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DYNAMICS OF HYPERBOLIC WEIGHTED COMPOSITION OPERATORS

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Abstract: In the present paper we investigate conditions under which a hyperbolic self-map of the open unit disk induces a hypercyclic weighted composition operator in the space of holomorphic functions on the unit ball in $\mathbb{C}^N$.

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1. Introduction

For $z = (z_1, ..., z_N)$ and $w = (w_1, ..., w_N)$ in $\mathbb{C}^N$, write $< z, w >$ for the Euclidean inner product $\sum_{j=1}^{N} z_j \bar{w}_j$ and let $|z| =< z, z >^{1/2}$. With this notation, the unit ball in $\mathbb{C}^N$ is the set $B_N = \{z \in \mathbb{C}^N : |z| < 1\}$ and the unit sphere in $\mathbb{C}^N$ is the set $S_N = \{z \in \mathbb{C}^N : |z| = 1\}$, analogously to the unit disc and circle for $N = 1$. The space $H(B_N)$, is the set of all holomorphic fuctions on B_N, can be made into a F-space by a complete metric for which a sequence $\{f_n\}$ in $H(B_N)$ converges to $f \in H(B_N)$ if and only if $f_n \longrightarrow f$ uniformly on every compact subsets of B_N. Each $\varphi \in H(B_N)$ and holomorphic self-map ψ of B_N induces a linear weighted composition operator $C_{\varphi,\psi} : H(B_N) \longrightarrow H(B_N)$ de-

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fined by $C_{\varphi,\psi}(f)(z) = \varphi(z)f(\psi(z))$ for every $f \in H(B_N)$ and $z \in B_N$. Indeed, $C_{\varphi,\psi} = M_\varphi C_\psi$ where M_φ denotes the operator of multiplication by φ and C_ψ is a composition operator by means of the definition $C_\psi(f) = f \circ \psi$ for every $f \in H(B_N)$.

A bounded linear operator T on a F-space X is said to be hypercyclic if there exists a vector $x \in X$ for which the orbit $Orb(T, x) = \{T^n x : n \in \mathbb{N}\}$ is dense in X and in this case we refer to x as a hypercyclic vector for T.

The holomorphic self maps of B_N are divided into classes of elliptic and non-elliptic. The elliptic type is an automorphism and has a fixed point in B_N. It is well known that this map is conjugate to a rotation.

For simplicity, throughout this paper we use the notation " $\overset{k}{\to}$ " for indicating uniform convergence on compact subsets of B_N. Also, by ψ_n we denote the nth iterate of ψ. To state the main result of the paper, we need the following theorems from [3].

Theorem 1.1. (Denjoy-Wolff Iteration in B_N) *Suppose ψ is a holomorphic self-map of the open unit ball B_N without interior fixed point. Then there is a point $w \in \partial B_N$ such that $\psi_n \overset{k}{\to} w$ and $0 < d(w) \leq 1$ where*

$$d(w) = \liminf_{|z| \to 1^-} \frac{1 - |\psi(z)|^2}{1 - |z|^2}.$$

The boundary point w is called the Denjoy-Wollf point of ψ.

Theorem 1.2. (Julia's Lemma in B_N) *Let ψ be an analytic map of the unit ball into itself with Denjoy-Wolff point $w \in \partial B_N$. Then for every $z \in B_N$,*

$$\frac{|1 - <\psi(z), w>|^2}{1 - |\psi(z)|^2} \leq d(w) \, \frac{|1 - <z, w>|^2}{1 - |z|^2}.$$

Recall that a holomorphic self-map ψ of B_N is called elliptic if ψ has a fixed point in B_N. Also, if ψ has no interior fixed point, then it is called hyperbolic whenever $d(w) < 1$, and is called parabolic if $d(w) = 1$.

For simplicity, we call a weighted composition operator $C_{\varphi,\psi}$, a hyperbolic weighted composition operator whenever the compositional symbol ψ is hyperbolic.

Definition 1.3. We say that a mapping $\varphi : B_N \to \mathbf{C}$ is semi-nonexpansive provided there exists a neighborhood U_w of w such that $|\varphi(z) - \varphi(w)| \leq |z - w|$ for all z in $U_w \cap B_N$.

The next section of the present paper shows that weighted composition operators with non-constant weight function and hyperbolic compositional symbol can be hypercyclic on $H(B_N)$. For some sources see [1]-[7].

2. Main Result

In this section we investigate the hypercyclicity of a hyperbolic weighted composition operator acting on $H(B_N)$.

Proposition 2.1. *Let φ be a nonzero holomorphic map on B_N and ψ be a hyperbolic map of B_N with w the Denjoy-Wolff point such that $\varphi(w) \neq 0$. If φ is semi-nonexpansive, then $C^*_{\varphi,\psi}$ is not hypercyclic, but $C_{\varphi,\psi}$ is hypercyclic whenever φ never vanishes on B_N, C_ψ is hypercyclic and $|\varphi(w)| = 1$.*

Proof. Let K be a compact subset of B_N. By Theorem 1.2, there exists a constant $c > 0$ such that

$$|1- < \psi_n(z), w > |^2 \leq c(1 - |\psi_n(z)|^2)$$

for every $z \in K$ and every $n \in \mathbb{N}$. But $|1- < \psi_n(z), w > |^2 = |w - \psi_n(z)|^2$, thus $|w - \psi_n(z)|^2 \leq c(1 - |\psi_n(z)|^2)$ for every $z \in K$ and every $n \in \mathbb{N}$. On the otherhand, since φ is semi-nonexpansive, there exists a neighborhood U_w of w satisfying $|\varphi(w) - \varphi(z)| \leq |w - z|$ for every z in $U_w \cap B_N$. Since $\psi_n \overset{k}{\to} w$, there exists N such that for all $n > N$, $\psi_n(z) \in U_w$. Substituting $\psi_n(z)$ instead of z in the previous relation we get

$$\begin{aligned} |\varphi(w) - \varphi(\psi_n(z))| \leq & |w - \psi_n(z)| \\ = & |1- < \psi_n(z), w > | \\ \leq & c^{1/2}(1 - |\psi_n(z)|^2)^{1/2}, \end{aligned} \qquad (*)$$

for every $n > N$. Now we apply the techniques used in [7]. Since ψ is hyperbolic, thus $0 < d(w) < 1$ and by Theorem 1.2 we have

$$\frac{|1- < \psi(z), w > |^2}{1 - |\psi(z)|^2} \leq d(w) \, \frac{|1- < z, w > |^2}{1 - |z|^2}$$

for all $z \in B_N$. By substituting $\psi_n(z)$ for $\psi(z)$ in the above inequality, we get

$$\frac{|1- < \psi_n(z), w > |^2}{1 - |\psi_n(z)|^2} \leq d(w)^n \, \frac{|1- < z, w > |^2}{1 - |z|^2}$$

for every $z \in B_N$ and $n \in \mathbb{N}$. Also, note that since K is compact, then there exists a constant $\beta > 0$ such that $4\frac{|1-<z,w>|^2}{1-|z|^2} < \beta$ for all z in K. So it follows that

$$\begin{aligned} 1-|\psi_n(z)|^2 &= (1-|\psi_n(z)|)(1+\psi_n(z)|) \\ &\leq 2|1-<\psi_n(z),w>| \\ &\leq 4\frac{|1-<\psi_n(z),w>|^2}{1-|\psi_n(z)|^2} \\ &\leq 4\frac{|1-<z,w>|^2}{1-|z|^2}d(w)^n \\ &< \beta\, d(w)^n. \end{aligned}$$

Hence we obtain

$$1-|\psi_n(z)|^2 \leq \beta d(w)^n. \qquad (**)$$

Now by using the relations $(*)$ and $(**)$, we get

$$\begin{aligned} |1-\frac{1}{\varphi(w)}\varphi(\psi_n(z))| &< \frac{c^{\frac{1}{2}}}{|\varphi(w)|}(1-|\psi_n(z)^2|)^{1/2} \\ &\leq \frac{c^{\frac{1}{2}}\beta^{\frac{1}{2}}}{|\varphi(w)|}d(w)^{n/2}. \end{aligned}$$

Since $0 < d(w) < 1$, thus $\sum_{n=0}^{\infty}|1-\frac{1}{\varphi(w)}\varphi(\psi_n(z))|$ and so $\prod_{n=0}^{\infty}\frac{1}{\varphi(w)}\varphi(\psi_n(z))$ converges uniformly on K. Define

$$g(z) = \prod_{n=0}^{\infty}\frac{1}{\varphi(w)}\varphi(\psi_n(z)).$$

Since $\varphi(w) \neq 0$ and $\psi_n \xrightarrow{k} w$, so there exists a neighborhood U_w of w such that $\varphi \circ \psi_n \neq 0$ on U_w for all n large enough. Let $z = (z_1, z') \in B_N$ where $z' = (z_2, ..., z_N) \in \mathbb{C}^{N-1}$. Define $f(z_1) = g(z_1, z')$, then f is a nonzero holomorphic function. Thus g is also a nonzero holomorphic function with respect to z_1. By the same method we can see that g is holomorphic with respect to other variables $z_2, ..., z_n$. This implies that g is a nonzero holomorphic function on B_N. Clearly, $C_{\varphi,\psi}g = \varphi(w)g$, and so $\varphi(w)$ is an eigenvalue of $C_{\varphi,\psi}$. But it is well-known that the adjoint of a hypercyclic operator has no eigenvector, thus $C^*_{\varphi,\psi}$ fails to be hypercyclic. Also, note that $C_{\varphi,\psi}M_g = M_g(\varphi(w)C_\psi)$, and g

has no zero in B_N whenever φ never vanishes. Thus, M_g is one to one and has dense range and so $C_{\varphi,\psi}$ is quasisimilar to $\varphi(w)C_\psi$. Now if $|\varphi(w)| = 1$ and C_ψ is hypercyclic, then $\varphi(w)C_\psi$ and so $C_{\varphi,\psi}$ is also hypercyclic on $H(B_N)$. This completes the proof. □

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