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RELATIVE HEAT LOSS REDUCTION FORMULA FOR WINDOWS WITH MULTIPLE PANES

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Abstract: We derive an expression for the heat flux through a window constructed with multiple panes of glass separated pairwise by air layers of a given thickness. We compute the relative heat loss reduction achieved in comparison to a window with no air gap and the same amount of glass. We examine how the relative heat loss reduction function behaves when we scale the number of panes up to a large value.

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1. Introduction

In this model, warm air escapes from a heated building through a window. The window comprises n glass panes of thickness d separated pairwise by a layer of air of width L.

We assume that the inside temperature T_1 is greater than the outside temperature T_2, reflecting the fact that heat flows from the inside to the outside. In the case $n = 2$, we follow Mesterton-Gibbons [1]. If we indicate by $x = 0$ the interface of the inside air and the inside window pane, then the outside pane and outside air meet at $x = 2d + L$. Let $T(x)$ be the temperature at x. Thus we have $T(0) = T_1$ and $T(2d + L) = T_2$. We also write $T(d) = T_A$ and $T(d + L) = T_B$ for the other two air-glass interfaces in between.

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The heat flows along x according to Fick's law

$$F(x) = -k(x)\frac{dT}{dx},$$

where $F(x)$ stands for the heat flux per unit area across the window at point x and the function $k(x)$ represents the thermal conductivity of the medium at x. Fick's law simply states that heat flux at a point x is proportional to the temperature gradient at x; The steeper the gradient, the higher the heat flow.

In the two-pane model, for example, k is a piecewise constant function that takes the value K_A for x between d and $d+L$, the points x that lie in the air gap. For x in the remaining intervals $0 < x < d$, $d+L < x < 2d+L$, in the glass medium, we assign to k the value k_G.

In this paper, we are interested in determining the heat flux across the glass and air for an n-glazed window once the flow has reached a constant equilibrium. In Section 2, some straight-forward computations as found in [1] yield the equilibrium value the flux F settles to in the case $n = 2$. We use these computations in Section 3 to treat the case $n = 3$. Formulas for the flux and relative heat flux reduction for a window with n panes follow in Section 4. We conclude our investigation with Section 5.

2. Double-Glazed Window

As in [1], integrating with respect to x the expression for Fick's law over the interval $0 < x < d$, we obtain

$$F(x) = K_G\frac{T_1 - T_A}{d}.$$

Repeating the process over the remaining intervals and equating the expressions for the respective heat fluxes so obtained, we have the chain of inequalities

$$K_G\frac{T_1 - T_A}{d} = K_A\frac{T_A - T_B}{L} = K_G\frac{T_B - T_2}{d} = F.$$

Solving for T_A and T_B in terms of T_1 and T_2 gives

$$T_A = \frac{\left(\frac{K_G L}{K_A d} + 1\right)T_1 + T_2}{\frac{K_G L}{K_A d} + 2}$$

$$T_B = \frac{T_1 + \left(\frac{K_G L}{K_A d} + 1\right)T_2}{\frac{K_G L}{K_A d} + 2}.$$

Therefore, the flux across across the double-paned window eventuallys settles down to an equilibrium value

$$F = F_2 = \frac{K_G}{d} \frac{T_1 - T_2}{2 + \frac{K_G L}{K_A d}},$$

which, in the $L = 0$ limit, becomes

$$F_{2s} = \frac{K_G}{d} \frac{T_1 - T_2}{2},$$

where F_{2s} is the rate of heat loss for a single-paned window with the same amount of glass.

Thus, by setting the glass panes a distance of L units apart, we achieve a relative reduction of heat loss

$$\Delta_2 = \frac{F_{2S} - F_2}{F_{2S}} = \frac{\frac{K_G L}{K_A d}}{2 + \frac{K_G L}{K_A d}}.$$

The thermal conductivity of glass at room temperature varies between 4×10^{-3} and 8×10^{-3} J/cm.sec.$^\circ$C [2], while the thermal conductivity of dry air is approximately 2.5×10^{-4} J/cm.sec.$^\circ$C [3]. Therefore, the ratio $\frac{K_G}{K_A}$ varies between 16 and 32, which means that a window with a conductivity ratio of 16 and an air gap of four pane-widths already achieves a heat loss reduction of 97%. The heat reduction function Δ_2 is a strictly increasing function of the gap aspect ratio L/d, and approaches rapidly the limiting value 1.

Manufacturers know this, which is why we do not often see windows with 4 or 5 glass panes on the market. Nevertheless, how the heat loss reduction function scale for a window with a large number, n, of glass panes remains an interesting theoretical question. How does Δ depend on n and the gap aspect ratio L/d?

Specifically, we address the following situation. Instead of two, we take n glass panes of thickness d each and put an air gap of thickness L between each successive pair to form an $n-$glazed window. Compared to a single-paned window of thickness nd, how much heat loss reduction is achieved? What form does the function Δ_n take?

We take an inductive approach to this question. Determining Δ_3 will allow us to infer the form of Δ_n for a general n.

3. Triple-Glazed Window

With a triple-glazed window, we have air-glass interfaces at $x = 0$, $d, d+L, 2d+L, 2d+2L, 3d+2L$ that are kept at temperatures $T_1, T_A, T_B, T_C, T_D, T_3$, respectively. Integrating Fick's law as we did in the last section leads us this time to the string of equations

$$K_G \frac{T_1 - T_A}{d} = K_A \frac{T_A - T_B}{L} = K_G \frac{T_B - T_c}{d} = \\ K_A \frac{T_C - T_D}{L} = K_G \frac{T_D - T_3}{d} = F_3, \tag{1}$$

where F_3 denotes the heat flux through the three-pane window.

Part of Equations (1) can be written as

$$K_G \frac{T_3 - T_D}{d} = K_A \frac{T_D - T_C}{L} = K_G \frac{T_C - T_B}{d} = -F_3,$$

which we can, as we did in the previous section, solve for T_D and T_C to find

$$T_D = \frac{(\rho + 1) T_3 + T_B}{\rho + 2}$$
$$T_C = \frac{T_3 + (\rho + 1) T_B}{\rho + 2},$$

if we set $\rho = \frac{K_G L}{K_A d}$.

On the other hand, in (1) we also have

$$K_G \frac{T_1 - T_A}{d} = K_A \frac{T_A - T_B}{L} = K_G \frac{T_B - T_c}{d} = F_3,$$

from which we deduce

$$T_A = \frac{(\rho + 1) T_1 + T_C}{\rho + 2},$$
$$T_B = \frac{T_1 + (\rho + 1) T_C}{\rho + 2}.$$

Next, we use the expressions for T_B and T_C to obtain

$$(\rho + 2) T_B = T_1 + (\rho + 1) T_C = T_1 + \frac{\rho + 1}{\rho + 2} T_3 + \frac{(\rho + 1)^2}{\rho + 2} T_B,$$

which yields

$$T_B = \frac{\rho+2}{2\rho+3}T_1 + \frac{\rho+1}{2\rho+3}T_3.$$

This allows us to find

$$T_D = \frac{T_1 + 2(\rho+1)\,T_3}{2\rho+3}.$$

Finally, inserting T_D into the last of the equations in (1), we arrive at the formula for the heat flux through a three-pane window:

$$F_3 = \frac{K_G}{d}\left[\frac{T_1 + 2T_3\left(1+\frac{K_G L}{K_A d}\right)}{3 + 2\frac{K_G L}{K_A d}} - T_3\right].$$

It is, as one would expect, a function of the the inside and outside air temperatures. This time again, the temperatures at the inner air-glass interfaces do not influence the flow at equilibrium.

4. Window with n Panes

From the expression obtained in the last section for the heat flux for a window with 3 panes, we infer the flux for an $n-$glazed window to be of the form

$$F_n = \frac{K_G}{d}\left[\frac{T_1 + (n-1)\,T_n\left(1+\frac{K_G L}{K_A d}\right)}{n + (n-1)\,\frac{K_G L}{K_A d}} - T_n\right],$$

where T_1 and T_n are the inside and outside air temperatures, respectively.

In the $L = 0$ limit, F_n reduces to

$$F_{ns} = \frac{K_G}{d}\frac{T_1 - T_n}{n},$$

with F_{ns} denoting the flux across a single-paned window with the same amount of glass.

The relative heat loss reduction is given by

$$\Delta_n = \frac{F_{nS} - F_n}{F_{nS}} = \frac{(n-1)\,\frac{K_G L}{K_A d}}{n + (n-1)\,\frac{K_G L}{K_A d}}.$$

As a sequence of functions of the gap aspect ratio $\frac{L}{d}$, Δ_n converges to the limit $\frac{\frac{K_G L}{K_A d}}{1+\frac{K_G L}{K_A d}}$ as n gets large. As for the behavior of that sequence, that limit is reached in a monotonic fashion, since we have

$$\frac{d}{dn}\Delta_n = \frac{\frac{K_G L}{K_A d}}{\left[n + (n-1)\frac{K_G L}{K_A d}\right]^2},$$

which remains strictly positive for all positive values of the gap aspect ratio $\frac{L}{d}$. Therefore, the absolute ceiling on the amount of relative heat loss reduction achievable is $\frac{\frac{K_G L}{K_A d}}{1+\frac{K_G L}{K_A d}}$, no matter the number of panes used. The monotonic convergence of the sequence Δ_n conforms to one's intuition that the more panes and the wider the gaps, the more energy is saved.

5. Conclusion

In an inductive approach, using the expression for the heat flux through a double-glazed, then the heat flux through a triple-glazed window, we inferred the expression for the heat flux through a window with n panes of thickness d set a distance of L units apart. We obtained a formula for the relative heat loss reduction Δ_n as the relative difference between the heat flux through an n-glazed window and that of a single-paned window with the same amount of glass. As a function of the gap aspect ratio $\frac{L}{d}$, we found that the amount of relative heat loss reduction Δ_n achieved starts at $\frac{\frac{K_G L}{K_A d}}{2+\frac{K_G L}{K_A d}}$ when n equals 2, increases monotonically with n, and approaches the limiting value $\frac{\frac{K_G L}{K_A d}}{1+\frac{K_G L}{K_A d}}$.

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