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EXPECTED NUMBER OF REAL ZEROS OF A CLASS OF RANDOM HYPERBOLIC POLYNOMIAL

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Abstract: Let $y_1(\omega), y_2(\omega), \dots, y_n(\omega)$ be independent and normally distributed random variables with mean zero and variance one. For large values of n, it is proved that the the expected number of times the random hyperbolic polynomial $y_1(\omega)sinht + y_2(\omega)sinh2t + \cdots + y_n(\omega)sinhnt$ crosses the line y=K is $(1/\pi)$ log n +O(1) as long as $0 \leq K \leq \sqrt{n}$.

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1. Introduction

Let $y_1(\omega), y_2(\omega), \dots, y_n(\omega)$ be a sequence of independent and normally distributed random variables with mean zero, variance one and defined in the probability space (Ω,A, P).Consider the random hyperbolic polynomial

$$y = f_n(t) \equiv f_n(t, \omega) = y_1(\omega) \sinh t + y_2(\omega) \sinh 2t + \cdots + y_n(\omega) \sinh nt.$$

Let $EN_{n,K}(\alpha, \beta)$ be the expected number of times $f_n(t)$ crosses the line $t = K$.

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A lot of information about the behavior of a polynomial can be obtained by determining $EN_{n,K}(\alpha, \beta)$ in different subintevals (α, β) of R.To calculate the expected number of crossings of different type of random polynomial the famous Kac-Rice formula(see[1],[2] and [4]) is a widely used technique.Applying the formula, Das[3] calculated the expected number of axis crossings of a random polynomial having hyperbolic elements, namely

$$g_n(t) = y_1(\omega)\cosh t + y_2(\omega) cosh2t + \cdots + y_n(\omega)\cosh nt$$

in a neighbourhood that does not contain zero. It is to be noted that Das's approach for calculation of expected number of real zeros of $g_n(t)$ cannot be emulated for our random polynomial $f_n(t)$ in a neighbourhood of the origin. We still do not have a satisfactory estimate of $EN_{n,0}(-\infty, \infty)$. Later using the Kac-Rice formula, Farahmand and Hannigan [5] have calculated $EN_{n,K}(-\infty, \infty)$. In their estimation they have tried to get over the obstacle one may face near zero by including an upper bound of expected number of level crossings near zero in their result. This move succeeds in providing the asymptotic estimate of $EN_{n,K}(-\infty, \infty)$ as $(1/\pi)\log n$, but brings about a rather unwelcome large error term of the order $\sqrt{\log n}$. Clearly,by reducing the error term, a more accurate picture of random polynomial can be obtained. Thus our objective in this paper is twofold. Firstly we calculate $EN_{n,0}(-\infty, \infty)$ and then we show that the error terem in the asyymptotic estimate of $EN_{n,K}(-\infty, \infty)$ is infact O(1). We prove the following two theorems.

Theorem 1. *If the coefficients of $f_n(t)$ are normally distributed independent random variables with mean zero and variance one, then for large values of n:*

$$EN_{n,0}(-\infty, \infty) = (1/\pi)\log n + C_1 + o(1),$$

where $1.7731.. < C_1 < 1.806772$.

Theorem 2. *If $K^2/n \to 0$ as $n \to \infty$, then*

$$EN_{n,K}(-\infty, \infty) = (1/\pi)\log n + C_2 + o(1),$$

where $0.7731.. < C_2 < 0.806772$.

It is to be observed that if we calculate $EN_{n,0}(-\infty, \infty)$ from $EN_{n,K}(-\infty, \infty)$, which has been found out in [5], by replacing K by zero, then we obtain a large error term $\sqrt{\log n}$. We apply a little calculus to overcome this obstacle in the proof of the Theorem 1.

2. Proof of the Theorem 1

Let us consider an interval (α, β) which does not include zero. The Kac-Rice formula for expected number of zeros of $f_n(t)$ in this interval is

$$EN_{n,0}(\alpha, \beta) = \frac{1}{\pi} \int_{\alpha}^{\beta} X_n^{-1} \sqrt{D_n} dt, \tag{1}$$

where:

$$X_n = X_n(t) = \sum_{k=1}^{n} \sinh^2 kt = g_m(t) - m,$$

$$Y_n = Y_n(t) = \sum_{k=1}^{n} k \sinh kt \cosh kt = g_m^{'}(t),$$

$$Z_n = Z_n(t) = \sum_{k=1}^{n} k^2 \cosh^2 kt = 3g_m^{''}(t) + 4(m^3 - m),$$

$$m = (2n + 1), \; g_m(t) = \sinh mt / \sinh t \text{ and } D_n = D_n(t) = X_n Z_n - Y_n^2.$$

For the ease of computation involved, we shall calculate the integral of (1) after obtaining the dominant terms in X_n, Y_n and Z_n in some suitably chosen subintervals of (-∞,∞). Let us consider the interval $(2m^{-1} \log m, \infty)$ first.In this interval we find that $n^s \sinh^s t (\sinh nt)^{-1}$, where s is a finite positive number, is a monotonically decreasing function of t and tends to zero for sufficiently large values of n. With the help of this information, we can obtain that

$$4X_n = g_m(t) \left(1 + O(n^{-4} \log n)\right), \tag{2}$$

$$8Y_n = g_m(t) \left(m - \coth t + O(n^{-7})\right) \tag{3}$$

and

$$16Z_n = g_m(t) \left((m - \coth t)^2 + \cos ech^2 t + O(n^{-2} \log n)\right). \tag{4}$$

Therefore

$$X_n^{-2} D_n = \cos ech^2 t + O\left(n^{-2} \log n\right). \tag{5}$$

Thus by (1) we have

$$EN_{n,0}(2m^{-1} \log m, \infty) = (2\pi)^{-1} \log\left(m / \log m\right) + O(n^{-1} \log n)^2. \tag{6}$$

Our next aim is to calculate $EN_{n,0}(1/2^n, 2m^{-1} \log m)$. To this end, we write $g_m(t)$ in the following manner

$$g_m(t) = m\tau^{-1} \sinh \tau + \eta(\tau) \sinh \tau,$$

where $\tau = mt$ and $\eta(\tau) = 1/\sinh t - 1/t$.

Then X_n,Y_n and Z_n can be written as

$$4X_n = \tau^{-1}\sinh\tau\,(ma_1 + a_0)\,, 8Y_n = \tau^{-1}\sinh\tau\left(m^2b_2 + mb_1 + b_0\right)$$

$$48Z_n = \tau^{-1}\sinh\tau\left(m^3c_3 + m^2c_2 + mc_1 + c_0\right),$$

where

$$a_0 = \tau\eta(\tau)\,,\ a_1 = 1 - \tau(\sinh\tau)^{-1}\,,\ b_0 = \tau\eta'(\tau)\,,$$

$$b_1 = \tau\eta(\tau)\coth\tau,\ b_2 = \coth\tau - \tau^{-1},\ c_0 = 3\tau\eta''(\tau)\,,$$

$$c_1 = \tau\left(6\eta'(\tau)\coth\tau - 3(\sinh\tau)^{-1}\right),$$

$$c_2 = 3\tau\eta(\tau)\,,\ c_3 = 3 + \tau(\sinh\tau)^{-1} - 6\tau^{-1}\coth\tau + 6\tau^{-2}.$$

We note that $\eta(\tau)$ has an absolutely and uniformly convergent power series representation in $(0,\,\pi)$ [6, page 35]. Hence $\eta(\tau)$, $\eta'(\tau)$,and $\eta''(\tau)$ are bounded in (0,1). As a consequence a_i, b_i and c_i are bounded in (0,1) and we obtain the following relations in $\left(1/2^n, 2m^{-1}\log m\right)$

$$4\tau(\sinh\tau)^{-1}X_n = ma_1(1 + o(1)), \tag{7}$$

$$8\tau(\sinh\tau)^{-1}Y_n = m^2b_2(1 + o(1)) \tag{8}$$

and

$$8\tau(\sinh\tau)^{-1}Z_n = m^3c_3(1 + o(1)). \tag{9}$$

It is to be noted that the o(1) terms mentioned in (7)-(9) are $O(n^{-2})$ if $O(t) \le n^{-1}$ and $O(\tau/m)^2$ if $O(t) \ge n^{-1}$. It follows from (7)-(9) that if $(\alpha,\beta) \subset (1/2^n, 2m^{-1}\log m)$

$$2X_n^{-1}\sqrt{D_n} = \sqrt{\delta_n(\tau)}\,(1 + o(1))\,, \tag{10}$$

where $\delta_n(\tau) = (c_3a_1 - 3b_2^2)/(3a_1^2)$.

We now calculate $EN_{n,0}(3/m, 2m^{-1}\log m)$ using (10).The following inequality, which can be shown to be valid for $\tau \ge 3$ if a little algebra is applied, helps to find the bounds of of $\sqrt{\delta_n(\tau)}$. Thus

$$\begin{aligned} -\frac{\tau^2 - 3\tau\cdot + 3}{3\sinh\tau} - \frac{\tau^3 - 3\tau^2 + 6\tau\cdot}{3\sinh^2\tau} &< \sqrt{\delta_n(\tau)} - \tau^{-1} \\ &< -\frac{\tau^2 - 3\tau}{3\sinh\tau} - \frac{\tau^3 - 3\tau^2 + 3\tau}{3\sinh^2\tau} - \frac{2\tau^4 - 6\lambda^2 + 6\lambda^2}{3\sinh^3\tau}. \end{aligned} \tag{11}$$

Using (11), (10) and (1) we can obtain with the help of standard results of integration and numeric integration that

$$EN_{n,0}(3/m, 2m^{-1}\log m) = (2\pi)^{-1})\log\log m + L + O(m^{-1}\log m)^2, \qquad (12)$$

where where $-0.114472 < L < -0.097636$.

Since $\sqrt{\delta_n(\tau)}$ is bounded for $t \in (1/2^n, 3/m)$, it is also possible to calculate $EN_n\,(1/2^n, 3/m)$ with the help of (10) using numeric integration. Thus

$$EN_{n,0}(1/2^n, 3/m) = 0.501022 + O\,(1/n)\,. \qquad (13)$$

We next calculate $EN_{n,0}\,(-1/2^n, 1/2^n)$. The formula (1) cannot be applied in this interval.However, an upper bound of

$$EN_{n,0}\,(-1/2^n, 1/2^n)$$

will be sufficient for our purpose. To this end, we shall first calculate the expected number of real zeros of $f_n^{'}\,(t)$ which we shall denote as

$$EN^1_{n,0}(-1/2^n, 1/2^n).$$

Using Kac-Rice formula for $f_n^{'}\,(t)$, we note that

$$EN^1_{n,0}(-1/2^n, 1/2^n) = \pi^{-1}\int_{-1/2^n}^{1/2^n} Z_n^{-1}\sqrt{Z_nB_n - A_n^2}dt, \qquad (14)$$

where $A_n = A_n(t) = \sum_{k=1}^n k^3\cosh kt \sinh kt$,$B_n = B_n(t) = \sum_{k=1}^n k^4\,\sinh^2 kt$.

We observe that

$$A_n = O(m^4/2^n),\;\; B_n = O(m^5/2^n),\;\; Z_n = m^3(1/24 + m^3/2^n)$$

in $(-1/2^n, 1/2^n)$. Hence, from (14) we have

$$EN^1_{n,0}(-1/2^n, 1/2^n) = O(m/2^n). \qquad (15)$$

We note that $f_n(t)$ has at least one zero in $(-1/2^n, 1/2^n)$, since $f_n(t)$ vanishes at t $= 0$. Since (15) is true, by Roll's theorem, $f_n(t)$ can have at most one zero in $(-1/2^n, 1/2^n)$. Thus, we conclude that

$$EN_{n,0}(-1/2^n, 1/2^n) = 1 + O(m/2^n). \qquad (16)$$

Finally, we observe from (1) that the expected number of zeros of (α, β) and $(-\beta, -\alpha)$ are equal. Therefore, the expected number of zeros in $(-\infty, -1/2^n)$, is same as those in $(1/2^n, \infty)$. The proof of the Theorem 1 now follows from (6), (12), (13) and (16).

3. Proof of Theorem 2

Proceeding in a manner similar to that used by Farahamand[4], we can obtain the extended Kac-Rice formula for expected number of zeros of $f_n(t) = K$ in (a, b) as

$$EN_{n,K}(a,b) = \pi^{-1} I_1(a,b) + \left(\sqrt{2}/\pi\right) I_2(a,b),$$

where

$$I_1(a,b) = \int_a^b X_n^{-1}\sqrt{D_n}\exp\{-K^2 Z_n\,(2D_n)^{-1}\}dt,$$

$$I_2(a,b) = \int_a^b KY_n X_n^{-3/2}\exp\{-K^2\,(2X_n)^{-1}\}erf\left(KY_n\,(2X_nD_n)^{-1/2}\right)dt.$$

Consider the interval $\left(2m^{-1}\log m, \infty\right)$. We observe from (2)-(4) that

$$Z_n/D_n = O\left(m^{-2}\,(\log m)^3\right)$$

and decreases in magnitude as t increases. Therefore, using (5), we have

$$I_1\left(2m^{-1}\log m, \infty\right) = (2\pi)^{-1}\log\left(m/\log m\right) + o\left(Kn^{-2}\,(\log n)^4\right). \quad (17)$$

If $t \in \left(3/m, 2m^{-1}\log m\right)$, it can be seen from (7)-(9) that

$$Z_n/D_n = \mu\left(\tau\right)\left(1 + o\left(1\right)\right),$$

where $\mu\left(\tau\right) = (3m)^{-1}\left(\tau/\sinh\tau\right)c_3/\left(c_3a_1 - 3b_2^2\right)$.After taking the derivative, we find that $\mu\left(\tau\right)$ is a monotonically decreasing function of τ and, therefore,in $(3/m, 2m^{-1}\log m)$,the maximum value of $\exp\{-K^2 Z_n\,(2D_n)^{-1}\}$ occurs at $t = 2m^{-1}\log m$. Hence, using (10) and (11) we find that

$$I_1\left(3/m, 2m^{-1}\log m\right) = (2\pi)^{-1}\log\log m + L + O\left(Kn^{-2}\,(\log m)^3\log\log n\right). \quad (18)$$

We note that $\mu\left(\tau\right)$ is $O(n^{-1})$ in (0, 3/m). Therefore applying numeric integration to the integral in (10) we have

$$I_1\left(1/2^n, 3/m\right) = 0.501022... + O\left(K^2/n\right). \quad (19)$$

Till now we have calculated $I_1\left(1/2^n, \infty\right)$. It follows from the defination that $I_1\left(\alpha, \beta\right) = I_1\left(-\beta, -\alpha\right)$. Therefore,

$$I_1\left(1/2^n, \infty\right) = I_1\left(-\infty, -1/2^n\right). \quad (20)$$

We can obtain $I_2\left(1/2^n, \infty\right)$ and $I_2\left(-\infty, -1/2^n\right)$ by noting that

$$I_2\left(1/2^n, \infty\right) = I_2\left(-\infty, -1/2^n\right) \leq \int_{1/2^n}^{\infty} K X_n^{3/2} X_n' dt = O\left(K/\sqrt{n}\right). \tag{21}$$

We observe that $f_n(t) - K$ does not have a zero at t = 0. Taking into account (14) and applying Roll's theorem, we can conclude that

$$EN_{n,K}(-1/2^n, 1/2^n) = O\left(m/2^n\right). \tag{22}$$

The proof of the Theorem 2 follows from (17)-(22).

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