\beta}{m}},$$

where A_λ is defined as above and m denotes the distinct numbers of $\{\alpha_i\}$.

We shall recall the following lemma due to Chen and Ding.

Lemma 5. *[7]. Let $\phi \in \mathcal{S}(\boldsymbol{R}^n)$, $\widehat{\Phi}(\xi) = \phi(\rho(\xi))$ and $\Phi_t(\xi) = t^{-\alpha}\phi(A_{t^{-1}}\xi)$ for $t > 0$. For $j \in \mathbf{Z}$, define the multiplier $\widehat{S_j f}(\xi) = \phi(2^j\rho(\xi))\hat{f}(\xi)$. Then for $1 < p < \infty$, there exists a constant C such that*

$$\left\|\left(\sum_j |\ S_j f\ |^2\right)^{1/2}\right\|_{L^p(\boldsymbol{R}^n)} \leq C\|f\|_{L^p(\boldsymbol{R}^n)}.$$

Let $\theta \geq 2$. For a measurable function $h : \mathbf{R}^+ \to \mathbf{C}$ and $\Omega : \mathbf{S}^{n-1} \to \mathbf{R}$, we define the family of measures $\{\sigma_{\Omega,h,t} : t \in \mathbf{R}^+\}$ and its corresponding maximal operators $\sigma^*_{\Omega,h,t}$ and $M_{h,\theta}$ on $\mathbf{R}^n$ by

$$\int_{\mathbf{R}^n} f d\sigma_{\Omega,h,t} = \frac{1}{t^\tau}\int_{1/2t\leq\rho(u)\leq t} f(u)\frac{h(\rho(u))\Omega(u)}{\rho(u)^{\alpha-\tau}}du,$$

$$\sigma^*_{\Omega,h}f(x) = \sup_{t\in\mathbf{R}^+}||\sigma_{\Omega,h,t}| * f(x)|,$$

and

$$M_{h,\theta}f(x) = \sup_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}||\sigma_{\Omega,h,t}| * f(x)|\frac{dt}{t},$$

where $|\sigma_{\Omega,h,t}|$ is defined in the same way as $\sigma_{\Omega,h,t}$, but with replacing h by $|h|$ and Ω by $|\Omega|$. We write $\|\nu_{t,s}\|$ for the total variation of $\nu_{t,s}$.

Lemma 6. *Let $\theta \geq 2$, $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $\gamma > 1$, $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$. Then there exist constants C and β with $0 < \beta < \min\{\frac{1}{2}, \frac{m}{2q'}, \frac{m}{\alpha}\}$ such that*

$$\|\sigma_{\Omega,h,t}\| \leq C(\ln\theta)\,\|h\|^2_{\Delta_\gamma(\mathbf{R}^+)}\,\|\Omega\|^2_{L^q(\mathbf{S}^{n-1})}; \tag{4}$$

$$\int_{\theta^k}^{\theta^{k+1}} |\hat{\sigma}_{\Omega,h,t}(\xi)|^2 \frac{dt}{t} \leq C(\ln\theta) |\xi A_{\theta^k}|^{\pm\frac{2\beta}{mq'\gamma'}} \|h\|^2_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|^2_{L^q(\mathbf{S}^{n-1})} \quad (5)$$

hold for all $k \in \mathbf{Z}$*, where* $|\xi A_{\theta^k}|^{\pm\frac{2\beta}{mq'\gamma'}} = \inf\left\{|\xi A_{\theta^k}|^{+\frac{2\beta}{mq'\gamma'}}, |\xi A_{\theta^k}|^{-\frac{2\beta}{mq'\gamma')}}\right\}$. *The constant* C *is independent of* k *and* ξ.

Proof. We prove this lemma only for the case $1 < q \leq 2$, since $L^q\left(\mathbf{S}^{n-1}\right) \subseteq L^2\left(\mathbf{S}^{n-1}\right)$ for $q \geq 2$. By Hölder's inequality, we obtain that

$$\begin{aligned} |\hat{\sigma}_{\Omega,h,t}(\xi)| &\leq \int_{\frac{1}{2}t}^{t} |h(\rho)| \left| \int_{\mathbf{S}^{n-1}} e^{-iA_\rho x\cdot\xi}\, \Omega(x)\, J(x) d\sigma(x)\right| \frac{d\rho}{\rho} \\ &\leq \|h\|_{\Delta_\gamma(\mathbf{R}^+)} \left(\int_{\frac{1}{2}t}^{t} \left| \int_{\mathbf{S}^{n-1}} e^{-iA_\rho \xi\cdot x}\Omega(x)\, J(x) d\sigma(x)\right|^{\gamma'} \frac{d\rho}{\rho}\right)^{1/\gamma'}. \end{aligned}$$

On one hand, if $1 < \gamma \leq 2$, then by a change of variable we get that

$$\begin{aligned} |\hat{\sigma}_{\Omega,h,t}(\xi)| &\leq \|h\|_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|^{(1-2/\gamma')}_{L^1(\mathbf{S}^{n-1})} \left(\int_{\frac{1}{2}t}^{t} \left| \int_{\mathbf{S}^{n-1}} e^{-iA_\rho \xi\cdot x}\Omega(x) J(x) d\sigma(x)\right|^{2} \frac{d\rho}{\rho}\right)^{1/\gamma'} \\ &\leq C\|h\|_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|^{(1-2/\gamma')}_{L^1(\mathbf{S}^{n-1})} \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \Omega(x)\overline{\Omega(y)} I(\xi,x,y) d\sigma(x)\, d\sigma(y)\right)^{1/\gamma'}, \end{aligned}$$

where $I(\xi,x,y) = \int_1^2 e^{-iA_{\frac{t}{2}\rho}\xi\cdot(x-y)} \frac{d\rho}{\rho}$. Let $\eta = \frac{A_{\frac{t}{2}}\xi}{|A_{\frac{t}{2}}\xi|}$. Since $\beta < m/\alpha$, then by lemma 4 we get that

$$\begin{aligned} I(\xi,x,y) &\leq C \mid A_{\frac{t}{2}}\xi\cdot(x-y)\mid^{-\beta/m} = C2^{\alpha\beta/m} \mid (\mid \eta\cdot(x-y)\mid\mid A_t\xi \mid)^{-\beta/m} \\ &\leq C \mid A_t\xi \mid^{-\beta/m} (\mid \eta\cdot(x-y)\mid)^{-\beta/m}, \end{aligned}$$

which leads to

$$|\hat{\sigma}_{\Omega,h,t}(\xi)| \leq C\|h\|_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|^{(1-2/\gamma')}_{L^1(\mathbf{S}^{n-1})}$$

$$\times \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \Omega(x)\overline{\Omega(y)}\,(|\,\eta\cdot(x-y)\,||\,A_t\xi\,|)^{-\beta/m}\,d\sigma\,(x)\,d\sigma(y) \right)^{1/\gamma'}$$

$$\le C\,\|h\|_{\Delta_\gamma(\mathbf{R}^+)}\,\|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(1-2/\gamma')}\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}^{2/\gamma'}\,|\,A_t\xi\,|^{-\beta/(q'm\gamma')}$$

$$\times \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} (|\,\eta\cdot(x-y)\,|)^{-\beta q'/m}\,d\sigma\,(x)\,d\sigma(y) \right)^{1/(q'\gamma')}.$$

As $0<\beta<\frac{m}{2q'}$, we get that the last integral is finite, and hence

$$\begin{aligned} |\hat{\sigma}_{\Omega,h,t}(\xi)| &\le C\,\|h\|_{\Delta_\gamma(\mathbf{R}^+)}\,\|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(1-2/\gamma')}\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}^{2/\gamma'}\,|\,A_t\xi\,|^{-\beta/(q'm\gamma')} \\ &\le C\,\|h\|_{\Delta_\gamma(\mathbf{R}^+)}\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\,|\,A_t\xi\,|^{-\beta/(q'm\gamma')}. \end{aligned}$$

Thus, we derive

$$\int_{\theta^k}^{\theta^{k+1}} |\hat{\sigma}_{\Omega,h,t}(\xi)|^2\,\frac{dt}{t} \le C(\ln\theta)\,|\,A_{\theta^k}\xi\,|^{-\frac{2\beta}{q'm\gamma'}}\,\|h\|_{\Delta_\gamma(\mathbf{R}^+)}^2\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2. \tag{6}$$

On the other hand, if $\gamma>2$, then by using Hölder's inequality, we get that

$$|\hat{\sigma}_{\Omega,h,t}(\xi)| \le C\,\|h\|_{\Delta_\gamma(\mathbf{R}^+)} \left(\int_{\frac{1}{2}t}^{t} \left| \int_{\mathbf{S}^{n-1}} e^{-iA_\rho x\cdot\xi}\Omega(x)d\sigma\,(x) \right|^2 \frac{d\rho}{\rho} \right)^{1/2}$$

$$\le \|h\|_{\Delta_\gamma(\mathbf{R}^+)} \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \Omega(x)\overline{\Omega(y)} \int_{\frac{1}{2}}^{1} e^{-iA_{t\rho}(x-y)\cdot\xi}\frac{d\rho}{\rho}d\sigma\,(x)\,d\sigma\,(y) \right)^{1/2}.$$

Following the above procedure give

$$|\hat{\sigma}_{\Omega,h,t}(\xi)| \le C\,|\,A_t\xi\,|^{-\beta/(q'm\gamma')}\,\|h\|_{\Delta_\gamma(\mathbf{R}^+)}\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})},$$

and therefore,

$$\int_{\theta^k}^{\theta^{k+1}} |\hat{\sigma}_{\Omega,h,t}(\xi)|^2\,\frac{dt}{t} \le C(\ln\theta)\,|\,A_{\theta^k}\xi\,|^{-2\beta/(q'm\gamma')}\,\|h\|_{\Delta_\gamma(\mathbf{R}^+)}^2\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2. \tag{7}$$

To prove the other estimate in Lemma 6, we use the cancelation property of Ω.

$$\int_{\theta^k}^{\theta^{k+1}} |\hat{\sigma}_{\Omega,h,t}(\xi)|^2 \frac{dt}{t} \leq \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \int_{\mathbf{S}^{n-1}} |\Omega(x)|\, |J(x)|^2$$

$$\times \int_{\theta^k}^{\theta^{k+1}} \left| \int_{\frac{1}{2}}^{1} |e^{-iA_{t\rho}\xi\cdot x} - 1||h(t\rho) \frac{d\rho}{\rho} \right|^2 \frac{dt}{t} d\sigma\,(x)$$

$$\leq C \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \int_{\mathbf{S}^{n-1}} |\Omega(x)| \int_{1}^{\theta} \left| \int_{\frac{1}{2}}^{1} |A_{\theta^k t\rho}\xi \cdot x||h(\theta^k t\rho) \frac{d\rho}{\rho} \right|^2 \frac{dt}{t} d\sigma\,(x)$$

$$\leq C \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \int_{\mathbf{S}^{n-1}} |\Omega(x)| \int_{1}^{\theta} \left| \int_{\frac{1}{2}}^{1} |A_{\theta^k \rho}\xi||h(\theta^k t\rho) \frac{d\rho}{\rho} \right|^2 \frac{dt}{t} d\sigma\,(x)\,.$$

Since $\gamma > 1$ and $\frac{1}{2} < \rho < 1$, we obtain that

$$\int_{\theta^k}^{\theta^{k+1}} |\hat{\sigma}_{\Omega,h,t}(\xi)|^2 \frac{dt}{t} \leq C(\ln\theta) \mid A_{\theta^k}\xi \mid^2 \|h\|^2_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|^2_{L^q(\mathbf{S}^{n-1})}\,,$$

which when combine with the trivial estimate

$$\int_{\theta^k}^{\theta^{k+1}} |\hat{\sigma}_{\Omega,h,t}(\xi)|^2 \frac{dt}{t} \leq C(\ln\theta) \|h\|^2_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|^2_{L^q(\mathbf{S}^{n-1})}$$

provides

$$\int_{\theta^k}^{\theta^{k+1}} |\hat{\sigma}_{\Omega,h,t}(\xi)|^2 \frac{dt}{t} \leq C(\ln\theta) \mid A_{\theta^k}\xi \mid^{\frac{2\beta}{mq'\gamma'}} \|h\|^2_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|^2_{L^q(\mathbf{S}^{n-1})}\,. \tag{8}$$

The proof is complete. □

Lemma 7. *Let $\Omega \in L^q\left(\mathbf{S}^{n-1}\right)$ for some $1 < q \leq 2$ and $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $\gamma > 1$. Then for any $f \in L^p(\mathbf{R}^n)$ with $\gamma' < p \leq \infty$, there exists a constant C_p (independent of Ω, h and f) such that*

$$\|\sigma^*_{\Omega,h} f(x)\|_{L^p(\mathbf{R}^n)} \leq C_p \|h\|_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbf{R}^n)}. \tag{9}$$

Proof. By Hölder's inequality, we have

$$\begin{aligned} ||\sigma_{\Omega,h,t}| * f(x)| &\leq \|h\|_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{1/\gamma} \\ &\times \sup_{t\in\mathbf{R}^+} \left(\frac{1}{t} \int_{\frac{t}{2}}^{t} \int_{\mathbf{S}^{n-1}} J|(y)||\Omega(y)||f(x - A_\rho y)|^{\gamma'} d\sigma(y) d\rho \right)^{1/\gamma'}. \end{aligned}$$

Using Minkowski's inequality for integrals gives

$$\begin{aligned} \|\sigma^*_{\Omega,h} f(x)\|_{L^p(\mathbf{R}^n)} &\leq C\|h\|_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{1/\gamma} \\ &\times \left(\int_{\mathbf{S}^{n-1}} |\Omega(y)| \left(\|\mathcal{M}_\Gamma \left(|f|^{\gamma'} \right) \|_{L^{p/\gamma'}(\mathbf{R}^n)} \right) d\sigma(y) \right)^{1/\gamma'}. \end{aligned}$$

Consequently, by using Lemma 3, we finish the proof of lemma 7. □

Lemma 8. *Let $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $1 < \gamma \leq 2$, $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \leq 2$ and $\theta = 2^{q'\gamma'}$. Then for any p satisfying $|1/p - 1/2| < 1/\gamma'$ and $f \in L^p(\mathbf{R}^n)$, there is a positive constant C_p such that*

$$\begin{aligned} &\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} |\sigma_{\Omega,h,t} * g_k|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^p(\mathbf{R}^n)} \\ &\qquad \leq C_p \frac{\|h\|_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})}}{[(q-1)(\gamma-1)]^{1/2}} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^n)} \end{aligned}$$

holds for arbitrary functions $\{g_k(\cdot), k \in \mathbf{Z}\}$ on $\mathbf{R}^n$.

Proof. We employ some ideas from [1], [4] and [14]. By Schwarz's inequality, we obtain

$$\begin{aligned} &|\sigma_{\Omega,h,t} * g_k(x)|^2 \leq C \|h\|_{\Delta_\gamma(\mathbf{R}^+)}^\gamma \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \\ &\quad \times \left(\int_{\frac{1}{2}t}^{t} \int_{\mathbf{S}^{n-1}} |g_k(x - A_\rho y)|^2 |\Omega(y)| J(y) |h(\rho)|^{2-\gamma} d\sigma(y) \frac{d\rho}{\rho} \right). \qquad (10) \end{aligned}$$

Let us first prove this lemma for the case $2 \le p < \frac{2\gamma}{2-\gamma}$. By duality, there is a non-negative function $\psi \in L^{(p/2)'}(\mathbf{R}^n)$ with $\|\psi\|_{L^{(p/2)'}(\mathbf{R}^n)} \le 1$ such that

$$\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} |\sigma_{\Omega,h,t} * g_k|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbf{R}^n)}^2 = \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} |\sigma_{\Omega,h,t} * g_k(x)|^2 \frac{dt}{t} \psi(x) dx.$$

By this, equation (10) and a change of variable, we derive

$$\begin{aligned} &\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} |\sigma_{\Omega,h,t} * g_k|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbf{R}^n)}^2 \\ \le\ & C \|h\|_{\Delta_\gamma(\mathbf{R}^+)}^{\gamma} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \int_{\mathbf{R}^n} \left(\sum_{k\in\mathbf{Z}} |g_k(x)|^2 \right) M_{\Omega,|h|^{2-\gamma},\theta} \widetilde{\psi}(-x) dx, \end{aligned}$$

where $\widetilde{\psi}(x) = \psi(-x)$. Since $h \in \Delta_\gamma(\mathbf{R}^+)$, then $|h(\cdot)|^{2-\gamma} \in \Delta_{\frac{\gamma}{2-\gamma}}(\mathbf{R}^+)$, and since $\left(\frac{p}{2}\right)' > \left(\frac{\gamma}{2-\gamma}\right)'$, then by Lemma 7, Hölder's inequality and the same arguments used in [4], we achieve that

$$\begin{aligned} &\left\| \left(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} |\sigma_{\Omega,h,t} * g_k|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbf{R}^n)}^2 \\ \le\ & C \|h\|_{\Delta_\gamma(\mathbf{R}^+)}^{\gamma} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^n)}^2 \left\| M_{\Omega,|h|^{2-\gamma},\theta} \widetilde{\psi} \right\|_{L^{(p/2)'}(\mathbf{R}^n)} \\ \le\ & C \ln(\theta) \|h\|_{\Delta_\gamma(\mathbf{R}^+)}^{\gamma} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^n)}^2 \left\| {\sigma^*}_{\Omega,|h|^{2-\gamma}} \widetilde{\psi} \right\|_{L^{(p/2)'}(\mathbf{R}^n)} \\ \le\ & C_p \frac{\|h\|_{\Delta_\gamma(\mathbf{R}^+,\frac{dt}{t})}^2 \|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2}{(q-1)(\gamma-1)} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^n)}^2 . \end{aligned}$$

For the case $\frac{2\gamma}{3\gamma-2} < p < 2$; by the duality, there are functions $\zeta = \zeta_k(x,t)$ defined on $\mathbf{R}^n \times \mathbf{R}^+$ with $\left\| \left\| \|\zeta_k\|_{L^2([\theta^k,\theta^{k+1}],\frac{dt}{t})} \right\|_{l^2} \right\|_{L^{p'}(\mathbf{R}^n)} \le 1$ such that

$$\left\|\left(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}\left|\sigma_{\Omega,h,t}*g_k\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}\right\|_{L^p(\mathbf{R}^n)}=\int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}\left(\sigma_{\Omega,h,t}*g_k(x)\right)\zeta_k(x,t)\frac{dt}{t}dx$$

$$\leq C_p\frac{\left\|(\Upsilon(\zeta))^{1/2}\right\|_{L^{p'}(\mathbf{R}^n)}}{\left[(q-1)(\gamma-1)\right]^{1/2}}\left\|\left(\sum_{k\in\mathbf{Z}}|g_k|^2\right)^{1/2}\right\|_{L^p(\mathbf{R}^n)}, \quad (11)$$

where

$$\Upsilon\zeta(x)=\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}\left|\sigma_{\Omega,h,t}*\zeta_k(x,t)\right|^2\frac{dt}{t}.$$

As $\frac{p'}{2}>1$, then by applying the above procedure, we reach that

$$\begin{aligned}\|\Upsilon(\zeta)\|_{L^{(p'/2)}(\mathbf{R}^n)}\leq & C\|h\|^{\gamma}_{\Delta_\gamma(\mathbf{R}^+)}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\left\|\left(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}\left|\zeta_k(\cdot,t)\right|^2\frac{dt}{t}\right)\right\|_{L^{(p'/2)}(\mathbf{R}^n)}\\ & \times\left\|{\sigma^*}_{\Omega,|h|^{2-\gamma}}(\varsigma)\right\|_{L^{(p'/2)'}(\mathbf{R}^n)}\\ \leq & C\,\|h\|^2_{\Delta_\gamma(\mathbf{R}^+)}\|\Omega\|^2_{L^q(\mathbf{S}^{n-1})} \qquad (12)\end{aligned}$$

where ς is a function in $L^{(p'/2)'}(\mathbf{R}^n)$ with $\|\varsigma\|_{L^{(p'/2)'}(\mathbf{R}^n)}\leq 1$. Thus, by equations (11) and (12), our estimate holds for $\frac{2\gamma}{3\gamma-2}\leq p<2$; and therefore the proof of Lemma 8 is complete. □

In the same manner, we achieve the following lemma.

Lemma 9. *Let $h\in\Delta_\gamma\left(\mathbf{R}^+\right)$ for some $\gamma\geq 2$, $\Omega\in L^q\left(\mathbf{S}^{n-1}\right)$ for some $1<q\leq 2$ and $\theta=2^{q'\gamma'}$. Then for any $p\in(1,\infty)$ and $f\in L^p(\boldsymbol{R}^n)$, there exists a constant C_p such that*

$$\left\|\left(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}\left|\sigma_{\Omega,h,t}*g_k\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}\right\|_{L^p(\mathbf{R}^n)}$$

$$\leq C_p\frac{\gamma^{1/2}\|h\|_{\Delta_\gamma(\mathbf{R}^+)}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}}{(q-1)^{1/2}}\left\|\left(\sum_{k\in\mathbf{Z}}|g_k|^2\right)^{1/2}\right\|^2_{L^p(\mathbf{R}^n)}$$

holds for arbitrary functions $\{g_k(\cdot),k\in\mathbf{Z}\}$ on $\mathbf{R}^n$.

3. Proof of Theorem 1

We prove Theorem 1 by applying the same approaches that Al-Qassem [1] as well as Fan and Pan [14] used. Let us first assume that $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $1 < \gamma \leq 2$. Then by Minkowski's inequality, we get that

$$\begin{aligned}
\mathcal{M}^{\tau}_{\Omega,h} f(x) &= \left(\int_0^\infty \left| \sum_{k=0}^{\infty} t^{-\tau} \int_{2^{-k-1}t<\rho(u)\leq 2^{-k}t} f(x-u) \frac{\Omega(u)h(\rho(u))}{\rho(u)^{\alpha-\tau}} du \right|^2 \frac{dt}{t} \right)^{1/2} \\
&\leq \sum_{k=0}^{\infty} \left(\int_0^\infty \left| t^{-\tau} \int_{2^{-k-1}t<\rho(u)\leq 2^{-k}t} f(x-u) \frac{\Omega(u)h(\rho(u))}{\rho(u)^{\alpha-\tau}} du \right|^2 \frac{dt}{t} \right)^{1/2} \\
&= \frac{2^a}{2^a-1} \left(\int_0^\infty \left| t^{-\tau} \int_{1/2t<\rho(u)\leq t} f(x-u) \frac{\Omega(u)h(\rho(u))}{\rho(u)^{\alpha-\tau}} du \right|^2 \frac{dt}{t} \right)^{1/2} \qquad (13)
\end{aligned}$$

Take $\theta = 2^{q'\gamma'}$, and choose a $C_0^\infty(\mathbf{R})$ function, $0 \leq \varphi \leq 1$, satisfying supp $\varphi \subset \{y : [1/(2q'\gamma') \leq \rho(y) < (2q'\gamma')\}$, and $\sum_k \varphi(\theta^k \rho(x)) = 1$, $x \in \mathbf{R}^n \setminus \{0\}$. Define $\Psi_k \in C^\infty(\mathbf{R}^n)$ by $\widehat{\Psi_k}(\xi) = \varphi(\theta^k \rho(\xi))$. Decompose $\sigma_{\Omega,h,t} * f(x) = \sum_{j\in\mathbf{Z}} Y_{\Omega,h,j,}(x,t)$, where

$$Y_{\Omega,h,j}(x,t) = \sum_{k\in\mathbf{Z}} \sigma_{\Omega,h,t} * \Psi_{k+j} * f(x) \chi_{[\theta^k,\theta^{k+1})}(t).$$

Define $S_{\Omega,h,j}(f)(x) = \left(\int\limits_0^\infty |Y_{\Omega,h,j}(x,t)|^2 \frac{dt}{t} \right)^{1/2}$. Then for any $f \in \mathcal{S}(\mathbf{R}^n)$,

$$\left(\int_0^\infty |\sigma_{\Omega,h,t} * f(x)|^2 \frac{dt}{t} \right) \leq \sum_{j\in\mathbf{Z}} S_{\Omega,h,j}(f)(x). \qquad (14)$$

Let us first compute the L^2-norm of $S_{\Omega,h,j}(f)$. By using Plancherel's theorem, Lemma 6 and the process used in [10], we obtain that

$$\begin{aligned}
\|S_{\Omega,h,j}(f)\|^2_{L^2(\mathbf{R}^n)} &\leq \sum_{k\in\mathbf{Z}} \int_{\Gamma_{k+j}} \left(\int\limits_{\theta^k}^{\theta^{k+1}} |\hat{\sigma}_{\Omega,h,t}(\xi)|^2 \frac{dt}{t} \right) \left| \hat{f}(\xi) \right|^2 d\xi \\
&\leq C_p (\ln\theta) \|h\|^2_{\Delta_\gamma(\mathbf{R}^+)} \|\Omega\|^2_{L^q(\mathbf{S}^{n-1})} 2^{-\varepsilon|j|} \sum_{k\in\mathbf{Z}} \int_{\Gamma_{k+j}} \left| \hat{f}(\xi) \right|^2 d\xi
\end{aligned}$$

$$\leq \quad C_p(\ln\theta)\left\|h\right\|^2_{\Delta_\gamma(\mathbf{R}^+)}\left\|\Omega\right\|^2_{L^q(\mathbf{S}^{n-1})}2^{-\varepsilon|j|}\left\|f\right\|^2_{L^2(\mathbf{R}^n)}$$

for some $0<\varepsilon<1$, where $\Gamma_{k+j}=[\theta^{-(j+k+1)},\theta^{-(j+k-1)}]$. Thus,

$$\left\|S_{\Omega,h,j}(f)\right\|_{L^2(\mathbf{R}^n)}\leq C_p\frac{\left\|h\right\|_{\Delta_\gamma(\mathbf{R}^+)}\left\|\Omega\right\|_{L^q(\mathbf{S}^{n-1})}}{\left[(q-1)(\gamma-1)\right]^{1/2}}2^{\frac{-\varepsilon|j|}{2}}\left\|f\right\|_{L^2(\mathbf{R}^n)}. \tag{15}$$

Now, let us compute the L^p-norm of $S_{\Omega,h,j}(f)$ for any p satisfying $\left|\frac{1}{p}-\frac{1}{2}\right|<\frac{1}{\gamma'}$ with $p\neq 2$. Applying Lemma 5 plus Lemma 8, we obtain

$$\left\|S_{\Omega,h,j}(f)\right\|_{L^p(\mathbf{R}^n)}\leq C_p\frac{\left\|h\right\|_{\Delta_\gamma(\mathbf{R}^+)}\left\|\Omega\right\|_{L^q(\mathbf{S}^{n-1})}}{\left[(q-1)(\gamma-1)\right]^{1/2}}\left\|f\right\|_{L^p(\mathbf{R}^n)}. \tag{16}$$

By interpolation between (15) and (16) we reach that there exists a constant $0<\mu<1$ such that

$$\left\|S_{\Omega,h,j}(f)\right\|_{L^p(\mathbf{R}^n)}\leq C_p 2^{\frac{-\mu|j|}{2}}\frac{\left\|h\right\|_{\Delta_\gamma(\mathbf{R}^+)}\left\|\Omega\right\|_{L^q(\mathbf{S}^{n-1})}}{\left[(q-1)(\gamma-1)\right]^{1/2}}\left\|f\right\|_{L^p(\mathbf{R}^n)}. \tag{17}$$

Consequently, by (13), (14) and (17), we get our result for the case $h\in\Delta_\gamma(\mathbf{R}^+)$ for some $1<\gamma\leq 2$.

The proof of our theorem for the case $h\in\Delta_\gamma(\mathbf{R}^+)$ for some $\gamma\geq 2$ is obtained by following the above argument except, we need to invoke Lemma 9 instead of Lemma 8. Therefore, the proof of Theorem 1 is complete.

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