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ON SUPERCYCLICITY CRITERIA

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Abstract: We prove that the Supercylicity Criterion for any operator T on a Hilbert space implies that T satisfies rank one *-commutator operator property.

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1. Introduction

A linear operator T on a Hilbert space $\mathcal{H}$ is hypercyclic if there exists an $x \in \mathcal{H}$ such that $\{x, Tx, T^2x, ...\}$ is dense in $\mathcal{H}$; that is, if there is a vector with dense orbit. An operator T is supercyclic if there exists an $x \in \mathcal{H}$ such that $\{\lambda T^n x : n \geq 0, \lambda \in \mathsf{C}\}$ is dense in $\mathcal{H}$; that is, if there exists a vector whose scaled orbit is dense. The first example of a hypercyclic operator on a Banach space was a multiple of the backward shift on $l^2(N)$; it was shown in 1969 by Rolewicz [4] that if B is the backward shift, then λB is hypercyclic if and only if $|\lambda| > 1$. It follows easily, then that B itself is supercyclic.

It was shown Later in 1974, by Hilden and Wallen [3] that all unilateral backward weighted shift are supercyclic, but there does not exist a vector that is supercyclic vector for all the unilateral backward weighted shifts. In [5] and [6], Salas characterized the bilateral weighted shifts in terms of their weight sequence to be hypercyclic and supercyclic.

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On the other side, some works studied how certain problems in quantum theory have motivated some recent reseach in pure Mathematics in matrix and operator theory. The mathematical key is that of a commutator. Given $T, S \in B(H)$. The operator C is said to be a commutator, if there exists an operator $X \in B(H)$ such that $TX - XT = C$. In general, if there exists an operator $X \in B(H)$ such that $TX - XS = C$, then C is said to be a generalized commutator. We are interested with the operator equation $TX - XT^* = C$ when C is of rank at most one. Given $T \in \mathbf{B}(\mathcal{H})$, By rank one *-commutator operator we mean a finite operator C of rank at most one such that $TX - XT^* = C$ for some operator $X \in \mathbf{B}(\mathcal{H})$. In this case T is called to satisfy the rank one *-commutator operator property. Remember that for $f, g \in \mathcal{H}$, $f \otimes g$ is the rank one operator defined by,
$(f \otimes g)x = \langle x, g\rangle f$ for each $x \in \mathcal{H}$.

In this paper the author, in an attempt to find a link between a supercyclicity criteria and rank one *-commutator. we prove that supercyclicity criteria implies rank one *-commutator operator property. Moreover, the solution X is compact operator. However, we give an example to showing that rank one *-commutator operator property does not give supercyclicity criteria.

2. Preliminaries

The principal tool used to show that operators are supercyclic are the following criteria.The criteria is developed by Salas [6]. A more general supercyclicity criteria was given in Feldman, Miller and Miller [2].

Theorem 1 (A Supercyclicity Criterion)**.** *Suppose that $T \in \mathbf{B}(\mathcal{H})$. If there exists a sequence $n_k \longrightarrow \infty$ and two dense sets Y and Z such that:*

1. *There exists a function $B : Z \longrightarrow Z$ such that $TBx = x$ for all $x \in Z$, and*
2. *If $y \in Y$ and $x \in Z$, then $\|T^{n_k}x\|\|B^{n_k}y\| \longrightarrow 0$ as $n \longrightarrow \infty$.*

Then T is supercyclic.

Note that the functions B, which are right inverses of T, need only be well defined maps; they may be, and usually are, discontinuous.

To conclude this section, we give one of the main spectral property of supercyclic operator. It was shown that, the adjoint of a supercyclic operator T^*

can have an eigenvalue. However, T^* cannot have more than one eigenvalue This is the content of the next result. For more details see Proposition 1.26 in [1].

Proposition 2. *Let $T \in \mathbf{B}(\mathcal{H})$ be supercyclic. Then either $\sigma_p(T^*) = \phi$ or $\sigma_p(T^*) = \{\lambda\}$ for some $\lambda \neq 0$. In the latter case, $Ker(T^* - \lambda)$ has dimension 1 and $Ker(T^* - \lambda)^n = Ker(T^* - \lambda)$ for all $n \geq 1$.*

3. Main Results

Theorem 3. *Suppose $T \in \mathbf{B}(\mathcal{H})$. If there is a sequence $n_k \longrightarrow \infty$ and a dense sets X and Y and functions $B_{n_k} : Y \longrightarrow \mathcal{H}$ s.t*

1. *If $y \in Y$, then $T^{n_k} B_{n_k} y \longrightarrow y$ as $k \longrightarrow \infty$ and*
2. *If $x \in X$ and $y \in Y$, then $\|T^{n_k} x\| \, \|B_{n_k} y\| \longrightarrow 0$ as $k \longrightarrow \infty$*

*i.e. T satisfies supercyclicity criterion, then. there exists a compact operator K such that $TK - KT^*$ has rank at most one, i.e., T satisfies rank one *-commutator operator property.*

Proof. Let $0 \neq x \in X, 0 \neq y \in Y$. Let $K_n = \sum_{k=0}^{n} T^k x \otimes B^k y$, where B^k satisfies the conditions (1) and (2), then K_n is of finite rank for each n,
if $K = \sum_{k=0}^{\infty} T^k x \otimes B^k y$ then by hypothesis of (supercycility criteria)

$$\|K - K_n\| = \left\| \sum_{k>n} T^k x \otimes B^k y \right\| \leq \sum_{k>n} \left\| T^k x \right\| \left\| B^k y \right\| \to 0.$$

Consequently, $K_n \longrightarrow K$, hence K is compact. Now a little bit calculation shows that

$$\begin{aligned} TK - KT^* &= T(\sum_{k=0}^{\infty} T^k x \otimes B^k y) - (\sum_{k=0}^{\infty} T^k x \otimes B^k y)T^* \\ &= \sum_{k=0}^{\infty} T^{k+1} x \otimes B^k y - \sum_{k=0}^{\infty} T^k x \otimes TB^k y \\ &= Tx \otimes y + T^2 x \otimes By + T^3 x \otimes B^2 y + \dots \\ &- x \otimes Ty - Tx \otimes y - T^2 x \otimes By - \dots \end{aligned}$$

$$= -x \otimes Ty.$$

Hence $TK - KT^*$ has rank at most one. □

Corollary 4. *If $T \in \mathbf{B}(\mathcal{H})$ is invertible and satisfies conditions in 3, then $TK - KT^*$ has rank one.*

Proof. Clearly if T is invertible then $Ty \neq 0$ in the proof of 3. Thus $TK - KT^*$ has rank one. □

As an illustration we have

Example 5. Let $\{e_n\}_{n\in\mathsf{Z}}$ be an orthonormal basis of $\mathcal{H}$. Define an operator T by $Te_n = w_{n+1}e_{n+1}$, $n \in \mathsf{Z}$, where

$$w_{n+1} = \begin{cases} \frac{1}{2} & n \geq 0, \\ 4 & n \leq 0. \end{cases}.$$

This operator is called weighted bilateral shift operator with weights w_n. It is well-known that T is invertible operator and $T^{-1}e_n = \frac{1}{w_n}e_{n-1}$, $n \in \mathsf{Z}$. Note also that $T^{-k}e_n \overset{k\to\infty}{\longrightarrow} 0$ for each $n \in \mathsf{Z}$ and $T^k e_n \overset{k\to\infty}{\longrightarrow} 0$ for each $n \in \mathsf{Z}$. Easily one can see that $\sum_{k=0}^{\infty} \left\|T^k e_n\right\| \left\|T^{-k}e_n\right\| < \infty$ for any fixed $n \in \mathsf{Z}$. Thus (4) implies that $TK - KT^*$ has rank one for some compact operator K.

The question arises whether rank one *-commutator operator property in Theorem 3 is sufficient, i.e., is it equivalent to supercyclic criterion. The following example satisfies rank one *-commutator property, but not supercyclicity criterion.

Example 6. Let $T \in l^2(N)$ be defined by

$$T(x_1, x_2, x_3, x_4, ...) = (x_1, -x_2, x_3, -x_4, ...).$$

It is easy to Show that 1 and -1 are eigenvalues of T with eigenvectors $(1, 0, 0, ...)$ and $(0, 1, 0, 0, ...)$ respectively.

This implies by proposition 2 that T is not supercyclic operator, hence does not satisfy supercyclicity criteria. On the other hand, we can construct a compact operator K such that $TK - KT^*$ has rank at most one. So T satisfies rank one *-commutator property. Let $f = (0,1,0,0,...)$ and $g = (1,0,0,...)$. Note that $Tg = g$ as g is the eigenvector of 1. Define $K = f \otimes g$, then

$$\begin{aligned} TK - KT^* &= T(f \otimes g) - (f \otimes g)T^* \\ &= Tf \otimes g - f \otimes Tg \\ &= Tf \otimes g - f \otimes g \\ &= (Tf - f) \otimes g. \end{aligned}$$

But $Tf \neq f$, hence $TK - KT^*$ has rank 1, and T satisfies rank one *-commutator operator property.

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