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DYNAMICS OF THE DILATION WEIGHTED COMPOSITION OPERATORS

Bahmann Yousefi[1], Fariba Ershad[2 §]

[1,2]Department of Mathematics
Payame Noor University
P.O. Box 19395-3697, Tehran, IRAN

Abstract: In this paper we investigate the hypercyclicity of adjoint of a special weighted composition operators on Hilbert spaces of analytic functions on the open unit disc.

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1. Introduction

A bounded linear operator T on a Hilbert space H is said to be hypercyclic if there exists a vector $x \in H$ for which the orbit $Orb(T, x) = \{T^n x : n \in \mathbb{N}\}$ is dense in H and in this case we refer to x as a hypercyclic vector for T. The holomorphic self maps of the open unit disk U are divided into classes of elliptic and non-elliptic. The elliptic type is an automorphism and has a fixed point in U. It is well known that this map is conjugate to a rotation $z \to \lambda z$ for some complex number λ with $|\lambda| = 1$. The maps of that are not elliptic are called of non-elliptic type. The iterate of a non-elliptic map can be characterized by the Grand Iteration Theorem (see [6], p. 78). By $\psi'(w)$ we denote the angular

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§Correspondence author

derivative of ψ at $w \in \partial U$. Note that if $w \in U$, then $\psi'(w)$ has the natural meaning of derivative. Also, by ψ_n we mean the nth iteration of the function ψ.

Proposition 1.1. (see [6]) *Suppose ψ is a holomorphic self-map of U that is not an elliptic automorphism. If ψ has a fixed point $w \in U$, then $\psi_n \to w$ uniformly on compact subsets of U and $|\psi'(w)| < 1$.*

The unique attracting point w in the above proposition is called the *Denjoy-Wolff point* of ψ, and ψ is called a dilation of the open unit disk.

Definition 1.2. Let H be a Hilbert space of analytic functions on the open unit disk U and let ψ be a dilation of the open unit disk U. We say that C_ψ is quasi-contraction on H if there exist a neighborhood U_w of w and $0 < \alpha < 1$ such that $|C_\psi f(z)| \leq \alpha |f(z)|$ for all $f,\ g \in H$, and every $z \in U_w$.

Let H be a Hilbert space of analytic functions on the open unit disk U. For each $\lambda \in U$, the evaluation function $e_\lambda : H \to C$ is defined by $e_\lambda(f) = f(\lambda)$, $f \in H$. A complex valued function φ on U for which $\varphi H \subseteq H$ is called a multiplier of H. The set of all multipliers of H is denoted by $M(H)$.

Consider the weighted composition operator $C_{\varphi,\psi}$ on a Hilbert space H of analytic functions on the open unit disk U defined by $C_{\varphi,\psi} f = \varphi . f \circ \psi \ \ (f \in H)$. We will investigate the hypercyclicity of the operator $C^*_{\varphi,\psi}$. For some sources we refer to [1]-[8].

2. Main Results

In this section ψ will denote a holomorphic self-map of U and φ is a nonzero holomorphic map on U.

Note that if the adjoint of a continuous operator T on a Banach space has an eigenvector, then T^* fails to be hypercyclic (see [4]).

Lemma 2.1. *Let H be a separable Hilbert space of analytic functions on the open unit disk U such that for each $\lambda \in U$, e_λ is bounded on H. Also, let $||M_z|| \leq 1$ on H, then $M(H) = H^\infty(U)$.*

Proof. Let $f \in H^\infty(U)$. Then there is a sequence $\{p_n\}_n$ of polynomials converging to f pointwise and for all n, $\|p_n\|_U \leq M$ for some $M > 0$. Since $||M_z|| \leq 1$ on H, by the Von-Neumann inequality, $\|M_q\| = \|q\|_U$ for all polynomials q. Hence we obtain $\|M_{p_n}\| \leq M$ for all n. But ball $B(H)$ is compact in the weak operator topology and so by passing to a subsequence if necessary, we may assume that for some $A \in B(H)$, $M_{p_n} \longrightarrow A$ in the weak operator topology. Using the fact that $M_{p_n}^{*} \longrightarrow A^{*}$ in the weak operator topology and acting these operators on e_λ we get $p_n(\lambda)e_\lambda = M_{p_n}^{*}e_\lambda \longrightarrow A^{*}e_\lambda$ weakly. Since $p_n(\lambda) \longrightarrow f(\lambda)$ we see that $A^{*}e_\lambda = f(\lambda)e_\lambda$ from which we can conclude that $A = M_f$ and this implies that $f \in M(H)$. Thus $H^\infty(U) \subset M(H)$. But it is well-known that $M(H) \subset H^\infty(U)$ (see [7]). Thus indeed, $M(H) = H^\infty(U)$. □

Theorem 2.2. *Let H be a separable Hilbert space of analytic functions on the open unit disk U such that $1 \in H$, for each $\lambda \in U$, e_λ is bounded, and $||M_z|| \leq 1$ on H. Furthermore, suppose that $\varphi \in M(H)$ and ψ is a dilation map of U with Denjoy-Wolff point w such that $\varphi(w) \neq 0$, and let C_ψ be a quasi-contraction. If $|\varphi \circ \psi_n(z)| \leq |\varphi(w)|$ eventually for all n or $\|\psi\|_U < 1$, then $\prod_{i=0}^{\infty} \frac{1}{\varphi(w)}\varphi \circ \psi_i$ is an eigenvector of $C_{\varphi,\psi}$ acting on H and $C_{\varphi,\psi}^{*}$ is not hypercyclic.*

Proof. First, let $|\varphi \circ \psi_n(z)| \leq |\varphi(w)|$ eventually for all n. Since C_ψ is quasi-contraction, there exist a neighborhood U_w of w and $0 < \alpha < 1$ such that

$$|C_\psi f(z) - C_\psi g(z)| \leq \alpha |f(z) - g(z)| \qquad (*)$$

for all $f,\ g \in H$, and every $z \in U_w$. Let $n > 2$ and define $f = \varphi \circ \psi_{n-1}$ and $g = \varphi(w)$. Note that $\varphi \circ \psi_n = C_\psi \varphi \circ \psi_{n-1}$, and since $\psi_n(z) \to w$, for n large enough we have $\psi_{n-1} \in U_w$. Clearly, f and g are in H and by using $(*)$ we have

$$\begin{aligned} |\varphi \circ \psi_n(z) - \varphi(w)| &\leq \alpha |\varphi \circ \psi_{n-1}(z) - \varphi(w)| \\ &\leq \alpha^n |\varphi(z) - \varphi(w)| \end{aligned}$$

for all $z \in U_w$ and n large enough. Therefore,

$$|1 - \frac{1}{\varphi(w)}\varphi(\psi_n(z))| \leq (1 + \frac{1}{|\varphi(w)|}|\varphi(z)|)\ \alpha^n.$$

Thus $\sum_{n=0}^{\infty} |1 - \frac{1}{\varphi(w)}\varphi(\psi_n(z))|$ converges uniformly on K. Hence $\prod_{n=0}^{\infty} \frac{1}{\varphi(w)}\varphi(\psi_n(z))$ also converges uniformly on K. Define $g(z) = \prod_{n=0}^{\infty} \frac{1}{\varphi(w)}\varphi(\psi_n(z))$. Thus g is nonzero holomorphic function on U and by our assumption, $g \in H^\infty(U)$. Now by the Lemma 2.1, $g \in M(H)$. But $1 \in H$, thus indeed $g \in H$. Note that $\varphi(z) \cdot g(\psi(z)) = \varphi(w)g(z)$, hence $\prod_{i=0}^{\infty} \frac{1}{\varphi(w)} \varphi \circ \psi_i$ is an eigenvector of $C_{\varphi,\psi}$. Now let $\|\psi\|_U < 1$ and note that $C^n_{\varphi,\psi} g = \varphi(w)^n g$. Thus

$$g = \varphi(w)^{-n} \prod_{j=0}^{n-1} \varphi(\psi_j) g \circ \psi_n.$$

Since $||\psi||_U < 1$, clearly $g \in H^{\infty}(U)$ and so $g \in H$. Also, $\prod_{i=0}^{\infty} \frac{1}{\varphi(w)} \varphi \circ \psi_i$ is an eigenvector of $C_{\varphi,\psi}$. Since in each case $\varphi(w)$ is an eigenvalue of $C_{\varphi,\psi}$, hence $C^{\square}_{\varphi,\psi}$ is not hypercyclic and so the proof is complete. $\square$

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