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CONIC REGIONS AND SYMMETRIC POINTS

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Abstract: In this note, the concept of N-symmetric points. Janowski functions and the conic regions are combined to define a class of functions in a new interesting domain which represents the conic type regions. certain interesting coefficient inequalities are deduced.

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1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

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$$\mathcal{U} = \{z : z \in \mathbb{C} \;\text{ and }\; |z| < 1\},$$

and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all function which are univalent in $\mathcal{U}$.

Definition 1.1. For two functions f and g analytic in $\mathcal{U}$, we say that the function f is subordinate to g in $\mathcal{U}$ and write $f(z) \prec g(z)$, if there exists a Schwarz function w, which is analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in \mathcal{U}$. If g is univalent in $\mathcal{U}$ then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Using the principle of the subordination to define the class P of functions with positive real part [4].

Definition 1.2. Let P denote the class of analytic functions of the form $p(z) = 1+\sum_{n=1}^{\infty} p_n z^n$ defined on $\mathcal{U}$ and satisfying $p(0) = 1$, $\Re\{p(z)\} > 0$, $z \in \mathcal{U}$.

Any function p in P has the representation $p(z) = \frac{1+w(z)}{1-w(z)}$ where $w(0) = 0$, $|w(z)| < 1$ on $\mathcal{U}$, the class of functions with positive real P plays a crucial role in geometric function theory. Its significance can be seen from the fact that simple subclasses like class of starlike $\mathcal{S}^*$, class of convex functions $\mathcal{C}$, class of starlike functions with respect to symmetric points have been defined by using the concept of class of functions with positive real part.
The definition of starlike functions with respect to N-symmetric points is as follows.

Definition 1.3. For a positive integer N, let $\varepsilon = \exp\left(\frac{2\pi i}{N}\right)$ denote the N^{th} root of unity for $f \in \mathcal{A}$, let

$$M_{f,N}(z) = \sum_{v=1}^{N-1} \varepsilon^{-v} f(\varepsilon^v z).\frac{1}{\sum_{v=1}^{N-1} \varepsilon^{-v}}, \tag{1.2}$$

be its N-weighed mean function.
A function f in $\mathcal{A}$ is said to belong to the class $\mathcal{S}^*_N$ if functions starlike with respect to N-symmetric points if for every r close to 1 , $r < 1$, the angular velocity of f about the point $M_{f_{N}(z_0)}$ positive at $z = z_0$ as z traverses the circle $|z| = r$ in the positive direction, that is $\Re\left\{\frac{zf'(z)}{f(z)-M_{f,N}(z_0)}\right\} > 0$ for $z = z_0$, $|z_0| = r$.

Definition 1.4. [2] A function f in $\mathcal{A}$ is univalent and starlike with respect to N-symmetric points, or briefly N-starlike if and only if

$$\Re\left\{\frac{zf'(z)}{f_N(z)}\right\} > o, \quad z \in \mathcal{U}, \tag{1.3}$$

where

$$f_N(z) = \frac{1}{N}(f(z) - M_{f,N}(z)) \tag{1.4}$$

If $f(z)$ defined by (1.1) then,

$$f_N(z) = z + \sum_{n=2}^{\infty} \lambda_N(n) a_n z^n,$$

where

$$\lambda_N(n) = \begin{cases} 1, & n = lN + 1, \quad l \in \mathbb{N}_0. \\ 0, & n \neq lN + 1 \end{cases} \tag{1.5}$$

Definition 1.5. [1]Let $P[A, B]$, where $-1 \leq B < A \leq 1$, denote the class of analytic function p defined on $\mathcal{U}$ with the representation $p(z) = \frac{1+Aw(z)}{1+Bw(z)}$, $z \in \mathcal{U}$, $w(0) = 0$, $|w(z)| < 1$.
$p \in P[A, B]$ if and only if $p(z) \prec \frac{1+Az}{1+Bz}$.

Geometrically, a function $p(z) \in P[A, B]$ maps the opine unit onto the disk defined by the domain,

$$\Omega[A, B] = \left\{ w : \left| w - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2} \right\}.$$

The class $P[A, B]$ is connected the class P of functions with positive real part by the relation,

$$p(z) \in P \Leftrightarrow \frac{(A + 1)p(z) - (A - 1)}{(B + 1)p(z) - (B - 1)} \in P[A, B].$$

This class was introduced by Janowski [1] and then studied by several authors. Kanas and Wisniowska [6,11] introduced and studied the class $k - UCV$ of k-uniformly convex functions and the corresponding class $k - ST$ of k-starlike functions. These classes were defined subject to the conic region Ω_k, $k \geq 0$ given by

$$\Omega_k = \{u + iv : u > k\sqrt{(u - 1)^2 + v^2}.$$

This domain represents the right half plane for $k = 0$, hyperbola for $0 < k < 1$, a parabola for $k = 1$ and ellipse for $k > 1$.
The functions $p_k(z)$ play the role of extremal functions for these conic regions

where

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k=0 \\ 1+\frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k=1. \\ 1+\frac{2}{1-k^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos\ k\right)\arctan h\sqrt{z}\right], & 0<k<1. \\ 1+\frac{2}{k^2-1}\sin\left[\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right]+\frac{1}{k^2-1}, & k>1, \end{cases} \tag{1.6}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t \in (0,1)$, $z \in \mathcal{U}$ and z is chosen such that $k = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, $R(t)$ is the Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral $R(t)$. $p_k(z) = 1 + \delta_k z + ...$, [10] where

$$\delta_k = \begin{cases} \frac{8(\arccos\ k)^2}{\pi^2(1-k^2)}, & 0 \le k < 1 \\ \frac{8}{\pi^2}, & k=1. \\ \frac{\pi^2}{4(k^2-1)\sqrt{t}(1+t)R^2(t)}, & k>1. \end{cases} \tag{1.7}$$

These conic regions are being studied by several authors see [5,12].
Following are the definitions of the classes $k-UCV$ and $k-ST$.

Definition 1.6. A function $f \in \mathcal{A}$ is said to be in the class $k-UCV$, if and only if,

$$\frac{(zf'(z))'}{f'(z)} \prec p_k(z), \quad z \in \mathcal{U}, \quad k \ge 0.$$

Definition 1.7. A function $f \in \mathcal{A}$ is said to be in the class $k-ST$, if and only if

$$\frac{zf'(z)}{f(z)} \prec p_k(z), \quad z \in \mathcal{U}, \quad k \ge 0.$$

Shams et al [7] further generalized the classes $k-ST$ and $k-UCV$ to $KD(k,\alpha)$ and $SD(k,\alpha)$ respectively subject to the conic domain $G(k,\alpha)$, $k \ge 0,\ 0 \le \alpha < 1$ which is

$$G(k,\alpha) = \{w : \Re w > k|w-1| + \alpha\}.$$

Now using the concepts of Janowski functions and the conic regions, we define the following.

Definition 1.8. A function p is said to be in the class $k-P[A,B]$, if and only if,

$$p(z) \prec \frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}, \quad k \ge 0,$$

where $p_k(z)$ is defined by (1.6) and $-1 \le B < A \le 1$.

Geometrically, the function $p \in k-P[A,B]$ takes all values from the domain $\Omega_k[A,B],\ -1 \le B < A \le 1,\ k \ge 0$ which is defined as

$$\Omega_k[A,B] = \left\{ w : \Re\left(\frac{(B-1)w(z)-(A-1)}{(B+1)w(z)-(A+1)}\right) > k\left|\frac{(B-1)w(z)-(A-1)}{(B+1)w(z)-(A+1)} - 1\right|\right\}$$

or equivalently

$$\Omega_k[A,B] = \{u+iv : [(B^2-1)(u^2+v^2) - 2(AB-1)u + (A^2-1)]^2$$
$$> k^2[(-2(B+1)(u^2+v^2) + 2(A+B+2)u - 2(A+1))^2 + 4(A-B)^2v^2]\}.$$

The domain $\Omega_k[A,B]$ retains the conic domain Ω_k inside the circular region defined by $\Omega[A,B]$. the impact of $\Omega[A,B]$ on the conic domain Ω_k changes the original shape of the conic regions . The ends of hyperbola and parabola get closer to each other but never meet anywhere and the ellipse gets the oval shape. When $A \to 1,\ B \to -1$, the radius of the circular disk defined by $\Omega[A,B]$ tends to infinity, consequently the arms of hyperbola and parabola expand and the oval turns into ellipse . we see that $\Omega_k[1,-1] = \Omega_k$, the conic domain defined by Kanas and Wisniowska[11].

Definition 1.9. A function $f \in \mathcal{A}$ is said to be in the class $k-ST[A,B,N]$, $k \ge 0,\ N \ge 2,\ \ -1 \le B < A \le 1$, if and only if

$$\Re\left(\frac{(B-1)\frac{zf'(z)}{f_N(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f_N(z)} - (A+1)}\right) > k\left|\frac{(B-1)\frac{zf'(z)}{f_N(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f_N(z)} - (A+1)} - 1\right|, \qquad (1.8)$$

where $f_N(z)$ is defined by (1.4).
Or equivalently,

$$\frac{zf'(z)}{f_N(z)} \in k - P[A,B].$$

Definition 1.10. A function $f \in \mathcal{A}$ is said to be in the class $k - UCV[A,B,N]$, $k \ge 0,\ N \ge 2,\ \ -1 \le B < A \le 1$, if and only if

$$\Re\left(\frac{(B-1)\frac{(zf'(z))'}{f_N'(z)} - (A-1)}{(B+1)\frac{(zf'(z))'}{f_N'(z)} - (A+1)}\right) > k\left|\frac{(B-1)\frac{(zf'(z))'}{f_N'(z)} - (A-1)}{(B+1)\frac{(zf'(z))'}{f_N'(z)} - (A+1)} - 1\right|, \qquad (1.9)$$

where $f_N(z)$ is defined by (1.4).
Or equivalently,

$$\frac{(zf'(z))'}{f_N'(z)} \in k - P[A,B].$$

It can be easily seen that

$$f(z) \in k - UCV[A,B,N] \Leftrightarrow zf'(z) \in k - ST[A,B,N]. \tag{1.10}$$

Special cases
we get the classes defined by Janowski [1] , Khalida Inyat Noor and Sarfraz Nawaz Malik [9], Kanas and Wisniowska [6], Shams [7] .

Lemma 1.11. *[13]Let $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be subordinate to $H(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$. If $H(z)$ is univalent in $\mathcal{U}$ and $H(z)$ is convex, then*

$$|c_n| \le |b_1|, \; n \ge 1.$$

2. Main Results

Theorem 2.1. *A function $f \in \mathcal{A}$ and of the form* (1.1) *is in the class $k - ST[A,B,N]$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} \{2(k+1)(n - \lambda_N(n)) + |n(B+1) - \lambda_N(n)(A+1)|\}|a_n| < |B - A|, \tag{2.1}$$

where $-1 \le B < A \le 1$, $k \ge 0$ and $\lambda_N(n)$ is defined by (1.5).

Proof. Assuming that (2.1) holds, then it suffices to show that

$$k\left|\frac{(B-1)\frac{zf'(z)}{f_N(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f_N(z)} - (A+1)} - 1\right| - \Re\left[\frac{(B-1)\frac{zf'(z)}{f_N(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f_N(z)} - (A+1)} - 1\right] < 1.$$

We get

$$\begin{aligned} k&\left|\frac{(B-1)\frac{zf'(z)}{f_N(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f_N(z)} - (A+1)} - 1\right| - \Re\left[\frac{(B-1)\frac{zf'(z)}{f_N(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f_N(z)} - (A+1)} - 1\right] \\ &\le (k+1)\left|\frac{(B-1)zf'(z) - (A-1)f_N(z)}{(B+1)zf'(z) - (A+1)f_N(z)} - 1\right| \\ &= 2(k+1)\left|\frac{f_N(z) - zf'(z)}{(B+1)zf'(z) - (A+1)f_N(z)}\right| \\ &= 2(k+1)\left|\frac{\sum_{n=2}^{\infty}(\lambda_N(n) - n)a_n z^n}{(B-A)z + \sum_{n=2}^{\infty}[n(B+1) - \lambda_N(n)(A+1)]a_n z^n}\right| \end{aligned}$$

$$\leq 2(k+1)\frac{\sum_{n=2}^{\infty}|\lambda_N(n)-n||a_n|}{|B-A|-\sum_{n=2}^{\infty}|n(B+1)-\lambda_N(n)(A+1)||a_n|}.$$

The last expression is bounded above by 1, then

$$\sum_{n=2}^{\infty}\{2(k+1)(n-\lambda_N(n))+|n(B+1)-\lambda_N(n)(A+1)|\}|a_n|<|B-A|,$$

and this completes the proof. □

When $N=1$, we have the following known result, proved by Khalida Inayat Noor and Sarfraz Nawaz Malik in [9].

Corollary 2.2. *A function $f \in \mathcal{A}$ and form* (1.1) *in the class $k-ST[A,B]$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}\{2(k+1)(n-1)+|n(B+1)-(A-1)|\}|a_n|<|B-A|, \tag{2.2}$$

where $-1 \leq B < A \leq 1$ and $k \geq 0$.

For $N=1,\ A=1$ *and* $B=-1$, we have following result due to Kanas and Wisniowska [6].

Corollary 2.3. *A function $f \in \mathcal{A}$ and form* (1.1) *in the class $k-ST$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}\{n+k(n-1)\}|a_n|<1, \quad k\geq 0. \tag{2.3}$$

For $N=1,\ A=1-2\alpha$ *and* $B=-1$ with $0\leq\alpha<1$, we arrive at Shams et result in [7].

Corollary 2.4. *A function $f \in \mathcal{A}$ and form* (1.1) *in the class $SD(k,\alpha)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}\{n(k+1)-(k+\alpha)\}|a_n|<1-\alpha, \tag{2.4}$$

where $0\leq\alpha<1$ and $k\geq 0$.

Also for $N=1,\ A=1-2\alpha$ *and* $B=-1,\ \ k=0$ with $0\leq\alpha<1$, then we get the well- known Silverman's results [8].

Corollary 2.5. *A function $f \in \mathcal{A}$ and form (1.1) in the class $S^*(\alpha)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}\{(n-\alpha)\}|a_n| < 1-\alpha, \tag{2.5}$$

where $0 \leq \alpha < 1$.

Using Theorem 2.1 and (1.10) we get the following Theorem.

Theorem 2.6. *A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $k-UCV[A,B,N]$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} n\{2(k+1)(n-\lambda_N(n)) + |n(B+1)-\lambda_N(n)(A+1)|\}|a_n| < |B-A|, \tag{2.6}$$

where $-1 \leq B < A \leq 1$, $k \geq 0$ and $\lambda_N(n)$ is defined by (1.5).

Theorem 2.7. *Let $f \in k-ST[A,B,N]$ and is of the form (1.1). Then for $n \geq 2$.*

$$|a_n| \leq \prod_{j=1}^{n-1} \frac{|\delta_k(A-B) - 2(j-\lambda_N(j))B|}{2(j+1-\lambda_N(j+1))}, \tag{2.7}$$

where δ_k is defined (1.7) and $\lambda_N(n)$ is defined by (1.5).

Proof. By the definition we have

$$\frac{zf'(z)}{f_N(z)} = p(z), \tag{2.8}$$

where

$$p(z) \prec \frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}$$

$$= [(A+1)p_k(z)-(A-1)][(B+1)p_k(z)-(B-1)]^{-1}$$

$$= \frac{A-1}{B-1}\left[1-\frac{A+1}{A-1}p_k(z)\right]\left[1+\sum_{n=1}^{\infty}\left(\frac{B+1}{B-1}p_k(z)\right)^n\right]$$

$$= \frac{A-1}{B-1} + \left(\frac{(A-1)(B+1)}{(B-1)^2} - \frac{A+1}{B-1}\right)p_k(z) + \left(\frac{(A-1)(B+1)^2}{(B-1)^3} - \frac{(A+1)(B+1)}{(B-1)^2}\right)(p_k(z))^2$$

$$+ \left(\frac{(A-1)(B+1)^3}{(B-1)^4} - \frac{(A+1)(B+1)^2}{(B-1)^3}\right)(p_k(z))^3 + \ldots\ldots\ldots\ldots$$

If $p_k(k) = 1 + \delta_k z + ...$, then we have after suitable simplification

$$p(z) \prec \sum_{n=1}^{\infty} \frac{-2(B+1)^{n-1}}{(B-1)^n} + \left\{ \sum_{n=1} \frac{2n(A-B)(B+1)^{n-1}}{(B-1)^{n+1}} \right\} \delta_k z + ...$$

Now we see that the series $\sum_{n=1}^{\infty} \frac{-2(B+1)^{n-1}}{(B-1)^n}$ and $\sum_{n=1} \frac{2n(A-B)(B+1)^{n-1}}{(B-1)^{n+1}}$ are convergent and converge to 1 and $\frac{A-B}{2}$ respectively.
Therefore.

$$p(z) \prec 1 + \frac{1}{2}(A-B)\delta_k z + ...$$

Now if $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, then by Lemma 1.11, we get

$$|c_n| \leq \frac{1}{2}(A-B)\delta_k, \quad n \geq 1 \tag{2.9}$$

Now from (2.8), we have

$$z + \sum_{n=2}^{\infty} n a_n z^n = \left[z + \sum_{n=2}^{\infty} \lambda_N(n) a_n z^n \right] \left[1 + \sum_{n=1}^{\infty} c_n z^n \right].$$

Equating coefficients of z^n on both sides, we have

$$(n - \lambda_N(n)) a_n = \sum_{j=1}^{n-1} \lambda_N(n-j) a_{n-j} c_j, \qquad a_1 = \lambda_N(1) = 1.$$

This implies that

$$|a_n| \leq \frac{1}{(n - \lambda_N(n))} \sum_{j=1}^{n-1} \lambda_N(n-j) a_{n-j} c_j, \qquad a_1 = \lambda_N(1) = 1.$$

By (2.9), we get

$$|a_n| \leq \frac{|\delta_k|(A-B)}{2(n - \lambda_N(n))} \sum_{j=1}^{n-1} \lambda_N(j) |a_j|, \qquad a_1 = \lambda_N(1) = 1. \tag{2.10}$$

Now we prove that

$$\frac{|\delta_k|(A-B)}{2(n - \lambda_N(n))} \sum_{j=1}^{n-1} \lambda_N(j) |a_j| \leq \prod_{j=1}^{n-1} \frac{|\delta_k(A-B) - 2(j - \lambda_N(j))B|}{2(j+1 - \lambda_N(j+1))}. \tag{2.11}$$

For this, we use the induction method.
For $n = 2$: from (2.10), we have

$$|a_2| \leq \frac{|\delta_k|(A-B)}{2(2-\lambda_N(2))}.$$

From (2.7), we have

$$|a_2| \leq \frac{|\delta_k|(A-B)}{2(2-\lambda_N(2))}.$$

For $n = 3$: from (2.10), we have

$$|a_3| \leq \frac{|\delta_k|(A-B)}{2(3-\lambda_N(3))}\left[1+\frac{|\delta_k|(A-B)}{2(2-\lambda_N(2))}\lambda_N(2)\right].$$

From (2.7), we have

$$\begin{aligned}|a_3| &\leq \frac{|\delta_k|(A-B)}{2(2-\lambda_N(2))}\frac{|\delta_k(A-B)-2(2-\lambda_N(2))B|}{2(3-\lambda_N(3))}\\ &\leq \frac{|\delta_k|(A-B)}{2(2-\lambda_N(2))}\frac{|\delta_k|(A-B)+2(2-\lambda_N(2))}{2(3-\lambda_N(3))}\\ &\leq \frac{|\delta_k|(A-B)}{2(3-\lambda_N(3))}\left[1+\frac{|\delta_k|(A-B)}{2(2-\lambda_N(2))}\right].\end{aligned}$$

Let the hypothesis be true for $n = m$. From (2.10), we have

$$|a_m| \leq \frac{|\delta_k|(A-B)}{2(m-\lambda_N(m))}\sum_{j=1}^{m-1}\lambda_N(j)|a_j|, \qquad a_1 = \lambda_N(1) = 1.$$

From (2.7), we have

$$\begin{aligned}|a_m| &\leq \prod_{j=1}^{m-1}\frac{|\delta_k(A-B)-2(j-\lambda_N(j))B|}{2(j+1-\lambda_N(j+1))}.\\ &\leq \prod_{j=1}^{m-1}\frac{|\delta_k|(A-B)+2(j-\lambda_N(j))}{2(j+1-\lambda_N(j+1))}.\end{aligned}$$

By the induction hypothesis, we have

$$\frac{|\delta_k|(A-B)}{2(m-\lambda_N(m))}\sum_{j=1}^{m-1}\lambda_N(j)|a_j| \leq \prod_{j=1}^{m-1}\frac{|\delta_k|(A-B)+2(j-\lambda_N(j))}{2(j+1-\lambda_N(j+1))}.$$

Multiplying both sides by $\frac{|\delta_k|(A-B)+2(m-\lambda_N(m))}{2(m+1-\lambda_N(m+1))}$, we have

$$\prod_{j=1}^{m}\frac{|\delta_k|(A-B)+2(j-\lambda_N(j))}{2(j+1-\lambda_N(j+1))}$$

$$\geq \frac{|\delta_k|(A-B)}{2(m-\lambda_N(m))}.\frac{|\delta_k|(A-B)+2(m-\lambda_N(m))}{2(m+1-\lambda_N(m+1))}\sum_{j=1}^{m-1}\lambda_N(j)|a_j|,$$

$$=\frac{|\delta_k|(A-B)}{2(m+1-\lambda_N(m+1))}\left[\frac{|\delta_k|(A-B)}{2(m-\lambda_N(m))}\sum_{j=1}^{m-1}\lambda_N(j)|a_j|+\sum_{j=1}^{m-1}\lambda_N(j)|a_j|\right],$$

$$\geq \frac{|\delta_k|(A-B)}{2(m+1-\lambda_N(m+1))}\left[\lambda_N(m)|a_m|+\sum_{j=1}^{m-1}\lambda_N(j)|a_j|\right],$$

$$=\frac{|\delta_k|(A-B)}{2(m+1-\lambda_N(m+1))}\sum_{j=1}^{m}\lambda_N(j)|a_j|.$$

That is

$$\frac{|\delta_k|(A-B)}{2(m+1-\lambda_N(m+1))}\sum_{j=1}^{m}\lambda_N(j)|a_j|\leq\prod_{j=1}^{m}\frac{|\delta_k|(A-B)+2(j-\lambda_N(j))}{2(j+1-\lambda_N(j+1))}.$$

Which shows that inequality (2.11) is true for $n=m+1$. Hence the required result. □

When $N=1$ we get result introduced by Khalida Inayat Noor and Sarfraz Nawaz Malik in [9].

Corollary 2.8. *Let $f\in k-ST[A,B]$ and is of the form* (1.1). *Then*

$$|a_n|\leq\prod_{j=0}^{n-2}\frac{|\delta_k(A-B)-2jB|}{2(j+1)},\quad -1\leq B<A\leq 1,\quad n\geq 2.$$

For $N=1,\ A=1\ B=-1$ we arrive at Kanas and Wisniowska result in [6].

Corollary 2.9. *Let $f\in k-ST$ and is of the form* (1.1). *Then*

$$|a_n|\leq\prod_{j=0}^{n-2}\frac{|\delta_k+j|}{(j+1)},\qquad n\geq 2.$$

Also for $N = 1,\ k = 0\ \delta_k = 2$, we get result due to Janowski in [1].

Corollary 2.10. *Let $f \in S^*[A, B]$ and is of the form* (1.1)*. Then*

$$|a_n| \leq \prod_{j=0}^{n-2} \frac{|(A - B) - jB|}{(j + 1)}, \quad -1 \leq B < A \leq 1, \quad n \geq 2.$$

By Theorem 2.7 and (1.10) we get the following Theorem.

Theorem 2.11. *Let $f(z) \in k - UCV[A, B, N]$ and is of the form* (1.1)*. Then for $n \geq 2$.*

$$|a_n| \leq \frac{1}{n} \prod_{j=1}^{n-1} \frac{|\delta_k(A - B) - 2(j - \lambda_N(j))B|}{2(j + 1 - \lambda_N(j + 1))}, \tag{2.12}$$

where δ_k is defined (1.7) *and $\lambda_N(n)$ is defined by* (1.5)*.*

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