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L-FUZZY UNIFORMITIES INDUCED BY *L*-NEIGHBORHOOD SYSTEMS AND *L*-FUZZY TOPOLOGIES

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Abstract: In this paper, we obtain L-fuzzy uniformities induced by L-neighborhood systems and L-fuzzy topologies in complete residuated lattices. Moreover, every N-continuous maps are fuzzy uniformly continuous and every fuzzy continuous maps are fuzzy uniformly continuous.

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1. Introduction

Lowen [9] introduced the notion of fuzzy uniformities as a viewpoint of the enourage approach. Many researchers [5-8,13] studied the different approach as powerset [6,13] or the uniform covering [5].

Using the Lowen neighborhood system [10], Katsaras proved that every linear fuzzy neighborhood space is uniformizable in the sense of Lowen uniformity.

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Kim [8] introduced the notion of fuzzy uniformities as an extension of Lowen in a stsc-quantale lattice L. Hájek [4] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Bělohlávek [2] investigated information systems and decision rules in complete residuated lattices.

In this paper, we obtain L-fuzzy uniformities induced by L-neighborhood systems and L-fuzzy topologies in complete residuated lattices. Moreover, every N-continuous maps are fuzzy uniformly continuous and every fuzzy continuous maps are fuzzy uniformly continuous.

2. Preliminaries

Definition 2.1. [2,4] An algebra $(L, \wedge, \vee, \odot, \to, \bot, \top)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, \bot, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\bot$;

(C2) $(L, \odot, \top)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \to z$ for $x, y, z \in L$.

For $\alpha \in L, \lambda \in L^X$, we denote $(\alpha \to \lambda), (\alpha \odot \lambda), \alpha_X \in L^X$ as $(\alpha \to \lambda)(x) = \alpha \to \lambda(x)$, $(\alpha \odot \lambda)(x) = \alpha \odot \lambda(x)$, $\alpha_X(x) = \alpha$.

In this paper, we assume that $(L, \vee, \wedge, \odot, \to, \bot, \top)$ is a complete residuated lattice.

Lemma 2.2. [2,4] *For each $x, y, z, w, x_i, y_i \in L$, the following properties hold:*

(1) *If $y \leq z$, then $x \odot y \leq x \odot z$.*

(2) *If $y \leq z$, then $x \to y \leq x \to z$ and $z \to x \leq y \to x$.*

(3) *$x \to y = \top$ iff $x \leq y$.*

(4) *$x \to \top = \top$ and $\top \to x = x$.*

(5) *$x \odot y \leq x \wedge y$.*

(6) *$x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \odot y = \bigvee_{i\in\Gamma}(x_i \odot y)$.*

(7) *$x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.*

(8) *$\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$ and $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$.*

(9) $(x \to y) \odot x \leq y$ and $(x \to y) \odot (y \to z) \leq (x \to z)$.

(10) $x \to y \leq (y \to z) \to (x \to z)$ and $x \to y \leq (z \to x) \to (z \to y)$.

(11) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.

(12) $y \to z \leq x \odot y \to x \odot z$ and $(x \to z) \odot (y \to w) \leq x \odot y \to z \odot w$.

Definition 2.3. [5] A map $\mathcal{T} : L^X \to L$ is called an L-fuzzy topology on X if it satisfies the following conditions:

(O1) $\mathcal{T}(\bot_X) = \mathcal{T}(\top_X) = \top$,

(O2) $\mathcal{T}(\lambda \odot \mu) \geq \mathcal{T}(\lambda) \odot \mathcal{T}(\mu), \ \forall \ \lambda, \mu \in L^X$,

(O3) $\mathcal{T}(\bigvee_i \lambda_i) \geq \bigwedge_i \mathcal{T}(\lambda_i), \ \forall \ \{\lambda_i\}_{i \in \Gamma} \subseteq L^X$.

An L-fuzzy topology is called enriched if

(R) $\mathcal{T}(\alpha \odot \lambda) \geq \mathcal{T}(\lambda)$ for all $\lambda \in L^X$ and $\alpha \in L$.

The pair $(X, \mathcal{T})$ is called an L-fuzzy topological space.

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L-fuzzy topological spaces. A mapping $\phi : X \to Y$ is said to be fuzzy continuous iff for each $\lambda \in L^Y$,

$$\mathcal{T}_2(\lambda) \leq \mathcal{T}_1(\phi^{\leftarrow}(\lambda)).$$

Definition 2.4. [8] A map $\mathcal{U} : L^{X \times X} \to L$ is called an L-fuzzy uniformity on X iff the following conditions hold:

(U1) There exists $u \in L^{X \times X}$ such that $\mathcal{U}(u) = \top$.

(U2) If $v \leq u$, then $\mathcal{U}(v) \leq \mathcal{U}(u)$.

(U3) For every $u, v \in L^{X \times X}, \mathcal{U}(u \odot v) \geq \mathcal{U}(u) \odot \mathcal{U}(v)$.

(U4) If $\mathcal{U}(u) \neq \bot$ then $\top_\triangle \leq u$ where

$$\top_\triangle(x, y) = \begin{cases} \top, & \text{if x = y} \\ \bot, & \text{if } x \neq y, \end{cases}$$

(U5) $\mathcal{U}(u) \leq \mathcal{U}(u^{-1})$, where $u^{-1}(x, y) = u(y, x)$.

(U6) $\mathcal{U} \leq \mathcal{U} \circ \mathcal{U}$, where $(\mathcal{U} \circ \mathcal{U})(w) = \bigvee \{\mathcal{U}(u) \odot \mathcal{U}(v) \mid u \circ v \leq w\}$,

$$u \circ v(x, z) = \bigvee_{y \in X} u(x, y) \odot v(y, z).$$

An L-fuzzy uniformity $\mathcal{U}$ on X is said to be stratified if

(R) $\mathcal{U}(\alpha \odot u) \geq \alpha \odot \mathcal{U}(u), \quad \forall\, u \in L^{X\times X}$.

The pair $(X,\mathcal{U})$ is called an L-fuzzy uniform space.

Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be L-fuzzy uniform spaces, and $\phi : X \to Y$ ba a mapping. Then ϕ is said to be fuzzy uniformly continuous if $\mathcal{V}(v) \leq \mathcal{U}((\phi\times\phi)^{\leftarrow}(v))$, for every $v \in L^{Y\times Y}$.

Remark 2.5. Let $(X,\mathcal{U})$ be an L-fuzzy uniform space.

(1) By (U1) and (U2), we have $\mathcal{U}(\top_{X\times X}) = \top$ because $u \leq \top_{X\times X}$ for all $u \in L^{X\times X}$.

(2) Since $\mathcal{U}(u) \leq \mathcal{U}(u^{-1}) \leq \mathcal{U}((u^{-1})^{-1}) = \mathcal{U}(u)$, then $\mathcal{U}(u) = \mathcal{U}(u^{-1})$.

Definition 2.6. [5] A map $N : X \to L^{L^X}$ is called an L-neighborhood system on X if N satisfies the following conditions

(N1) $N_x(\top_X) = \top$ and $N_x(\bot_X) = \bot$,

(N2) $N_x(\lambda \odot \mu) \geq N_x(\lambda) \odot N_x(\mu)$ for each $\lambda, \mu \in L^X$,

(N3) If $\lambda \leq \mu$, then $N_x(\lambda) \leq N_x(\mu)$,

(N4) $N_x(\lambda) \leq \lambda(x)$ for all $\lambda \in L^X$.

(N5) $N_x(\lambda) \leq \bigvee\{N_x(\mu) \mid \mu(y) \leq N_y(\lambda),\ \forall\, y \in X\}$.

The previous axiom can be reformulated in the following way
(N5) $\forall\, \lambda \in L^X$ and $x \in X$, $N_x(\lambda) \leq N_x(N_-(\lambda))$, where $N_-(\lambda) \in L^X$ is defined by

$$[N_-(\lambda)](y) = N_y(\lambda) \quad \forall\, y \in Y.$$

An L-neighborhood system is called stratified if

(R) $N_x(\alpha \odot \lambda) \geq \alpha \odot N_x(\lambda)$ for all $\lambda \in L^X$ and $\alpha \in L$.

The pair (X, N) is called an L-neighborhood space.
Let (X, N) and (Y, M) be two L-neighborhood spaces. A mapping $\phi : X \to Y$ is said to be N-continuous at $x \in X$ iff $M_{\phi(x)}(\lambda) \leq N_x(\phi^{\leftarrow}(\lambda))$ for each $\lambda \in L^Y, \phi$ is N-continuous if it is N-continuous at every $x \in X$.

3. L-Fuzzy Uniformities Induced by L-Neighborhood Systems and L-Fuzzy Topologies

Definition 3.1. For $\lambda \in L^X$, we define $u_\lambda \in L^{X\times X}$ associated with λ by

$$u_\lambda(x,y) = \begin{cases} \top, & \text{if } x = y \\ \lambda(x)\odot\lambda(y), & \text{if } x \neq y. \end{cases}$$

Theorem 3.2. *Let (X, N) be a stratified L-neighborhood space. Define a map $\mathcal{U}_N : L^{X\times X} \to L$ by*

$$\mathcal{U}_N(u) = \bigvee\{\bigvee_{x\in X} N_x(\lambda) \mid u_\lambda \le u\},\ \ \forall\, u \in L^{X\times X}.$$

Then $(X,\mathcal{U}_N)$ is a stratified L-fuzzy uniform space.

Proof. (U1) Since $u_{1_X} \le 1_{X\times X}$, we have

$$\mathcal{U}(1_{X\times X}) \ge \bigvee_{x\in X} N_x(\top_x) = \top.$$

(U2) If $u_1 \le u_2,\ u_1,u_2 \in L^{X\times X}$, then

$$\begin{aligned}\mathcal{U}(u_1) &= \bigvee\{\bigvee_{x\in X} N_x(\lambda) \mid u_\lambda \le u_1\}\\ &\le \bigvee\{\bigvee_{x\in X} N_x(\lambda) \mid u_\lambda \le u_2\} = \mathcal{U}(u_2).\end{aligned}$$

(U3) Since $u_\lambda \odot u_\rho = u_{\lambda\odot\rho}$, we have

$$\begin{aligned}&\bigvee\{N_x(\lambda) \mid u_\lambda \le u\} \odot \bigvee\{N_x(\rho) \mid u_\rho \le u\}\\ &\le \bigvee\{N_x(\lambda)\odot N_x(\rho) \mid u_\lambda\odot u_\rho \le u\odot v\}\\ &\le \bigvee\{\bigvee_{x\in X} N_x(\lambda\odot\rho) \mid u_{\lambda\odot\rho} \le u\odot v\}\\ &\le \mathcal{U}(u\odot v).\end{aligned}$$

By Lemma 2.2(6), $\mathcal{U}(u)\odot\mathcal{U}(v) \le \mathcal{U}(u\odot v)$.

(U4) If $\mathcal{U}(u) \neq \bot$, then there exists $\lambda \in L^X$ with $N_x(\lambda) \neq \bot$ such that $u_\lambda \le u$. Hence $\top_\triangle \le u_\lambda \le u$.

(U5) Let $u \in L^{X\times X}$, then

$$\begin{aligned}\mathcal{U}(u) &= \bigvee\{\bigvee_{x\in X} N_x(\lambda) \mid u_\lambda \le u\}\\ &= \bigvee\{\bigvee_{x\in X} N_x(\lambda) \mid u_\lambda^{-1} \le u^{-1}\} = \mathcal{U}(u^{-1}).\end{aligned}$$

(U6) Since $N_-(\lambda) = \bigvee_{z\in X}(N_z(\lambda)\odot \top_z)$ and N is stratified, then

$$N_x(N_-(\lambda)) = N_x(\bigvee_{z\in X}(N_z(\lambda)\odot \top_z) \geq \bigvee_{z\in X}(N_z(\lambda)\odot N_x(\top_z))$$

Thus,

$$\begin{aligned}
&\mathcal{U}_N(u) = \bigvee\{\bigvee_{x\in X} N_x(\lambda) \mid u_\lambda \leq u\}\\
&\leq \bigvee\{\bigvee_{x\in X} N_x(N_-(\lambda)) \mid u_\lambda \leq u\} \quad \text{(by (N5))}\\
&\leq \bigvee\{\bigvee_{x\in X}\bigvee_{z\in X} N_z(\lambda)\odot N_x(\top_z) \mid u_\lambda \circ u_{\top_z} \leq u_\lambda \leq u\}
\end{aligned}$$

Put $w(x,y) = \bigwedge_{p\in X} u_{\top_z}(y,p) \to u(x,p)$, then $w\circ u_{\top_z} \leq u$ because

$$\begin{aligned}
w\circ u_{\top_z}(x,p) &= \bigvee_{y\in X} w(x,y)\odot u_{\top_z}(y,p)\\
&= \bigvee_{y\in X}(\bigwedge_{s\in X} u_{\top_z}(y,s) \to u(x,s))\odot u_{\top_z}(y,p)\\
&\leq \bigvee_{y\in X}(u_{\top_z}(y,p) \to u(x,p))\odot u_{\top_z}(y,p)\\
&\leq \bigvee_{y\in X} u(x,p) = u(x,p).
\end{aligned}$$

$$\begin{aligned}
&\mathcal{U}_N(u) = \bigvee\{\bigvee_{x\in X} N_x(\lambda) \mid u_\lambda \leq u\}\\
&\leq \bigvee\{\bigvee_{x\in X}\bigvee_{z\in X} N_z(\lambda)\odot N_x(\top_z) \mid u_\lambda \circ u_{\top_z} \leq u_\lambda \leq u\}\\
&\leq \bigvee\{\bigvee_{x\in X}\bigvee_{z\in X} N_z(\lambda)\odot N_x(\top_z) \mid u_\lambda(x,y) \leq \bigwedge_{p\in X} u_{\top_z}(y,p) \to u(x,p)\}\\
&\leq \bigvee\{\bigvee_{x\in X}\bigvee_{z\in X} N_z(\lambda)\odot N_x(\top_z) \mid u_\lambda \leq w\}\\
&\leq \bigvee\{\bigvee_{z\in X} N_z(\lambda) \mid u_\lambda \leq w\}\odot\{\bigvee_{x\in X} N_x(\top_z) \mid u_{\top_z} \leq u_{\top_z}\}\\
&\leq \{\mathcal{U}(w)\odot\mathcal{U}(u_{\top_z}) \mid w\circ u_{\top_z} \leq u\}\\
&\leq \bigvee\{\mathcal{U}(u_1)\odot\mathcal{U}(u_2) \mid u_1\circ u_2 \leq u\} = \mathcal{U}\circ\mathcal{U}(u)
\end{aligned}$$

Finally, let $u \in L^{X\times X}$, $\lambda \in L^X$ and $\alpha \in L$, we have

$$\begin{aligned}
\mathcal{U}_N(\alpha\odot u) &= \bigvee\{\bigvee_{x\in X} N_x(\rho) \mid u_\rho \leq \alpha\odot u\}\\
&\geq \bigvee\{\bigvee_{x\in X} N_x(\alpha\odot\lambda) \mid u_{\alpha\odot\lambda} \leq \alpha\odot u\}\\
&\geq \bigvee\{\alpha\odot\bigvee_{x\in X} N_x(\lambda) \mid \alpha\odot u_\lambda \leq \alpha\odot u\}\\
&\geq \alpha\odot\bigvee\{\bigvee_{x\in X} N_x(\lambda) \mid u_\lambda \leq u\}\\
&= \alpha\odot\mathcal{U}_N(u).
\end{aligned}$$

Theorem 3.3. *Let (X,N) and (Y,M) be stratified L-neighborhood spaces. Let $\phi : (X,N) \to (Y,M)$ be N-continuous. Then $\phi : (X,\mathcal{U}_N) \to (Y,\mathcal{V}_M)$ is fuzzy uniformly continuous.*

Proof. We want to show that for all $v \in L^{Y\times Y}$, we have

$$\mathcal{U}_N((\phi\times\phi)^{\leftarrow}(v)) \geq \mathcal{V}_M(v).$$

$$\begin{aligned}
&(\phi\times\phi)^{\leftarrow}(v_\lambda)(x_1,x_2) = v_\lambda(\phi(x_1),\phi(x_2))\\
&= \begin{cases} 1, & \text{if } \phi(x_1)=\phi(x_2)\\ \lambda(\phi(x_1))\odot\lambda(\phi(x_2)), & \text{if } \phi(x_1)\neq\phi(x_2)\end{cases}\\
&\geq \begin{cases} 1, & \text{if } x_1=x_2\\ \phi^{\leftarrow}(\lambda)(x_1)\odot\phi^{\leftarrow}(\lambda)(x_2), & \text{if } x_1\neq x_2\end{cases}\\
&= v_{\phi^{\leftarrow}(\lambda)}(x_1,x_2).
\end{aligned}$$

Also, $v(\phi(x),\phi(y)) = (\phi\times\phi)^{\leftarrow}(v)(x,y)$. Thus:

$$\begin{aligned}
&\mathcal{V}_M(v) = \bigvee\{\bigvee_{y\in Y} M_y(\lambda) \mid v_\lambda\leq v\}\\
&= \bigvee\{\bigvee_{x\in X} M_{\phi(x)}(\lambda) \mid v_\lambda(\phi(x),\phi(y))\leq v(\phi(x),\phi(y))\}\\
&= \bigvee\{\bigvee_{x\in X} M_{\phi(x)}(\lambda) \mid (\phi\times\phi)^{\leftarrow}(v_\lambda)(x,y)\leq(\phi\times\phi)^{\leftarrow}(v)(x,y)\}\\
&\leq \bigvee\{\bigvee_{x\in X} N_x(\phi^{\leftarrow}(\lambda)) \mid v_{\phi^{\leftarrow}(\lambda)}\leq(\phi\times\phi)^{\leftarrow}(v)\}\\
&\leq \mathcal{U}_N((\phi\times\phi)^{\leftarrow}(v)).
\end{aligned}$$

Theorem 3.4. *Let $(X,\mathcal{T})$ be an L-fuzzy topological space. Define a map $\mathcal{U}_{\mathcal{T}}: L^{X\times X}\to L$ by*

$$\mathcal{U}_{\mathcal{T}}(u) = \bigvee\{\mathcal{T}(\lambda) \mid u_\lambda\leq u, \lambda\neq\bot_X\}.$$

Then (1) $(X,\mathcal{U}_{\mathcal{T}})$ is an L-fuzzy uniform space,

(2) If $(X,\mathcal{T})$ is enriched, then $\mathcal{U}_{\mathcal{T}}(\alpha\odot u)\geq\mathcal{U}_{\mathcal{T}}(u)$ for all $\alpha\in L$ and $u\in L^{X\times X}$.

Proof. (U1) Since $u_{1_X}\leq 1_{X\times X}$, we have

$$\mathcal{U}(1_{X\times X})\geq\mathcal{T}(\top_x)=\top.$$

(U2) If $u_1\leq u_2$, $u_1,u_2\in L^{X\times X}$, then

$$\begin{aligned}
\mathcal{U}(u_1) &= \bigvee\{\mathcal{T}(\lambda)\mid u_\lambda\leq u_1\}\\
&\leq \bigvee\{\mathcal{T}(\lambda)\mid u_\lambda\leq u_2\} = \mathcal{U}(u_2).
\end{aligned}$$

(U3) Let $\lambda,\mu\in L^X$ and $u,v\in L^{X\times X}$, then $u_\lambda\odot v_\mu = u_{\lambda\odot\mu}$ and

$$\begin{aligned}
&\mathcal{U}_{\mathcal{T}}(u)\odot\mathcal{U}_{\mathcal{T}}(w)\\
&= \bigvee\{\mathcal{T}(\lambda)\mid u_\lambda\leq u,\lambda\neq\bot_X\}\odot\bigvee\{\mathcal{T}(\rho)\mid u_\rho\leq u,\lambda\neq\bot_X\}\\
&\leq \bigvee\{\mathcal{T}(\lambda)\odot\mathcal{T}(\rho)\mid u_\lambda\odot u_\rho = u_{\lambda\odot\rho}\leq u,\lambda\odot\rho\neq\bot_X\}\\
&\leq \bigvee\{\mathcal{T}(\lambda\odot\rho)\mid u_\lambda\odot u_\rho = u_{\lambda\odot\rho}\leq u,\lambda\odot\rho\neq\bot_X\}\\
&\leq \mathcal{U}_{\mathcal{T}}(u\odot w).
\end{aligned}$$

(U4) If $\mathcal{U}(u) \neq \bot$, then there exists $\lambda \in L^X$ with $\mathcal{T}(\lambda) \neq \bot$ such that $u_\lambda \leq u$. Hence $\top_\triangle \leq u_\lambda \leq u$.

(U5) Let $u \in L^{X\times X}$, then

$$\begin{aligned}\mathcal{U}(u) &= \bigvee\{\mathcal{T}(\lambda) \mid u_\lambda \leq u\}\\ &= \bigvee\{\mathcal{T}(\lambda) \mid u_\lambda^{-1} \leq u^{-1}\} = \mathcal{U}(u^{-1}).\end{aligned}$$

(U6) Let $u \in L^{X\times X}$ and $\lambda \in L^X$. Choose $\mu, \rho \in L^X$ such that $u_\mu \circ u_\rho \leq u_\lambda$ and now, we have

$$\begin{aligned}&\mathcal{U}_\mathcal{T}(u) = \bigvee\{\mathcal{T}(\lambda) \mid u_\lambda \leq u, \lambda \neq \bot_X\}\\ &= \bigvee\{\mathcal{T}(\lambda) \odot \mathcal{T}(\top_X) \mid u_\lambda \circ u_{\top_X} \leq u, \lambda \neq \bot_X\}\\ &\leq \bigvee\{\mathcal{T}(\lambda_1) \odot \mathcal{T}(\lambda_2) \mid u_{\lambda_1} \circ u_{\lambda_2} \leq u, \lambda_i \neq \bot_X\}\\ &\leq \bigvee\{\mathcal{U}_\mathcal{T}(u_1) \odot \mathcal{U}_\mathcal{T}(u_2) \mid u_1 \circ u_2 \leq u\}\\ &= \mathcal{U}_\mathcal{T} \circ \mathcal{U}_\mathcal{T}(u).\end{aligned}$$

Finally, let $u \in L^{X\times X}$, $\lambda \in L^X$ and $\alpha \in L$, we have

$$\begin{aligned}\mathcal{U}_\mathcal{T}(\alpha \odot u) &= \bigvee\{\mathcal{T}(\lambda) \mid u_\lambda \leq \alpha \odot u\}\\ &\geq \bigvee\{\mathcal{T}(\alpha \odot \lambda) \mid u_{\alpha\odot\lambda} \leq \alpha \odot u\}\\ &\geq \bigvee\{\mathcal{T}(\lambda) \mid \alpha \odot u_\lambda \leq \alpha \odot u\}\\ &\geq \bigvee\{\mathcal{T}(\lambda) \mid u_\lambda \leq u\}\\ &= \mathcal{U}_\mathcal{T}(u).\end{aligned}$$

Theorem 3.5. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be L-fuzzy topological spaces. Let $\phi : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be fuzzy continuous. Then $\phi : (X, \mathcal{U}_{\mathcal{T}_X}) \to (Y, \mathcal{V}_{\mathcal{T}_Y})$ is fuzzy uniformly continuous.*

Proof. Since $(\phi \times \phi)^{\leftarrow}(v_\lambda)(x_1, x_2) = v_\lambda(\phi(x_1), \phi(x_2)) \geq v_{\phi^{\leftarrow}(\lambda)}(x_1, x_2)$ and $v(\phi(x), \phi(y)) = (\phi \times \phi)^{\leftarrow}(v)(x, y)$, we have

$$\begin{aligned}&\mathcal{V}_{\mathcal{T}_Y}(v) = \bigvee\{\mathcal{T}_Y(\lambda) \mid v_\lambda \leq v\}\\ &\leq= \bigvee\{\mathcal{T}_Y(\lambda) \mid (\phi \times \phi)^{\leftarrow}(v_\lambda) \leq (\phi \times \phi)^{\leftarrow}(v)\}\\ &\leq= \bigvee\{\mathcal{T}_X(\phi^{\leftarrow}(\lambda)) \mid v_{\phi^{\leftarrow}(\lambda)} \leq (\phi \times \phi)^{\leftarrow}(v)\}\\ &\leq \mathcal{U}_{\mathcal{T}_X}((\phi \times \phi)^{\leftarrow}(v)).\end{aligned}$$

Example 3.6. Let $(L = [0,1], \odot, \rightarrow, ^*)$ be a complete residuated lattice with a strong negation defined by

$$x \odot y = (x + y - 1) \vee 0, \;\; x \rightarrow y = (1 - x + y) \wedge 1, \; x^* = 1 - x.$$

Let $X = \{x, y, z\}$ be a set and $\rho \in L^X$ with $\rho(x) = 0.6, \rho(y) = 0.6, \rho(z) = 0.5$. Define $\mathcal{T} : L^X \rightarrow L$ as follows

$$\mathcal{T}(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{1_X, 0_X\}, \\ 0.6, & \text{if } \lambda = \rho, \\ 0.3, & \text{if } \; \lambda = \rho \odot \rho, \\ 0, & \text{otherwise.} \end{cases}$$

We obtain $u_{\rho \otimes \rho} = 1_\triangle$ and

$$u_\rho = \begin{pmatrix} 1 & 0.2 & 0.1 \\ 0.2 & 1 & 0.1 \\ 0.1 & 0.1 & 1 \end{pmatrix}.$$

By Theorem 3.4, we can construct $\mathcal{U} : L^{X \times X} \rightarrow L$ as follows

$$\mathcal{U}(u) = \begin{cases} 1, & \text{if } u = 1_{X \times X}, \\ 0.6, & \text{if } u_\rho \leq u \neq 1_{X \times X}, \\ 0.3, & \text{if } \; 1_\triangle \leq u \not\geq u_\rho, \\ 0, & \text{otherwise.} \end{cases}$$

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