BASIC PROPERTIES OF LINEAR RELATIVE $n$-WIDTHS

Sergei P. Sidorov
Department of Mechanics and Mathematics
Saratov State University
Saratov, RUSSIAN FEDERATION

Abstract: The paper examines the basic properties of linear relative widths as well as the different ways of finding their estimates.

AMS Subject Classification: 41A35, 41A29
Key Words: shape preserving approximation, linear $n$-widths

1. Introduction

The theory of shape-preserving approximation is one of the rapidly developing areas of the approximation theory. Results of this theory are actively used in computational mathematics, the theory of functions, computer-aided geometric design. This theory studies approximation properties of the different methods for approximation of functions preserving its shape properties (monotonicity, convexity). The most significant results were gathered in [1], [2]. Note that if a function $f$ has some shape properties, it usually means that element $f$ belongs to a cone (a convex set, closed under nonnegative scalar multiplication).

Definitions of relative linear widths were introduced in papers [3], [4], [5]. In this paper we are going to examine basic properties of this quantities. The paper organized into five sections. In section 2 one can find two different definitions of linear relative width, in the senses of Konovalov and Korovkin. In Section 3 one can find the list of basic properties of Konovalov linear relative widths. The
main question of this section how different the properties of Konovalov linear relative widths from linear width are. Section 4 describes basic properties of Korovkin linear relative width. Both in Section 3 and Section 4 we investigate different ways of finding the lower estimations of linear relative widths.

2. Definitions of Linear Relative Width

A nonempty subset $V$ of a linear space is said to be a cone if it satisfies the following properties:

1. $V$ is closed with respect to taking sums of its elements: if $f, g \in V$, then $f + g \in V$;
2. $V$ is a conic set: if $f \in V$, $\lambda \geq 0$, then $\lambda f \in V$.

First we remind the definition of relative width given by Konovalov in [6].

**Definition 1.** Let $X$ be a normed linear space, $V$ be a cone of $X$, $A \cap V$ be a nonempty subset of $X$. Relative $n$-width of $A \cap V$ in $X$ with the constraint $V$ is defined by

$$d_n(A \cap V, V)_X = \inf_{X_n} \sup_{f \in A \cap V} \inf_{g \in X_n \cap V} \|f - g\|_X,$$

the left-most infimum is taken over all affine subsets $X_n$ of dimension $\leq n$, such that $X_n \cap V \neq \emptyset$.

The definition of relative $n$-width was first introduced in 1984 by Konovalov [6]. Though he considered a problem not connected with preserving shapes, the concept of relative $n$-width arises in the theory of shape-preserving approximation naturally. Of course, it is impossible to obtain $d_n(A \cap V, V)_X$ and determine optimal subspaces $X_n$ (if they exist) for all $A, V, X$. Nevertheless, some estimations of relative shape-preserving $n$-widths have been obtained in papers [7], [8], [9].

Let $L : X \to X$ be a linear operator and $V$ be a cone in $X$, $V \neq \emptyset$. We will say that the operator $L$ is shape-preserving operator relative to the cone $V$, if $L(V) \subset V$.

Let $A$ be a subset of $X$ and $L : X \to X$. The value

$$e(A, L) := \sup_{f \in A} \|f - Lf\|_X = \sup_{f \in A} \|(I - L)f\|_X$$

is the error of approximation of identity operator $I$ by the operator $L$ on the set $A$. 

One might consider the problem of finding (if exists) a linear operator of finite rank $n$, which gives the minimal error of approximation of identity operator on some set over all finite rank $n$ linear operators $L$ preserving the shape in the sense $V$. It leads us naturally to the notion of linear relative $n$-width.

The definition of linear relative $n$-width was introduced in the papers [3], [4], [5]. The paper [3] finds estimates of linear relative $n$-widths for linear operators preserving an intersection of cones of $p$-monotonicity functions.

**Definition 2.** A linear operator $L : X \to X$ is called the operator with finite rank $n$, if the dimension of linear subspace $L(X)$ is equals to $n$, $\dim\{L(X)\} = n$.

Two definitions of linear relative widths are presented below. The first one is based on Konovalov’s ideas [6].

**Definition 3.** Let $X$ be a linear normed space. Let $V$ be a cone in $X$. Konovalov linear $n$-width of a set $A \cap V \subset X$ in $X$ with constraint $V$ is defined by [5]

$$\delta_n(A \cap V, V)_X := \inf_{L_n(V) \subset V} e(A \cap V, L_n)_X,$$

where infimum is taken over all linear continuous operators $L_n$ such that $L_n : X \to X$ is of finite rank $n$ and $L_n(V) \subset V$.

To determine the negative impact of the property of shape-preserving on the order of linear approximation, the following definition based on ideas of Korovkin was introduced in [3].

**Definition 4.** Let $X$ be a linear normed space. Let $V$ be a cone in $X$. Korovkin linear $n$-width of a set $A \subset X$ in the space $X$ with constraint $V$ is defined by

$$\delta_n(A, V)_X := \inf_{L_n(V) \subset V} e(A, L_n)_X,$$

where infimum is taken over all linear continuous operators $L_n$ such that $L_n : X \to X$ is of finite rank $n$ and $L_n(V) \subset V$.

Estimation of linear relative $n$-widths is of interest in the theory of shape-preserving approximation as, knowing the value of relative linear $n$-width, we can judge how good or bad (in terms of optimality) this or that finite-dimensional method preserving the shape in the sense $V$ is.
3. Basic Properties of Konovalov Linear Relative $n$-Widths

Let $X$ be a linear normed space, $A \subset X$, $A \neq \emptyset$. Let $\delta_n(A)_X$ denote the linear
$n$-width of the set $A$ in $X$,

$$\delta_n(A)_X := \inf_{L_n} e(A, L_n)_X,$$

where infimum is taken over all linear continuous operators $L_n : X \to X$ with
finite rank $n$.

The well-known basic properties of linear widths are listed below (see, for
example [10]).

**Proposition 1.** Let $X$ be a linear normed space, $A \subset X$, $A \neq \emptyset$. Then

1. $\delta_n(A)_X = \delta_n(\overline{A})_X$, where $\overline{A}$ denotes the closure of $A$.
2. For every $\alpha \in \mathbb{R}$, $\delta_n(\alpha A)_X = |\alpha| \delta_n(A)_X$.
3. If $b(A) := \{\alpha f : f \in A, |\alpha| \leq 1\}$ denotes the balanced hull of $A$, then
   $$\delta_n(b(A))_X = \delta_n(A)_X.$$  
4. $\delta_n(coA)_X = \delta_n(A)_X$.
5. $\delta_n(A)_X \geq \delta_{n+1}(A)_X$, $n = 0, 1, \ldots$.

As it will be shown in this section, some of the properties of linear widths
mentioned in Proposition 1 do not hold for Konovalov relative linear widths.

1. There exist $A, V, X, \alpha$, such that $\delta_n(\alpha A \cap V, V)_X \neq \alpha \delta_n(A \cap V, V)_X$ (Example 1).
2. There exist $A, V, X$, such that $\delta_n(b(A) \cap V, V)_X \neq \delta_n(A \cap V, V)_X$, (Example 2).
3. The inequality $\delta_n(coA \cap V, V)_X \neq \delta_n(A \cap V, V)_X$ holds for some choice of
   $X, A, V$ (Example 3).

First we will show that the analogue of property 2 of Proposition 1 does
not hold in the case of Konovalov linear widths.

**Example 1.** Let us consider $X = \mathbb{R}^2$ with Euclidean norm $|x| = \sqrt{x_1^2 + x_2^2}$, $x = (x_1, x_2)$. Let $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \leq 1, x_1, x_2 \geq 0\}$, $V = V_1 =
V = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$. Then $\delta_1(A \cap V, V)_X = \frac{1}{\sqrt{2}}$. We have
$-2A \cap V = \{(0,0)\}$ and $\delta_1(-2A \cap V, V)_X = 0.$
Example 2. Consider $X = \mathbb{R}^2$ with Euclidean norm $|x| = \sqrt{x_1^2 + x_2^2}$, $x = (x_1, x_2)$. Let $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1, x_1 + x_2 \geq 0\}$, $V = V = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 0, x_2 \leq 0\}$. Then $b(A) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$, $A \cap V = \{(0, 0)\}$, $b(A) \cap V = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 0, x_2 \leq 0, x_1^2 + x_2^2 \leq 1\}$, and $\delta_1(b(A) \cap V, V)_X = 2 - \sqrt{2}$, $\delta_1(A \cap V, V)_X = 0$.

Example 3. Consider $X = \mathbb{R}^2$ with Euclidean norm $|x| = \sqrt{x_1^2 + x_2^2}$, $x = (x_1, x_2)$. Let $A = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| + |x_2| \leq 1, x_1, x_2 \geq 0\} \cup \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| + |x_2| \leq 1, x_1, x_2 \leq 0\}$, $V = V = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \geq 0, x_1 \geq 0, x_2 \leq 0, x_1 - x_2 \leq 0\}$. Then $coA = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| + |x_2| \leq 1\}$, $A \cap V = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, x_2 = 0\}$, $coA \cap V = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \geq 0, x_1 \geq 0, x_2 \leq 0, x_1 - x_2 \leq 1\}$, and $\delta_1(coA \cap V, V)_X = 1/(2\sqrt{2})$, $\delta_1(A \cap V, V)_X = 0$.

We can establish the following properties of $\delta_n(A \cap V, V)_X$.

Theorem 1. Let $X$ be a linear normed space, $A \subset X$, $A \neq \emptyset$, $V \subset X$ is a non-empty cone. Then

1. $\delta_n(A \cap V, V)_X = \delta_n(\overline{A \cap V}, V)_X$, where $\overline{A \cap V}$ denotes the closure of $A \cap V$.

2. For every $\alpha \geq 0$, $\delta_n(\alpha A \cap V, V)_X = \alpha \delta_n(A \cap V, V)_X$.

3. If $b_+(A) := \{\alpha f : f \in A, \alpha \in [0, 1]\}$, then

$$\delta_n(b_+(A) \cap V, V)_X = \delta_n(A \cap V, V)_X.$$

4. $\delta_n(A \cap V, V)_X \geq \delta_{n+1}(A \cap V, V)_X$, $n = 0, 1, \ldots$.

Proof. It follows from the definition of cone that $\alpha V \in V$ for every $\alpha \geq 0$. Then 2) follows from equalities $\alpha A \cap V = \alpha A \cap \alpha V = \alpha(A \cap V)$ and definition 3.

Let us prove 3). Since $b_+(A) \supset A$, we have

$$\delta_n(b_+(A), V)_X \geq \delta_n(A, V)_X.$$

On the other hand, for every $f \in b_+(A)$, there exist $f^* \in A$ and $\alpha \in [0, 1]$, such that $f = \alpha f^*$. We can write $b_+(A) = \cup_{\alpha \in [0, 1]} b_\alpha(A)$, where $b_\alpha(A) := \{\alpha f : f \in A\}$. We have

$$\delta_n(b_+(A) \cap V, V)_X = \inf_{L_n(V) \subset V} \sup_{f \in b_+(A) \cap V} \|f - L_n f\|_X = \inf_{L_n(V) \subset V} \sup_{\alpha \in [0, 1]} \sup_{f \in b_\alpha(A) \cap V} \|f - L_n f\|_X =$$

$$= \inf_{L_n(V) \subset V} \sup_{\alpha \in [0, 1]} \sup_{f \in b_\alpha(A) \cap V} \|f - L_n f\|_X.$$
A continuous operator is a subspace of \( Y \) in the form \( L \).

Konovalov linear relative widths. holds for continuous linear operators of finite rank \( Y \) space \( \{0\} \subset X \) (not necessary a cone). The next example show that

\[ \text{Let } \alpha A \subset V, V \] \( \alpha \in \mathbb{R} \).

The following example shows that in the proposition 1) of Theorem 1 we can not write \( \overline{A} \cap V \) (instead of \( A \cap V \) as it is in Proposition 1).

**Example 4.** Let us consider \( X = R^3 \) with Euclidean norm

\[ |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad x = (x_1, x_2, x_3). \]

Let \( A = \{(x_1, x_2, x_3) \in R^3 : |x_1| + |x_2| + |x_3| \leq 1, \ x_1, x_2 \geq 0, \ x_3 > 0\} \cup \{(0, 0, 0)\}, \ V = V = \{(x_1, x_2, x_3) \in R^3 : x_1, x_2 \geq 0, x_3 = 0\}. \) Then \( \overline{A} \cap V = \{(x_1, x_2, x_3) \in R^3 : x_1, x_2 \geq 0, x_1 + x_2 \leq 1, \ x_3 = 0\} \), \( A \cap V = \{(0, 0, 0)\} \), and \( \delta_1(A \cap V, V)_X = 1/\sqrt{2}, \delta_1(A \cap V, V)_X = 0. \)

Note that the property 2) of Theorem 1 is not true if \( V \) is an arbitrary set (not necessary a cone). The next example show that \( \delta_1(\alpha A \cap V, V)_X \neq |\alpha| \delta_n(A \cap V, V)_X \), \( A, V \) and \( \alpha \in \mathbb{R} \).

**Example 5.** Let us consider \( X = R^2 \) with Euclidean norm \( |x| = \sqrt{x_1^2 + x_2^2}, \ x = (x_1, x_2) \).

Then \( \delta_1(A \cap V, V)_X = \delta_1(\alpha A \cap V, V)_X = 1/(2\sqrt{2}) \) for all \( \alpha \geq 1. \)

The next theorem points out several ways of finding lower estimations of Konovalov linear relative widths.

**Theorem 2.** If \( A \subset X \subset Y \), where \( X \) is the subspace of the normed linear space \( Y \), then

\[ \delta_n(A \cap V, V)_X \geq \delta_n(A \cap V, V)_Y. \] (1)

**Proof.** Every continuous linear operator \( L_n \) of finite rank \( n \) may be written in the form \( L_n f = \sum_{i=1}^{n} l_i(f) f_i \), where \( f_i \in X \ l_i \in X' \). The analogous property holds for continuous linear operators of finite rank \( n \), defined in \( Y \). Since \( X \) is a subspace of \( Y \), it follows from HahnBanach theorem that there exists a continuous operator \( \tilde{L}_n \) on \( Y \), for which \( \tilde{L}_n = L_n \) for all \( f \in X \) and for all \( f \in A. \)
Using on example [10, p. 10] it is possible to achieve a strong inequality in 2 for particular choices of \( X, A, V \).

Since linear cone-preserving approximation is not better than best cone-preserving approximation, we get the following proposition.

**Theorem 3.** Let \( X \) be a linear normed space and \( A \subset X, V \subset X \). Then

\[
\delta_n(A \cap V, V)_X \geq d_n(A \cap V, V)_X.
\]  

(2)

**Proof.** Denote \( X_n = \text{range } L_n \). Then

\[
\delta_n(A \cap V, V)_X = \inf_{L_n(V) \subset V} \sup_{f \in A \cap V} \| f - L_n f \|_X \geq \sup_{f \in A \cap V} \inf_{g \in X_n \cap V} \| f - g \|_X \geq d_n(A \cap V, V)_X.
\]

\[\square\]

It follows from 3 follows that one of the methods for estimation of Konovalov linear relative width \( \delta_n(A \cap V, V)_X \) from below is estimation of the corresponding relative \( n \)-width \( d_n(A \cap V, V)_X \).

The following example 6 shows that \( \delta_n(A \cap V, V)_X > d_n(A \cap V, V)_X \) for some choice of \( A, V \) in \( X = L^2_{2\pi}(R) \), where \( L^2_{2\pi}(R) \) denotes the Banach space of all (equivalence classes of) functions \( f : R \to R \) that are Lebesgue integrable to the 2-th power over \([-\pi, \pi]\) and that satisfy \( f(x+2\pi) = f(x) \) for a.e. \( x \in R \).

The space \( L^2_{2\pi}(R) \) is endowed with the norm

\[
\| f \|_2 = \frac{1}{2\pi} \left( \int_{-\pi}^{\pi} f^2(x) dx \right)^{\frac{1}{2}}.
\]

**Example 6.** Let \( X = L^2_{2\pi}(R), P := \text{span}\{1, \cos x, \sin x\}, A = \{ f \in P : \| f \|_2 \leq 1 \} \) and \( V = V = \{ f \in L^2_{2\pi}(R) : f \geq 0 \} \). Then every linear operator \( L_n \) of finite rank \( n \), such that \( L_n(V) \subset V \), may be written in the form

\[
L_n f = \sum_{i=1}^{n} \langle a_i, f \rangle g_i,
\]

where \( a_i, g_i \in L^2_{2\pi}(R), i = 1, \ldots, n, \) are a.e. non-negative functions.

It is obvious that \( d_n(A \cap V, V)_{L^2_{2\pi}(R)} = 0 \).

Our aim is to show that the set of all linear operators satisfying

1. \( \| f - L_n f \|_2 = 0 \) for all \( f \in P \);
2. $L_n(V) \subset V$

is empty. Denote $h_t(x) = \sin^2 \frac{t-x}{2}$. Then $h_t \in A$ and

$$\|h_t - L_n h\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( h_t(x) - \sum_{i=1}^{n} \langle a_i, h_t \rangle g_i(x) \right)^2 dx = 0.$$  

Thus

$$h_t = \sum_{i=1}^{n} \langle a_i, h_t \rangle g_i$$

It follows from $h_t(t) = 0$ follows that $\sum_{i=1}^{n} \langle a_i, h_t(t) \rangle = 0$. It is imposable for all $t \in [-\pi, \pi]$ because of $a_i, g_i \geq 0$ a.e.

**Theorem 4.** Let $X$ be a linear normed space and $V$ be a cone in $X$. Then

$$\delta_n(A \cap V, V)_X \geq \delta_n(A \cap V)_X,$$

where $\delta_n(A \cap V)_X$ denotes the linear $n$-width of $A \cap V$ in $X$.

**Proof.** We have

$$\delta_n(A \cap V, V)_X = \inf_{L_n(V) \subset V} \sup_{f \in A \cap V} \|f - L_n f\|_X \geq \inf_{L_n} \sup_{f \in A \cap V} \|f - L_n f\|_X,$$

where infimum is taken over all continuous linear operators $L_n : X \to X$ with finite rank $n$. The last quantity is the linear $n$-width of $A \cap V$ in $X$.

Note that if $V = X$ then relative $n$-width of $A$ in $X$ is equal to $n$-width of $A$ in $X$ for every $A$, i.e. $\delta_n(A \cap X, X)_X = \delta_n(A \cap X)_X = \delta_n(A)_X$.

The example 7 shows that there is a choice of $X, A, V$ for which strong inequality holds in (3).

**Example 7.** Let $X = R^4$, $\|y\|_X = \max_{0 \leq i \leq 3} |y_i|$, $y = (y_0, y_1, y_2, y_3)$, and let $V = V = R^4_+$ is non-negative octant of $R^4$. Denote $e_0(x) = 1$, $e_1(x) = x$, $e_2(x) = x^2$, $x \in R$. For $f \in C(R)$ denote $If = (f(0), f(1), f(2), f(3))$. Let $A = \{If : f = \sum_{j=0}^{2} a_j e_j, |a_2| \leq 1 \} \subset R^4$.

Consider the linear operator $L : R^4 \to R^4$ of finite rank 3, defined by

$$Ly = y_0 Ie_0 + \frac{1}{6} (9y_1 - 8y_0 - y_3) Ie_1 + \frac{1}{6} (2y_0 - 3y_1 + y_3) Ie_2,$$

$$y = (y_0, y_1, y_2, y_3). \quad (4)$$

Note that range range $L = \text{span}\{Ie_0, Ie_1, Ie_2\}$ and dimension of range range $L$ are equal to 3.
It follows from equalities \((Ly)_i = y_i, i = 0, 1, 3\), that \(Ly = y\) for all \(y \in A\). As \(\|y - Ly\|_X = 0\) for all \(y \in A\), we have \(\delta_3(A \cap V)_X = 0\).

It is worth noting that operator \(L\) defined above does not satisfy the shape-preserving property \(L(V) \subset V\). To prove this let us consider vectors \(y = (1, 0, 0, 0) \in V\) and \(z = Ie_0 - \frac{4}{3}Ie_1 + \frac{4}{3}Ie_2 = (1, 0, -\frac{1}{3}, 0)\). We have \(Ly = L(z + (0, 0, \frac{1}{3}, 0)) = Lz + L(0, 0, \frac{1}{3}, 0) = (1, 0, -\frac{1}{3}, 0) + (0, 0, 0, 0) = (1, 0, -\frac{1}{3}, 0) \notin V\).

Moreover, there does not exist a linear operator \(L : \mathbb{R}^4 \rightarrow \mathbb{R}^4\) with finite rank 3, such that \(Ly = y\) for all \(y \in A\) and for which the shape-preserving property \(L(V) \subset V\) holds.

Suppose that such operator \(L\) exists. Then \(L\) may be presented in the form \(L(y) = \sum_{i=0}^2 a_i(y)Ie_i\). Since \(L(Ie_0) = Ie_0, L(Ie_1) = Ie_1, L(Ie_2) = Ie_2\), we have

\[
\begin{align*}
& a_0e_0(y_0) + a_1e_0(y_1) + a_2e_0(y_3) = e_0(y_2) \\
& a_0e_1(y_0) + a_1e_1(y_1) + a_2e_1(y_3) = e_1(y_2) \\
& a_0e_2(y_0) + a_1e_2(y_1) + a_2e_2(y_3) = e_2(y_2)
\end{align*}
\]

Let us consider the vector \(y^* = I(y_2^2e_0 - 2y_2e_1 + e_2)\). We have

\[
y^*_i = \begin{cases} 
0, & i = 2 \\
> 0, & i = 0, 1, 3.
\end{cases}
\]

It follows from (5) that \(a_0y_0^* + a_1y_1^* + a_2y_2^* = 0\), and therefore one of the coefficients \(a_0, a_1, a_2\) is strongly less than 0, i.e. \(L\) is not positive and the property \(L(V) \subset V\) does not hold. Thus, \(\delta_3(A \cap V, V) \neq 0\).

All inequalities listed in Theorems 2, 3, 4 can not be improved. Easiest way to show this is to set \(X = Y\) in (1), \(V = X = R^2\) in (2), \(V = X\) in (3).

4. Basic Properties of Korovkin Relative Linear \(n\)-Widths

Some basic properties of Korovkin relative linear \(n\)-widths \(\delta_n(A,V)_X\) which follows from the definition are listed below.

**Theorem 5.** Let \(X\) be a linear normed space, \(A \subset X, A \neq \emptyset, V \subset X\) be a cone in \(X\). Then

1. \(\delta_n(A,V)_X = \delta_n(\overline{A},V)_X\), where \(\overline{A}\) denotes the closure of \(A\).

2. For every \(\alpha \in \mathbb{R}\), \(\delta_n(\alpha A,V)_X = |\alpha|\delta_n(A,V)_X\).
3. Let $b(A)$ denotes the balanced hull of $A$, then

$$\delta_n(b(A), V)_X = \delta_n(A, V)_X.$$ 

4. $\delta_n(coA, V)_X = \delta_n(A, V)_X$.

5. $\delta_n(A, V)_X \geq \delta_{n+1}(A, V)_X$, $n = 0, 1, \ldots$

Analogues of theorems 2, 3, 4 are holds and can be find below.

**Theorem 6.** If $A \subset X \subset Y$, where $X$ is the subspace of the normed linear space $Y$, then

$$\delta_n(A, V)_X \geq \delta_n(A, V)_Y.$$

**Theorem 7.** Let $X$ be a linear normed space and $A \subset X$, $V \subset X$. Then

$$\delta_n(A, V)_X \geq d_n(A, V)_X.$$

**Theorem 8.** Let $X$ be a linear normed space and $V$ be a cone in $X$. Then

$$\delta_n(A, V)_X \geq \delta_n(A)_X,$$

(6)

where $\delta_n(A)_X$ is the linear $n$-width of $A$ in $X$.

Of course, the value of Konovalov linear relative $n$-width is not greater than the value of Korovkin linear relative $n$-width:

$$\delta_n(A, V)_X \geq \delta_n(A \cap V, V)_X.$$

(7)

If we compare the value of Korovkin linear relative $n$–width $\delta_n(A, V)_X$ of set $A$ in $X$ with the constraint $V$ to the value of linear $n$–width $\delta_n(A)_X$ of the set $A$ in $X$ we can evaluate the negative impact of the shape-preserving constraint $L_n(V) \subset V$ on the intrinsic error of approximation by means of the *shape-preserving* linear operators of finite rank $n$ compare to the error of *unconstrained* linear finite-rank approximation on the same set.

It seems that the problem of estimation of Konovalov linear relative $n$-width $\delta_n(A \cap V, V)_X$ is much harder that the problem of estimation of Korovkin linear relative $n$-width $\delta_n(A, V)_X$, so it will be difficult to use (7) for getting lower estimations for the value of $\delta_n(A, V)_X$. On the other hand, (7) can potentially help for getting upper estimations for the value of Konovalov linear relative $n$-width $\delta_n(A \cap V, V)_X$. 


Acknowledgments

The results were obtained within the framework of the state task of Russian Ministry of Education and Science (project 1.1520.2014K).

References


