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# EXISTENCE AND UNIQUENESS OF WEAK SOLUTION FOR WEIGHTED LAPLACE DIRICHLET PROBLEM

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**Abstract:** This paper deals with the following equation

$$\begin{cases} -\operatorname{div}\left(h\left(x\right)\nabla u\right)+\lambda a\left(x\right)u=f\left(x,u\right) \text{ in } \Omega, \\ \\ u=0 \text{ on } \partial\Omega \end{cases}$$

in bounded domain $\Omega \in \mathbb{R}^N$ with Dirichlet boundary value condition. The existence and uniqueness results are obtained by Browder Theorem.

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**Key Words:** weak solution, nonlinear elliptic equation, Laplace operator, Browder theorem

## 1. Introduction

Consider the boundary value problem

$$\begin{cases} -\operatorname{div}\left(h\left(x\right)\nabla u\right)+\lambda a\left(x\right)u=f\left(x,u\right) \text{ in } \Omega, \\ u=0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \in C^{0,1}$ be a bounded domain in $\mathbb{R}^N$.Let $\lambda > 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a

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caratheodory function which is decreasing with respect to the second variable, i.e.,

$$f(x,t_1) \leq f(x,t_2) \tag{1.2}$$

for a.a $x \in \Omega$ and $t_1, t_2 \in \mathbb{R}$, $t_1 \geq t_2$ and there exist $\alpha$ and $\beta \in \mathbb{R}$ such that

$$0 < \alpha \leq a(x) \leq \beta$$

Assume, moreover, that there exists $f_0 \in L^2(\Omega)$, and $c > 0$ such that

$$|f(x,s)| \leq f_0(x) + c|s| \tag{1.3}$$

Throughout this work, we assume the weight $h$,$h \in L^1_{loc}(\Omega)$,$h^{-s} \in L^1$, $s \in \left(\frac{N}{2}, \infty\right) \cap [1, +\infty)$.With the number $s$ we define $2_s = \frac{2s}{s+1}, 2^*_s = \frac{N2_s}{N-2_s} > 2$.

We define the Hilbert spaces $W_0^{1,2}(h,\Omega)$ as the clusures of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_h^2 = \int_\Omega h(x) |\nabla u|^2 dx \tag{1.4}$$

for all $u \in C_0^\infty(\Omega)$.

**Definition 1.1.** We say that $u \in X$ is a weak solution to (1.1) if

$$\int_\Omega h(x) \nabla u \nabla v dx + \lambda \int_\Omega a(x) u.v dx = \int_\Omega f(x,u) v dx \tag{1.5}$$

for all $u, v \in W_0^{1,2}(h,\Omega)$.

**Lemma 1.2.** *The space $W_0^{1,2}(h,\Omega)$ is compactly imbedded into the space $L^2(\Omega)$ ,,i.e.*

$$W_0^{1,2}(h,\Omega) \hookrightarrow L^2(\Omega) \tag{1.6}$$

*which means that*

$$\|u\|_{L^2(\Omega)} \leq c_{emb} \|u\|_{W_0^{1,2}(h,\Omega)} \tag{1.7}$$

*i.e.*

$$\int_\Omega |u|^2 dx \leq c_{emb}^2 \int_\Omega h(x) |\nabla u|^2 dx \tag{1.8}$$

*where $c_{emb}$ is the constant of the embedding of $W_0^{1,2}(h,\Omega)$ into $L^2(\Omega)$.*

## 2. Preliminaries and Space Setting

**Definition 2.1**. Let $A : V \to V$ be an operator on a real Banach space $V$. We say that the operator $A$ is:

(i) *bounded*, iff it maps bounded sets into bounded i.e. for each $r > 0$ there exists $M > 0$ ($M$ depending on $r$) such that

$$\|u\| \leq r \Rightarrow \|A(u)\| \leq M, \forall u \in V$$

(ii) *coercive*, iff

$$\lim_{\|u\| \to \infty} \frac{\langle Au, u \rangle}{\|u\|} = \infty$$

(iii) *monotone*, iff $\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq 0$ for all $u_1, u_2 \in V$.

(iv) *strictly monotone*, iff $\langle A(u_1) - A(u_2), u_1 - u_2 \rangle > 0$ for all $u_1, u_2 \in V, u_1 \neq u_2$.

(v) *strongly monotone*, iff $\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq k \|u_1 - u_2\|$ for all $u_1, u_2 \in V, u_1 \neq u_2$.

(vi) *continuous*, iff $(u_n) \to^w u$ implies $Au_n \to A(u)$ for all $u_n, u \in V$.

(viii) *demicontinuous*, iff $(u_n) \to u$ implies $A(u_n) \to^w A(u)$ for all $u_n, u \in V$.

**Theorem 2.2.** (Browder [4]) *Let $A$ be a reflexive real Banach space. Moreover let $A : V \to V$ be an operator which is: bounded, demicontinuous, coercive, and monotone on the space $V$. Then, the system $A(u) = F$ has at least one solution $u \in V$ for each $F \in V'$: If moreover, $A$ is strictly monotone operator, then the system* (1.1) *has precisely one solution $u \in V$ for every $F \in V'$.*

*Proof.* We consider the Sobolev space $X = W_0^{1,2}(h, \Omega)$ with the norm

$$\|u\|_X = \left( \int_\Omega h(x) |\nabla u|^2 dx \right)^{\frac{1}{2}}$$

We define for $\lambda \in \mathbb{R}$ operators $J, G, : X \to X^*$ by

$$\langle J(u), v \rangle = \int_\Omega h(x) \nabla u \nabla v dx$$

$$\langle G(u), v\rangle = \int_\Omega a(x)\, u.vdx$$

and $F : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\langle F(u), v\rangle = \int_\Omega [f(x,u)\, v]\, dx$$

for all $u, v \in X$.

We say that $u$ is a weak solution of (1.1) if

$$\langle A(u), v\rangle = \langle J(u), v\rangle + \lambda \langle G(u), v\rangle - \langle F(u), v\rangle = 0$$

holds for any $v \in X$.Thus, to find a weak solution of (1.1) is equivalent to finding $u \in X$ which satisfies the operator equation $A(u) = 0$.

Now, we have the following properties of the operators $J, G$ and $F$ :

a) $J, G$ and $F$ are well defined. Using Holder's inequality, we have

$$\begin{aligned} |\langle J(u), v\rangle| &= \left| \int_\Omega h(x)\, \nabla u \nabla v dx \right| \\ &\leq \left( \int_\Omega h(x) |\nabla u|^2 dx \right)^{\frac{1}{2}} \left( \int_\Omega h(x) |\nabla v|^2 dx \right)^{\frac{1}{2}} < \infty \end{aligned}$$

$$\begin{aligned} |\langle G(u), v\rangle| &= \left| \int_\Omega a(x)\, u.vx \right| \\ &\leq \int_\Omega |a(x)\, u.vdx| \\ &\leq \beta \left( \int_\Omega |u|^2 dx \right)^{\frac{1}{2}} \left( \int_\Omega |v|^2 dx \right)^{\frac{1}{2}} < \infty \end{aligned}$$

$$|\langle F(u), v\rangle| = \left| \int_\Omega f(x,u)\, vdx \right|$$

$$
\begin{aligned}
&\leq \int_{\Omega} |f_0(x) + c|u|| \, |v| \, dx \\
&\leq \left( \int_{\Omega} |f_0(x)|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |v|^2 \, dx \right)^{\frac{1}{2}} \\
&\quad + c \left( \int_{\Omega} |u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |v|^2 \, dx \right)^{\frac{1}{2}} < \infty
\end{aligned}
$$

and hence $J, G$ and $F$ are well defined.

b) $J, G$ and $F$ are bounded operators. Indeed, for every $u$ such that $\|u\|_X \leq M$; we have

$$
\begin{aligned}
\|J(u)\|_{X^*} &= \sup_{\|v\|_{X^* \leq 1}} |\langle J(u), v \rangle| \\
&\leq \sup_{\|v\|_{X^* \leq 1}} \int_{\Omega} h(x) |\nabla u| \, |\nabla v| \, dx
\end{aligned}
$$

Using Holder's inequality, we obtain

$$
\|J(u)\|_{X^*} \leq \sup_{\|v\|_{X^* \leq 1}} \left[ \left( \int_{\Omega} h(x) |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} h(x) |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \right] \leq M
$$

Similarly, we have

$$
\begin{aligned}
\|G(u)\|_{X^*} &= \sup_{\|v\|_{X^* \leq 1}} |\langle G(u), v \rangle| \\
&\leq \beta \sup_{\|v\|_{X^* \leq 1}} \left( \int_{\Omega} |u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |v|^2 \, dx \right)^{\frac{1}{2}} \\
&\leq \beta c_{emb}^2 M
\end{aligned}
$$

Also, we get

$$
\|F(u)\|_{X^*} = \sup_{\|v\|_{X^* \leq 1}} |\langle f(x, u), v \rangle|
$$

$$
\begin{aligned}
&\le \sup_{\|v\|_{X^*\le 1}} \int_\Omega \left(f_0(x) + c|u|\right)|v|\,dx \\
&\le \sup_{\|v\|_{X^*\le 1}} \left[\left(\int_\Omega |f_0(x)|^2\,dx\right)^{\frac{1}{2}} + \left(\int_\Omega |u|^2\,dx\right)^{\frac{1}{2}}\right]\left(\int_\Omega |v|^2\,dx\right)^{\frac{1}{2}} \\
&\le c_{emb}\left(\|f_0\|_{L^2(\Omega)} + c_{emb}\|u\|_X\right) \\
&\le c_{emb}\left(\|f_0\|_{L^2(\Omega)} + c_{emb}M\right)
\end{aligned}
$$

c) $J, G$ and $F$ are continuous operators. If $u_n \to u$ in $X$: Then, we have $\|u_n - u\|_X \to 0$,so that$\|u_n - u\|_{L^2(\Omega)} \to 0$.

Applying Dominated Convergence Theorem, we obtain

$$\|h(x)(\nabla u_n - \nabla u)\|_{L^2(\Omega)} \to 0.$$

Hence

$$
\begin{aligned}
\|J(u_n) - J(u)\|_{X^*} &= \sup_{\|v\|_{X^*\le 1}} |J(u_n) - J(u), v\rangle| \\
&\le \sup_{\|v\|_{X^*\le 1}} \left(\int_\Omega \left(h(x)[\nabla u_n - \nabla u]^2\right)dx\right)^{\frac{1}{2}} \left(\int_\Omega |v|^2\,dx\right)^{\frac{1}{2}} \\
&\le c_{emb}\left(\int_\Omega \left(h(x)[\nabla u_n - \nabla u]^2\right)dx\right)^{\frac{1}{2}} \to 0 \text{ for } n \to \infty.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\|G(u_n) - G(u)\|_{X^*} &= \sup_{\|v\|_{X^*\le 1}} |G(u_n) - G(u), v\rangle| \\
&\le \beta c_{emb}\left(\int_\Omega (u_n - u)^2\,dx\right)^{\frac{1}{2}} \to 0 \text{ for } n \to \infty.
\end{aligned}
$$

Also, we get

$$\|F(u_n) - F(u)\|_{X^*} = \sup_{\|v\|_{X^*\le 1}} |F(u_n) - F(u), v\rangle|$$

$$\leq \quad c_{emb} \left( \int_{\Omega} |f(x,u_n) - f(x,u)|^2 \, dx \right)^{\frac{1}{2}} \to 0 \text{ for } n \to \infty.$$

d) So

$$\begin{aligned} \langle J(u) - J(v), u - v \rangle &= \int_{\Omega} h(x) (\nabla u - \nabla v)^2 \, dx \\ &= \int_{\Omega} h(x) \nabla u (\nabla u - \nabla v) \, dx - \int_{\Omega} h(x) \nabla v (\nabla u - \nabla v) \, dx \\ &= I_1 + I_2. \end{aligned}$$

$$I_1 + I_2 \geq \int_{\Omega} h(x) |\nabla u - \nabla v|^2 = C \|u - v\|_X^2 .$$

So

$$\langle J(u) - J(v), u - v \rangle \geq C \|u - v\|_X^2 . \tag{2.1}$$

Similarly, we have

$$\begin{aligned} \langle G(u) - G(v), u - v \rangle &= \int_{\Omega} a(x) (u - v)^2 \, dx \\ &\geq \int_{\Omega} a(x) |u - v|^2 \, dx \\ &\geq \alpha C \|u - v\|_{L^2(\Omega)}^2 \geq 0. \end{aligned}$$

Hence

$$\langle G(u) - G(v), u - v \rangle \geq 0. \tag{2.2}$$

Also, we get

$$\langle F(u) - F(v), u - v \rangle = \int_{\Omega} [f(x,u) - f(x,v)] (u - v) \, dx.$$

Since $f$ is decreasing with respect to the second variable, we have

$$[f(x,u) - f(x,v)] (u - v) \leq 0$$

consequently

$$\langle F(u)-F(v),u-v\rangle=\int\limits_{\Omega}[f(x,u)-f(x,v)](u-v)\,dx\leq 0 \tag{2.3}$$

Equations (2.1), (2.2) and (2.3) imply that

$$\langle A(u)-A(v),u-v\rangle\geq C\|u-v\|_X^2. \tag{2.4}$$

So $A$ is strongly monotone.

Now, to apply Browder Theorem, it remains to prove that $A$ is a coercive operator.

From (2.4), we have

$$\langle A(u),u\rangle\geq\langle A(0),u\rangle+C\|u\|_X^2.$$

On the other hand

$$\begin{aligned}\langle A(0),u\rangle &= \langle J(0),u\rangle+\lambda\langle G(0),u\rangle-\langle F(0),u\rangle\\ &= -\int\limits_{\Omega}[f(x,0)u]\,dx\geq-\int\limits_{\Omega}f_0u\,dx\\ &\geq -\left[\int\limits_{\Omega}(f_0(x))^2\right]^{\frac{1}{2}}\left(\int\limits_{\Omega}|u|^2\,dx\right)^{\frac{1}{2}}\\ &\geq c_{emb}\|f_0\|_{L^2(\Omega)}\|u\|_X.\end{aligned}$$

then

$$\langle A(u),u\rangle\geq C\|u\|_X^2-c_{emb}\|f_0\|_{L^2(\Omega)}\|u\|_X.$$

So,

$$\lim_{\|u\|_X\to\infty}\frac{\langle A(u),u\rangle}{\|u\|_X}=\infty.$$

This proves the coercivity condition and so, the existence of weak solution for (1.1).

The uniqueness of weak solution of (1.1), is a direct consequence of (2.4). Suppose that $u,v$ be a weak solutions of (1.1) such that $u\neq v$.

Now, from (2.4), we have

$$0=\langle A(u)-A(v),u-v\rangle\geq C\|u-v\|_X^2\geq 0.$$

Therefore $u=v$. This completes the proof. □

## References

[1] G.A. Afrouzi, S. Mahdavi, Z. Naghizadeh, Existence and uniqueness of solution for $p$-Laplacian Dirichlet problem, *International Journal of Nonlinear Science*, **8**, No. 3 (2009), 274-278.

[2] P. Drabek, A. Kufner, F. Nicolosi, *Quasilinear Elliptic Equation with Degenerations and Singularities*, Walter de Gruyter, Bertin, New York, 1997.

[3] J. Francu. Solvability of operator equations, survey Directed to differential equations, In: *Lecture Notes of IMAMM 94*, Proc. of the Seminar "Industerial Mathematics and Mathematical Modelling", Rybnik, Univ. West Bohemia in Pilsen, Faculty of Applied Sciences, Dept of Math., July, 4-8, 1994.

[4] J. Leray, J.L. Lions, Quelques resultats de Visik sur les proble'mes elliptiques nonlineaires par les methodes de Minty Browder, *Bull. Soc. Math. France*, **107** (1965), 93-97.

[5] P. Lindqvist. On the equation $\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \lambda |u|^{p-2} u = 0$, *Proc. Amer. Math Soc.*, **109** (1990), 157-164.

[6] H. Serag, S. Khafagy, Existence of weak solutions for $n \times n$ nonlinear systems involving different degenerated p-Laplacian operators, *New Zealand Journal of Mathematics*, **38** (2008), 75-86.

[7] H. Serag, S. Khafagy, Existence of weak solutions for nonlinear systems involving degenerated $p$-Laplacian operators, *Sarajevo Journal of Mathematics*, **5**, No. 1 (2009), 41-54.

[8] H. Serag, S. Khafagy, On maximum principle and existence of positive weak solutions for $n \times n$ nonlinear systems involving degenerated p-Laplacian operator, *Turk. J.M.*, **34** (2010), 59-71.

[9] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, Volumes I, II/A, II/B, III and IV, Springer Verlag, Berlin, Heidelberg, New York, 1986.