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ON THE SEGRE UPPER BOUND OF THE REGULARITY FOR FAT POINTS IN $\mathbb{P}^4$, II

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Abstract: We study the generalized Segre bound in $\mathbb{P}^4$ for fat points schemes. In this second part we prove the quartic case and an inductive lemma. We also discuss stronger vanishing theorems for sets with additional properties.

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1. Introduction

Fix a finite set $S \subset \mathbb{P}^4$, $S \neq \emptyset$. For each $P \in S$ fix an integer $m_P > 0$ and set $Z := \sum_{P \in S} m_P P$. Set $s := \sharp(S)$ and let $m_1 \geq \cdots \geq m_s$ be the multiplicities of the points of Z.

We recall that in $\mathbb{P}^4$ the Segre conjecture (or the generalized Segre conjecture) may be restated in the following way: $h^1(\mathcal{I}_Z(d)) = 0$ if $m_1 \leq d$ and $w(L) \leq cd+1$ for every c-dimensional linear subspace $L \subseteq \mathbb{P}^4$, $c = 1, 2, 3, 4$. We recall that this conjecture is true in $\mathbb{P}^2$ ([4], [8]) in $\mathbb{P}^3$ ([7], [9]), if S is in linearly

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general position ([5]) and if $\sharp(S)$ is small ([3], [12], [2]). We fix an integer $d > 0$ and we take Z satisfying the Segre condition in degree d. The aim (not yet reached) of our series of papers (of which this is the second part) is to prove that $h^1(\mathcal{I}_Z(d)) = 0$. This was proved if $d = 3$ in [1]. In this paper we make a part of an inductive steps $d-2, d-1 \Longrightarrow d$ (we assume the full conjecture for degrees $< d$, but we prove that $h^1(\mathcal{I}_Z(d)) = 0$ only if Z satisfies another numerical condition, i.e. we prove the following result.

Lemma 1. *Fix an integer $d \geq 4$ and assume that the Segre condition is true in degrees $d-2$ and $d-1$, hat Z satisfies the Segre conditions in degree d and that $m_1 + m_2 \leq d$. Then $h^1(\mathcal{I}_Z(d)) = 0$.*

We also prove the case $d = 4$ (Proposition 2) and a critical case for $d = 5$ (Proposition 3). See Section 2 for the introduction of notation needed in the third and fourth parts of this series. We introduce it to point out why some steps of the proof of Lemma 1 work in other cases and avoid to repeat them in the other parts.

We also prove the following result for an arbitrary $\mathbb{P}^n$ when S satisfies a strong condition.

Proposition 1. *Assume the existence of an integer x such that $2 \leq x < d$ and for any $S' \subseteq S$ we have $h^0(\mathcal{I}_{S'}(x)) = \max\{0, \binom{n+x}{n} - \sharp(S')\}$. Set $r := -1+\min\{\binom{n+x}{n}, \sharp(S)\}$. Assume $m_1+m_2 \leq d+1$ and $m_1+\cdots+m_s \leq r\lfloor d/x \rfloor+1$. Then $h^1(\mathcal{I}_Z(d)) = 0$.*

The condition for S is weaker than the Uniform Position Property used in [11] (it does not even implies that S is in linearly general position), but the proof of Proposition 1 is just a reduction to the linearly general position case proved in [5, Theorem 1.4].

2. Preliminaries

We fix an integer $d > 0$ and assume that the generalized Segre conjecture is true in $\mathbb{P}^4$ for all positive integers $< d$. We fix $Z = \sum_{p \in S} m_p p$ and assume that Z satisfies the assumptions of the generalized Segre conjecture in degree d, i.e. we assume $d \geq m_1$, $w(\mathbb{P}^4) \leq 4d+1$ and $w(E) \leq id + i - 2$ for $i = 1, 2, 3$ and each i-dimensional linear subspace $E \subset \mathbb{P}^4$. We use induction on the integer $w(\mathbb{P}^4) = \sum_{p \in S} m_p$. In particular we assume $h^1(\mathcal{I}_W(d)) = 0$ if $W = \sum_{p \in S'} m_p p$, where $S' \subsetneq S$. For each $p \in \mathbb{P}^4$ let m_p denote the multiplicity of p in Z. Hence $m_p = 0$ if and only if $p \notin S$. We say that Z satisfies $\diamond\spadesuit$ if $m_1 + m_2 \geq d+1$. We assume that $\diamond\spadesuit$ fails in sections 3, 4, 5, 6, 7. We say that Z satisfies $\spadesuit$ if

$m_1 > m/2+1$. We say that Z satisfies $\diamond\diamond$ if d is odd and $m_1 = m_2 = (d+1)/2$.

Notation 1. For any set $A \subseteq \mathbb{P}^4$ set $w(A) := \sum_{P \in S \cap A} m_P$. Write $w_Z(A) := w(A)$. If $U \subset \mathbb{P}^4$ is any finite subset and for each $P \in U$ we take a non-negative integer n_P, then set $w_W(A) := \sum_{P \in U \cap A} n_P$, where $W := \sum_{P \in U} n_P P$. Let B_1 (resp. B_1') be the set of all lines $L \subset \mathbb{P}^4$ such that $w(L) = d+1$ (resp. $w(L) = d$). Let B_2 (resp. B_2') be the set of all planes $L \subset \mathbb{P}^4$ such that $2d \le w(L) \le 2d+1$ (resp. $2d-2 \le w(L) \le 2d-1$). Let B_3 (resp. B_3') be the set of all hyperplanes $L \subset \mathbb{P}^4$ such that $3d-1 \le w(L) \le 3d+1$ (resp. $3d-4 \le w(L) \le 3d-2$).

Proof. Since $U \in B_3$, we have $\sharp(U \cap S) \ge 4$ and hence $\mathbb{P}^4$ satisfies the Segre condition for $\mathbb{P}^4$. Since $w(\mathbb{P}^4 \setminus U) \le d$, $\mathrm{Res}_U(Z)$ satisfies the Segre condition with respect to all lines, all planes A with $w(A) = 2d$ and all hyperplanes with $w(A) = 3d-1$. Fix a plane A such that $w(A) = 2d+1$. Since $m_1 \le d$, we have $\sharp(A \cap H \cap S) \ge 2$. Fix a hyperplane G with $w(G) \ge 3d$. Since $m_1 + m_2 \le d+1$ and $w(\mathbb{P}^4 \setminus U) + d + 1 < 3d$, we get $\sharp(G \cap U \cap S) \ge 3$. □

For any set $E \subseteq \mathbb{P}^4$ let $\langle E \rangle$ denote the linear span of E, i.e. the minimal linear subspace of $\mathbb{P}^4$ containing E. We use the following lemmas ([1], Lemmas 3 and 1).

Lemma 2. *If there is a hyperplane $H \subset \mathbb{P}^4$ such that $\sharp(S \cap H) \ge \sharp(S) - 1$, then $h^1(\mathcal{I}_Z(d)) = 0$.*

Lemma 3. *If $m_1 = d$, then $h^1(\mathcal{I}_Z(d)) = 0$.*

By Lemma 3 we may assume $m_1 \le d-1$.

Proof of Proposition 1: We will even prove that $h^1(\mathcal{I}_Z(x\lfloor d/x \rfloor)) = 0$. Let $\nu_x : \mathbb{P}^n \to \mathbb{P}^N$, $N := \binom{n+x}{n} - 1$, the Veronese embedding of $\mathbb{P}^n$ by the projective space of all degree x forms in n variable. If $h^1(\mathcal{I}_{\nu_x(Z)}(\lfloor d/x \rfloor)) = 0$, then $h^1(\mathcal{I}_Z(x\lfloor d/x \rfloor)) = 0$. Let $W \subset \mathbb{P}^N$ be the fat point scheme of $\mathbb{P}^N$ with support $\nu_x(S)$ and multiplicity m_p at the point $\nu_x(p)$, $p \in S$. Since W is zero-dimension, $h^1(W, \mathcal{I}_{Z,W}(d)) = 0$ and so the restriction map $H^0(\mathcal{O}_W(\lfloor d/x \rfloor)) \to H^0(\mathcal{O}_{\nu_x(Z)}(\lfloor d/x \rfloor))$ is surjective. Hence if

$$h^1(\mathbb{P}^N, \mathcal{I}_W(\lfloor d/x \rfloor)) = 0,$$

then $h^1(\mathcal{I}_{\nu_x(Z)}(\lfloor d/x \rfloor)) = 0$. The linear span $M := \langle \nu_x(S) \rangle$ has dimension r. By the case $S \subset H$ of Lemma 2 applied $N - \dim(N)$ times to $\mathbb{P}^N$ it is sufficient to prove that $h^1(M, \mathcal{I}_{W \cap M}(\lfloor d/x \rfloor)) = 0$. This is true by [5, Theorem 1.4], because $\nu_x(S)$ is by assumption in linearly general position in M. □

For any hypersurface $T \subset \mathbb{P}^4$ and any scheme $A \subset \mathbb{P}^4$ the residual scheme $\mathrm{Res}_T(A)$ of A with respect to T is the closed subscheme of $\mathbb{P}^4$ with $\mathcal{I}_A : \mathcal{I}_T$ as its ideal scheme. Set $a := \deg(T)$. We have an exact sequence (called the residual exact sequence of T):

$$0 \to \mathcal{I}_{\mathrm{Res}_T(A)}(t-a) \to \mathcal{I}_A(t) \to \mathcal{I}_{T\cap A,T}(t) \to 0 \qquad (1)$$

Remark 1. By (1) to prove that $h^1(\mathcal{I}_Z(d)) = 0$ it is sufficient to find T such that $h^1(T, \mathcal{I}_{T\cap Z,T}(d)) = 0$ and $h^1(\mathcal{I}_{\mathrm{Res}_T(Z)}(t-a)) = 0$. We will use the cases $a = 1, 2$. In the case of hyperplanes we always have $h^1(T, \mathcal{I}_{Z\cap T,T}(d)) = 0$ ([9], [7]). Fix a hyperplane $H \subset \mathbb{P}^4$ and a quadric $Q \subset \mathbb{P}^4$. To check that $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree $d-1$ it is sufficient to check it for $\mathbb{P}^4$ (where it is true if $\sharp(S \cap H) \ge 4$, for all $B \in B_1$ (we only need to prove that $B \cap H \cap S \ne \emptyset$), for all $A \in B_2$ (we only need to prove that $\sharp(A \cap H \cap S) \ge w(A) - 2d - 1$) and for all $U \in B_3$ (we only need to prove that $\sharp(U \cap H \cap S) \ge w(U) - 3d - 4$). To check that $\mathrm{Res}_Q(Z)$ satisfies the Segre conditions in degree $d-1$ it is sufficient to check it for $\mathbb{P}^4$ (it is true if $\sharp(S\cap Q) \ge 8$), for all lines, all planes and all hyperplanes. To check it for all lines it is sufficient to check it for the elements $B \in B_1 \cup B_1'$; if $B \in B_1'$ it is sufficient to have $B \cap S \cap Q \ne \emptyset$, if $B \in B_1$ it is sufficient to have $\sharp(B \cap S \cap Q) \ge 2$. To check it for all planes it is sufficient to test if for all $A \in B_2 \cup B_2'$; if $A \in B_2 \cup B_2'$ it is sufficient to check that $\sharp(S \cap Q) \ge w(A) - 2d + 3$. To check it for all hyperplanes it is sufficient to prove that $\sharp(U \cap S \cap Q) \ge w(U) - 3d - 7$ for all $U \in B_3 \cup B_3'$. If $S \cap Q \ne \emptyset$, then the inductive assumption on the weight gives $h^1(\mathcal{I}_{\sum_{p\in S\cap Q} m_p p}(d)) = 0$; since $Q \cap (\sum_{p\in S\cap Q} m_p p) = Q \cap Z$, we get $h^1(\mathcal{I}_{Z\cap Q}(d)) = 0$ and hence $h^1(Q, \mathcal{I}_{Z\cap Q,Q}(d)) = 0$.

We use the following remarks ([1]).

Remark 2. If $A \in B_i \cup B_i'$, then $A \cap S$ spans A (use the Segre conditions for the proper linear subspaces of A, for B_1' we also need the assumption $m_1 \le d-1$).

Remark 3. Tale $L, R \in B_1$ such that $L \cap R \ne \emptyset$. Since $w(E) \le 2d+1$ and $w(L) = w(R) = d+1$, we have $L \cap R \cap S \ne \emptyset$.

Remark 4. Fix an integer $t > 0$. Let $A \subset \mathbb{P}^4$ be a zero-dimensional scheme such that $h^1(\mathcal{I}_A(t)) = 0$. Let B be a subscheme of A. Since the restriction map $H^0(\mathcal{O}_A(t)) \to H^0(\mathcal{O}_B(t))$ is surjective we have $h^1(\mathcal{I}_B(t)) = 0$. In particular for every hypersurface $T \subset \mathbb{P}^4$ we have $h^1(\mathcal{I}_{A\cap T}(t)) = 0$. Hence $h^1(T, \mathcal{I}_{T\cap A}(t)) = 0$.

Remark 5. Assume the existence of $L_i \in B_1$, $i = 1, 2, 3$, such that $L_i \cap L_j = \emptyset$ for all $i \ne j$. Let $L \subset \mathbb{P}^4$ be a line with $w(L) \ge d-1$. Since

$w(\mathbb{P}^4) \le 4d+1$, then $S \cap L \cap (L_1 \cup L_2 \cup L_3) \neq \emptyset$.

Remark 6. We have $m_1+m_2 \le d$, i.e. not $\diamond\spadesuit$, if and only if $\sharp(B\cap S) \ge 3$ for all $B \in B_1$. Assume $m_1+m_2 \le d$. We have $\sharp(B \cap S) \ge 3$ for all $B \in B_1'$, except the case d even and $m_1 = m_2 = d/2$. Note that if $A \in B_2 \cup B_2'$ (resp. $A \in B_3 \cup B_3'$), then $\sharp(A\cap S)) > 3$ (resp. $\sharp(A\cap S) > 4$) and that if $w(A) = 2d+1$, then $\sharp(A \cap S) \ge 5$, because $m_1+m_2+m_3+m_4 \le \min\{4m_1, 3d-2m_1\}$. Take a line $L \subset N$. If $w(N) = 2d+1$ and $m_1 \le d-1$, then $\sharp(S \cap (N \setminus L)) \ge 2$.

Remark 7. Take lines L_1, L_2, L_3 such that $L_i \cap L_j = \emptyset$ for all $i \neq j$. If $w(L_1)+w(L_2)+w(L_3) \ge 3d+1$, then every element of B_1 meets $S \cap (L_1 \cup L_2 \cup L_3)$. If $w(L_1)+w(L_2)+w(L_3) \ge 3d+2$, then every element of B_1' meets $S \cap (L_1 \cup L_2 \cup L_3)$.

Lemma 4. *Assume the existence of a hyperplane U such that $w(U) = 3d+1$. Then $\mathrm{Res}_U(Z)$ satisfies the Segre condition in degree $d-1$.*

Lemma 5. *Let $E, A \subset \mathbb{P}^4$ such that $E \cap A$ is a single point, o. Then $h^1(\mathcal{I}_{A\cup E}(t)) = 0$ for all $t \ge 2$ and $\mathcal{I}_{A\cup E}(2)$ is spanned.*

Proof. Fix lines L, D of $\mathbb{P}^3$ such that $L \cap D = \emptyset$. We have $h^1(\mathcal{I}_{L\cup D}(2)) = 0$, $h^0(\mathcal{I}_{A\cup D}(2)) = 4$ and $\mathcal{I}_{L\cup D}(2)$ is spanned. Since $|\mathcal{I}_{A\cup D}|$ is the linear system of all quadric cones with vertex containing o, we get $h^1(\mathcal{I}_{A\cup D}(2)) = 0$ and that $\mathcal{I}_{A\cup E}(2)$ is spanned outside o. Take affine coordinates x_1, x_2, x_3, x_4 centered at o such that the affine part of A (resp. E) has equations $x_1 = x_2 = 0$ (resp. $x_3 = x_4 = 0$). Since the quadratic equations x_ix_j, $i = 1, 2$, $j = 3, 4$, gives at o a reduced scheme, we get that $\mathcal{I}_{A\cup E}(2)$ is spanned at o. The case $t > 2$ follows from Castelnuovo-Mumford's lemma. □

Lemma 6. *Fix a plane $E \subset \mathbb{P}^4$ and lines L, D such that $E \cap L = E \cap D = L \cap D = \emptyset$. Then $h^1(\mathcal{I}_{E\cup L\cup D}(2)) = 0$.*

Proof. Set $H := \langle D \cup L\rangle$. Since $H \cap E$, L and D are 3 disjoint lines of H, we have $h^0(H, \mathcal{I}_{(E\cap H)\cup L\cup D\cup\{p\}}(2)) = 0$. For a general $p \in H$. To prove the lemma it is sufficient to prove that $h^0(\mathcal{I}_{E\cup L\cup D\cup\{p\}}(2)) = 0$. $|\mathcal{I}_{E\cup L\cup D\cup\{p\}}(2)|$ is the linear system of all quadrics $H \cup M$ with M a hyperplane containing p. □

We say that Z or $(m_1, \dots, m_s)$ is not in $\diamond\spadesuit$ (or not with $\diamond\spadesuit$), if for every $L \in B_1$ we have $\sharp(L \cap S) \ge 3$, i.e. if $m_1 + m_2 \le d$.

In sections 3, 4, 5, 6 and 7 we prove Lemma 1. From now on we fix an integer $d \ge 4$ and assume that the generalized Segre condition is true in $\mathbb{P}^4$ with respect to the integers $d-1$ and $d-2$.We use induction on the integer $w(\mathbb{P}^4)$, i.e. we assume that the lemma is true for all (S', Z') satisfying the

Segre conditions and not in $\diamond\spadesuit$ with $w_{Z'}(\mathbb{P}^4) < w(\mathbb{P}^4)$. We do not prove it in this paper, because we prove that it holds with respect to d only for certain multiplicities (i.e. we exclude $\diamond$ and $\spadesuit$), but we adapt the steps so that many of them may be used even in case $\spadesuit$ and some of them for $\diamond$.

Remark 8. Let $L_i \subset \mathbb{P}^4$, $i = 1, 2, 3$, be lines such that $L_i \cap L_j = \emptyset$ for all $i \neq j$. There is a unique line J intersecting every line L_1, L_2, L_3 and $J \cup L_1 \cup L_2 \cup L_3$ is the base locus of $|\mathcal{I}_{L_1 \cup L_2 \cup L_3}(2)|$.

3. Extremal Hyperplanes and Lines

In this section we assume the existence of $M \in B_3$ and $L \in B_1$ such that $M \cap L \cap S = \emptyset$. We assume $m_1 + m_2 \leq d$, but we point out the single steps that work with weaker assumptions.

Since $w(\mathbb{P}^4) \leq 4d + 1$, we have $w(M) \leq 3d$. Until step (k) we assume $S \subset M \cup L$ (this is always the case if $w(M) = 3d$). Fix any $A \in B_c \cup B'_c$, $A \neq L$, which is not contained in M. If $S \subset M \cup L$, then $A \cap S = (A \cap M \cap S) \sqcup (A \cap L \cap S)$ with $A \cap M \cap S$ contained in the hyperplane $M \cap A$ of A.

(a) Assume $c = 2$ and $A \supset L$. We get $w(A \cap M \cap S) = w(A) - d - 1$ and hence $A \cap M \cap S \neq \emptyset$. If $w(A) = 2d + 1$, we get $A \cap M \in B'_1$. If $w(A) = 2d$, then $w(A \cap M) = d - 1$.

(b) Assume $c = 3$ and $A \supset L$. The plane $A \cap M$ satisfies $w(A \cap M) = w(A) - d - 1$.

(c) Assume $c = 1$ and $A \neq L$. Since $m_1 \leq d - 1$ (Lemma 3), we get $\sharp(A \cap M \cap S) = 1$, $\sharp(A \cap L \cap S) = 1$ and $m_{A \cap M \cap S} + m_{A \cap L \cap S}$. Hence $m_1 + m_2 = d + 1$. This case may occur only if $m_1 + m_2 = d + 1$.

(d) Assume $c = 2$ and $A \nsubseteq L$. Since $A \cap M$ is a line, we have $w(A \cap M) \leq d + 1$. Since $A \cap S$ spans A, we get $A \cap L \neq \emptyset$ and that $A \cap L$ is a point of S. We get $w(A) = w(A \cap M) + m_{A \cap L}$. We exclude the case $w(A) = 2d + 1$, because $h^1(\mathcal{I}_Z(d)) = 0$ if $m_{A \cap L} = d$ (Lemma 3). If $w(A) = 2d$, we get $m_{A \cap L} \geq d - 1$. Since $d \geq 4$, we are in the case $\spadesuit$ and $A \cap L$ is the only point of S with multiplicity $m_1 = d - 1$.

(e) Assume $c = 3$ and $A \nsupseteq L$. Since $A \cap S$ spans A, we get that $A \cap L$ is a point of L. We get $w(A) = w(A \cap M) + m_{A \cap L}$. Write $w(A) = 3d + e$ with $e \in \{-1, 0, 1\}$. Since $w(A \cap M) \leq 2d + 1$, we get $m_{A \cap L} \geq d + e - 1$. Lemma 3 solves the case $e = 1$. If $e \leq 0$ we get $m_{A \cap L} \geq d - 1 + e$. If either $d \geq 5$ or $d = 4$ and $e = 0$, we are in case $\spadesuit$ and $A \cap L$ is the only point of S with multiplicity m_1. If $d = 4$ and $e = -1$ we get $m_{A \cap L} \geq 2$.

(f) Assume the existence of a plane $E \subset M$ with $w(E) = 2d+1$. We only exclude the case d odd, $m_1 = m_2 = m_3 = (d+1)/2$ with $E \cap S$ formed by 3 points with multiplicity $(d+1)/2$ and one point with multiplicity $(d-1)/2$ and the case d even, $m_1 = d/2+1$ and E containing the point of multiplicity $d/2+1$ and 3 points of multiplicity $d/2$. With these exclusions we have $\sharp(E \cap S) \ge 5$. We prove that $h^1(\mathcal{I}_Z(d)) = 0$. Fix $q \in L \cap S$ with maximal multiplicity among the points of $L \cap S$ and set $H := \langle E \cup \{q\} \rangle$. We check that $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree $d-1$, and hence that $h^1(\mathcal{I}_Z(d)) = 0$ by induction on d. We have $w_{\mathrm{Res}_H(Z)}(\mathbb{P}^4) \le 4d+1-\sharp(S \cap L \cap H) - \sharp(E \cap S)$ and hence to check the Segre condition in degree $d-1$ for $\mathrm{Res}_H(Z)$ and $\mathbb{P}^4$ it is sufficient to use that $\sharp(E \cap S) \ge 3$. M satisfies the Segre condition in degree $d-1$ for $\mathrm{Res}_H(Z)$, because $w(M) \le 3d$ and $\sharp(M \cap H \cap S) = \sharp(E \cap S) \ge 3$. Since $L \cap H \cap S \ne \emptyset$, L satisfies the Segre condition in degree $d-1$ for $\mathrm{Res}_H(Z)$. If $R \in B_1$, $R \ne L$, and $R \nsubseteq M$, then (c) shows that R satisfies the Segre condition in degree $d-1$ for $\mathrm{Res}_H(Z)$. Fix $A \in B_2$ with $A \supset L$. Since $S \cap H \cap L \ne \emptyset$, to check the Segre condition for A we may assume $w(A) = 2d+1$; in this case it is sufficient to check that $A \cap E \cap S \ne \emptyset$; we have $w(A \setminus L) = d$; since $w(M \setminus E) \le d-1$, we get $A \cap E \cap S \ne \emptyset$. Fix $A \in B_3$ such that $A \supset L$. Since $L \cap H \cap A \cap S \ne \emptyset$, it is sufficient to prove that $\sharp(E \cap A \cap S) \ge w(A) - 3d + 1$. We have $w(A \cap M \cap S) = w(A) - d - 1$ and hence it is sufficient to use that $w(M \setminus E) \le d-1$ and that if $w(A) = 3d+1$, then $w(A \cap E \cap S) \ge d+1$ and hence $\sharp(A \cap E \cap S) \ge 2$.

Fix $D \in B_1$ such that $D \subset M$. Since $w(M \setminus D) > w(E)$, we have $D \cap E \cap S \ne \emptyset$. Fix $A \in B_2$ with $A \subset M$. Since $w(A) \ge d + w(M \setminus E)$, we have $\sharp(A \cap S) \ge 2$, concluding the proof that $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree $d-1$.

(g) Assume the existence of $R, D \in B_1$ such that $R \subset M$, $D \subset M$ and $R \cap D = \emptyset$. We assume that either $m_1 + m_2 \le d$ or $m_1 > m_2$ and the only point with multiplicity $m_1 = d+1-m_2$ is contained in L. Hence $\sharp(B \cap S) \ge 3$ for each $B \in B_1$ with $B \subset M$. Fix $p, q \in L \cap S$ with $p \ne q$ and with maximal multiplicity among the points of $L \cap S$. Let $J \subset M$ be the only line containing the point $M \cap L$ and intersecting both R and D (it is the only line of $\mathbb{P}^4$ intersecting the 3 lines L, D, R). $L \cup D \cup R \cup J$ is the base locus $|\mathcal{I}_{L \cup D \cup R}(2)|$. Since $M \cap L \notin S$, $D \cup R \cup \{p, q\}$ is the base locus of $|\mathcal{I}_{D \cup R \cup \{p,q\}}(2)|$. Therefore there is $Q \in |\mathcal{I}_{D \cup R \cup \{p,q\}}(2)|$ such that $Q \cap S \nsubseteq S$. By the inductive assumption we get $h^1(\mathcal{I}_{\sum_{o \in S \cap Q} m_o o}(d)) = 0$ and hence $h^1(Q, \mathcal{I}_{Z \cap Q}(d)) = 0$. Therefore it is sufficient to check the Segre condition for $\mathrm{Res}_Q(Z)$ in degree $d-2$. Since $p, q \in S$, to check the condition for $\mathbb{P}^4$ it is sufficient to have $\sharp(D \cap S) + \sharp(R \cap S) \ge 6$. This is true by our assumptions on m_1, m_2. By construction $\mathrm{Res}_Q(Z)$ satisfies the Segre condition in degree $d-2$ for L, R, D and, since $\sharp(D \cap S) + \sharp(R \cap S) \ge 6$, for M. Every $B \in B_1 \setminus \{L\}$ is contained in M and meets both D and R, so that $w_{\mathrm{Res}_Q(Z)}(B) \le d-1$; by Remark 5 every $B \in B_1'$ with $B \subset M$ meets $D \cup R$ and hence it meets $(D \cup R) \cap S$ (Remark 3), so that $w_{\mathrm{Res}_Q(Z)}(B) \le d-1$. Take $B \in B_1'$ with $B \not\supseteq M$; since $B \cap S = (M \cap B \cap S) \sqcup (L \cap M \cap S)$, there are $o \in M \cap S$ and $u \in L \cap S$ with $B = \langle\{o, u\}\rangle$ and $m_o + m_u = d$; we have $u \in \{p, q\}$, because in case $m_1 + m_2 = d+1$ we assumed $m_1 > m_2$ and that the only point with multiplicity m_1 is contained in L, while if d is even and $m_o = m_u = d/2$, then u is one of the points (at most two) with maximal multiplicity in $L \cap S$.

(g1) Fix $A \in B_2 \cup B_2'$ such that $A \supset L$. Since $p, q \in A \cap Q \cap S$, to check the Segre condition for A we may assume $A \in B_2$. We have $w(A \setminus L) = w(A) - d - 1 > w(M \setminus (D \cup R))$. Hence $A \cap S \cap (D \cup R) \neq \emptyset$, say $A \cap D \cap S \neq \emptyset$. Therefore to check the Segre condition for A we may assume $w(A) = 2d+1$; we only need to check the case $\sharp(L \cap S) = 2$, i.e. ♠, $m_1 + m_2 = d+1$, the point with maximal multiplicity is in L and $\sharp(A \cap (R \cup D) \cap S) = 1$, say $R \cap A \cap S = \emptyset$; since $w(\langle A \cup R\rangle) \ge 3d+2$, the hyperplane $\langle R \cup A\rangle$ contradicts the Segre conditions for Z.

(g2) Fix $A \in B_2 \cup B_2'$ such that $A \not\subseteq L$ and $A \not\subseteq M$. A is spanned by $A \cap M \cap S$ and $A \cap L \cap S$ and hence $A \cap L \in S$ and $m_{A \cap L} + w(A \cap M) = w(A)$. We use Proposition 2 for the case $d = 4$ and $m_{A\cap L} = 1$. Now assume that either $d \ge 5$ or $m_{A \cap L} > 1$. Since $w(A) - w(A \cap M) \ge 2d - 2 - d - 1 = d - 3$, if $d \ge 5$ we get that $m_{A\cap L} \in \{p, q\}$, while this is true if $d = 4$, because p, q have maximal multiplicity. By Lemma 3 we may assume $m_{A \cap L} \le d-1$. Since $w(A \cap M) \le d+1$, we get $w(A) \le 2d$ and that if $w(A) = 2d$. First assume $w(A) = 2d$. We get $m_{A \cap L} = d-1$ and $B := A \cap M \in B_1$. Hence $m_2 \le 2$. Since $Q \supset D \cup R$, it is sufficient to prove that $D \cap B \cap S \neq \emptyset$ and $R \cap B \cap S \neq \emptyset$. Assume for instance $D \cap B \cap S = \emptyset$. Since the point of $D \cap B \cap S$ has multiplicity ≤ 2, we get $3d \ge w(M) \ge 3(d+1) - 2$, a contradiction. If $w(A) = 2d-2$, it is sufficient to use that $A \cap L \in \{p, q\}$. Now assume $w(A) = 2d-1$. Since $A \cap L \in \{p, q\}$, it is sufficient to prove that $A \cap (D \cup R) \cap S \neq \emptyset$. Since $m_1 \le d-1$, we have $w(A \cap M) \ge d$ and (since $M \cap L) = \emptyset$, $A \cap (D \cup R) \cap S \neq \emptyset$ by Remark 7.

(g3) Fix $A \in B_2 \cup B_2'$ contained in M. We need to check that $\sharp(A \cap S \cap (R \cup D)) \ge w(A) + 3 - 2d$. Since $w(M \setminus (D \cup R)) \le d-2$, we have $A \cap S \cap (R \cup D) \ge w(A) - d + 2$. Since $w(A) \ge 2d-2$, then we get $A \cap S \cap (R \cup SD) \neq \emptyset$. If $w(A) = 2d-1$, we get $\sharp(A \cap S \cap (R \cup D)) \ge 2$. If $w(A) = 2d$, we get $\sharp(A \cap S \cap (R \cup D)) \ge 3$. Now assume $w(A) = 2d+1$. This case is solved in step (f).

(g4) Fix $A \in B_3 \cup B_3'$. If $A = M$, we use that $\sharp(D \cap S) \ge 3$ and $\sharp(R \cap S) \ge 3$. Assume $A \supset L$; since $\{p, q\} \in L$, we may assume $w(A) \ge$

$3d-1$; since $w(A \setminus L) = w(A) - d - 1$ and $w(M \setminus (D \cup R)) \le d-2$, we get $w(A \cap (D \cup R)) \ge w(A) - 2d + 1 > 0$; hence we may assume $w(A) \ge 3d$; we get $\sharp(S \cap A \cap (D \cup R)) \ge 2$, and hence we may assume $w(A) = 3d+1$; we get $\sharp(S \cap (D \cup R)) \ge 3$, because either $m_1 \le d/2$ or ♠ with L containing the point with multiplicity m_1; hence A contains at least two points of one of the lines D, R (say D); since $\sharp(S \cap D) \ge 3$, it is sufficient to check that $S \cap R \cap A \ne \emptyset$; this is true, because $m_p + m_q + w(D) + w(M \setminus (R \cup D)) \le d + d + 1 + d - 2 < w(A)$.

Now assume $A \ne M$ and $A \not\supseteq L$. We have $w(A) = w(A \cap M) + m_{A \cap L}$. By step (f) we may assume $w(A \cap M) \le 2d$. Since $A \cap S$ spans S (Remark 2), we have $m_{A\cap L} > 0$. If $w(A) \ge 3d$, then $w(A \cap M) \ge 2d+1$, because $m_1 \le d-1$ (Lemma 3) and hence we conclude by step (f). If $w(A) = 3d-1$, then we get $m_1 = d-1$, $m_2 \le 2$, $w(A \cap M) = 2d$ and $A \cap L \in \{p, q\}$; since $w(M \setminus (D \cup R)) \le d-2$, we get $w(A \cap (D \cup R)) \ge d+2$ and hence $\sharp(A \cap (R \cup D)) \ge 3$, so that $w_{\mathrm{Res}_Q(Z)}(A) \le 3d-5$. Hence we may assume $3d-4 \le w(A) \le 3d-2$. First assume $A \cap L \notin \{p, q\}$; we get $m_{A\cap L} \le \lfloor (d+1)/3 \rfloor$; since $w(A \cap M) \le 2d$, we get $d < 4$ (a contradiction) if $w(A) = 3d-2$, $4 \le d \le 5$ if $w(A) = 3d-3$ and $4 \le d \le 6$ if $w(A) = 3d-4$; in these exceptional cases we have $w(A \cap M) = 2d$ and so $w(A \cap (D \cup R)) \ge d+2$ and $\sharp(A \cap (D \cup R) \cap S) \ge 3$ (since only L may have a point with multiplicity $> d/2$, and hence $w_{\mathrm{Res}_Q(Z)}(A) \le 3d-5$. Since $\{p, q\} \in A$, we may assume $3d-3 \le w(A) \le 3d-2$; in the first case we need to prove that $A \cap (D \cup R) \cap S \ne \emptyset$, while in the second case we need $\sharp(A \cap (D \cup R) \cap S) \ge 2$. If $w(A) = 3d-3$, it is sufficient to have $m_1 + d - 2 \le 3d-4$, which is true for all $d \ge 4$. Assume $w(A) = 3d-2$. It is sufficient to use that $2d-1 \le 3d-2$ and hence $m_1 + m_2 < w(A) - w(M \setminus (D \cup R))$.

(g5) Assume the existence of $T \in B_1 \cup B_1'$ with $T \ne L$ and $T \nsubseteq M$; we get either ♠ with ◇♠ or d even, $m_1 = m_2 = d/2$ and $m_p = d/2$; in the latter case $p \in B$ and hence $w_{\mathrm{Res}_Q(Z)}(T) \le d-1$.

(g6) Fix $T \in B_1 \cup B_1'$ such that $T \subset M$; since R and D satisfy the Segre condition for $\mathrm{Res}_Q(Z)$ in degree $d-2$, we may assume $T \ne D$ and $T \ne R$; since $w(M \setminus (R \cup D)) \le d-2$, we get $T \cap S \cap (R \cup D) \ne \emptyset$; hence if either $T \in B_1'$ or $T \cap S$ meets both D and R, then T satisfies the Segre condition in degree $d-2$, but not if either $T \cap D \cap S = \emptyset$ or $T \cap R \cap S = \emptyset$. With no loss of generality we may assume $T \cap R \cap S = \emptyset$. Since $T, R \in B_1$, we have $T \cap R = \emptyset$ (Remark 3).

Claim 1: Fix any $p_1, p_2 \in L \setminus (L \cap M)$. Then $h^1(\mathcal{I}_{T \cup D \cup R \cup \{p_1,p_2\}}(2)) = 0$, i.e. $h^0(\mathcal{I}_{T \cup D \cup R \cup \{p_1,p_2\}}(2)) = 5$.

Proof of Claim 1: Any two configurations T, D, R inside a projective space M are projectively equivalent and $h^1(M, \mathcal{I}_{T \cup D \cup R, M}(2)) = 0$. Hence

$h^1(\mathcal{I}_{T\cup D\cup M}(2)) = 0$. We have $h^1(\mathcal{I}_{\{p_1,p_2\}}(1)) = 0$. Since $p_i \neq L\cap M$, using reducible quadrics with M as a component, we get $h^1(\mathcal{I}_{T\cup D\cup R\cup\{p_1,p_2\}}(2)) = 0$. $\square$

Fix $o \in T\cap S\setminus D\cap S$. We have $h^0(\mathcal{I}_{T\cup D\cup R\cup\{p,q\}}(2)) = h^0(\mathcal{I}_{D\cup R\cup\{o,p,q\}}(2))-1$ (Claim 1) and hence there is $Q' \in |\mathcal{I}_{D\cup R\cup\{o,p,q\}}(2)|$ with $T \nsubseteq Q'$. Since $T\cap Q'$ contains two points of T, o and $D\cap T$, we get $T\cap Q' = \{o, D\cap T\}$. Since $\sharp(T\cap S) \geq 3$, we have $Q'\cap S \neq S$. Therefore $h^1(\mathcal{I}_{Q'\cap S}(2)) = 0$. We may repeat the construction just given and see that $\mathrm{Res}_{Q'}(Z)$ satisfies the Segre conditions in degree $d-2$, except perhaps for the lines like T, i.e. the lines $A \in B_1$ with $A \subset M$ and A intersecting only one of the lines R, D. The choice of o gives that T satisfies the Segre condition in degree $d-2$. Take another line $A \neq T$. Since every point of M has multiplicity $\leq d/2$, we have $w(T\setminus T\cap(D\cup R)) \geq d/2+1$ and the same inequality is true for A. Since $w(M\setminus(D\cup R)) \leq d-2 < w(T\setminus T\cap(D\cup R)) + w(A\setminus A\cap(D\cup R))$, T and A meets in a point of S not contained in $D\cap T$. We may take as o this point and handle simultaneously A and T. So the only problem arises if there is a third line T' and $o \notin T'$. We just saw that $T\cap T' \neq \emptyset$ and $T'\cap A \neq \emptyset$, so that T' (and any element of B_1 with the same properties, but not containing o) is contained in the plane $\langle A\cup T\rangle$. We first note that $T\cap T'\cap(D\cup R) = \emptyset$, because T and T' are different lines containing a point outside $D\cup R$. Hence the plane $\langle A\cup T\rangle$ contains at least two points of one of the lines D, R, say of D, and hence it contains D. Therefore $\langle A\cup T\rangle = \langle D\cup T\rangle$ and R is disjoint from T, A, T' and all other elements of B_1 with the same properties. Take a general $Q_1 \in |\mathcal{I}_{T\cup D\cup R\cup\{p_1,p_2\}}(2)|$. We have $\sharp(A\cap S\cap Q_1) \geq 2$, because $A\cap D \neq A\cap T$ and any two mutually intersecting elements of B_1 meets in a point of S (Remark 3). Hence A (and any other element of B_1 with similar properties) satisfies the Segre conditions in degree $d-2$ with respect to $\mathrm{Res}_{Q_1}(Z)$. The problem is that usually $Q_1\cap S = S$, because (since R meets the hyperplane $\langle A\cup T\rangle$ of M) we have $\langle A\cup T\rangle \subset Q_1$. This is the base locus of $|\mathcal{I}_{\langle T\cup D\rangle\cup L\cup R}(2)|$ and so we conclude unless $S \subset \langle T\cup D\rangle\cup L\cup R$. From now on we assume $S \subset \langle T\cup D\rangle\cup L\cup R$.

Let Q_2 be the reducible quadric hypersurface union of the hyperplane $\langle T\cup D\cup\{p\}\rangle$ and the hyperplane $\langle\{q,o\}\cup R\rangle$. Since $\{p,q\}\cup R\cup\langle T\cup D\rangle \subset Q_2$ the proofs just given for $\mathrm{Res}_Q(Z)$ works (even for all $B \in B_1$ with $B \subset \langle T\cup D\rangle$). We have $Q_2\cap S \neq S$, because $Q_2\cap L = \{L\cap M, p, q\}$, unless ♠ and the only point of S with multiplicity m_1 is contained in L; in this case we call p this point and instead of Q_2 we use the reducible quadric Q_3 union of the hyperplane $\langle T\cup D\cup\{p\}\rangle$ and the hyperplane $\langle\{p,o\}\cup R\rangle$; the point p has multiplicity m_p-2 in $\mathrm{Res}_{Q_3}(Z)$ and hence the proof just given works even in the case $A \supseteq L$.

(h) Assume the existence of $R \in B_1$ such that all $B \in B_1$ and contained in M meet R. Set $H := \langle L\cup R\rangle$. It is sufficient to prove that $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree $d-1$. We have $w_{\mathrm{Res}_H(Z)}(\mathbb{P}^4) \le 4d+1-\sharp(S\cap R)-\sharp(S\cap L)$. We have $w_{\mathrm{Res}_H(Z)}(B) \le d+1-\sharp(B\cap S)$ if $B \in \{R, L\}$. Now take $B \in B_1 \setminus \{R, L\}$. If $B \subset M$, then $B \cap R \ne \emptyset$ by assumption and $B\cap R\cap S \ne \emptyset$ (Remark 3) and so $w_{\mathrm{Res}_H(Z)}(B) \le d$. If $B \nsubseteq M$, then $B\cap L\cap S \ne \emptyset$, because $S \subset M \cup L$ and so $w_{\mathrm{Res}_H(Z)}(B) \le d$.

(h1) Fix $A \in B_2$. If $A \subset M$, then we may assume $w(A) = 2d$ by step (f). We have $A\cap R\cap S \ne \emptyset$, because $w(M) \le 3d$ and hence $w(M \setminus R) \le 2d-1$. If $A \nsubseteq M$, then we have $w(A\cap M) \le d+1$ and if equality holds, then $A\cap R\cap S \ne \emptyset$ by our assumption on B_1. If $A \supset L$, then we use that $\sharp(L\cap S) \ge 2$. If $A \not\supseteq L$, we get $A\cap L \in S$ and $m_{A\cap L} + w(A\cap M) = w(A)$; since $m_{A\cap L} \le d-1$, we get $w(A\cap M) = d+1$, $w(A) = 2d$ and $A\cap S\cap H \ne \emptyset$; hence A satisfies the Segre condition for $\mathrm{Res}_H(Z)$.

(h2) Take $A \in B_3$. If $A = M$, then we use that $R\cap S \subset H$. Now assume $A \supset L$; since $\sharp(L\cap S) \ge 2$, we are done, unless $w(A) = 3d+1$, ♠ and ⋄♠ and L contains a point with maximal multiplicity; since $S \subset M\cup L$, we get $w(A\cap M) = 2d$; since $w(M\setminus R) = w(M) - d - 1 \le 2d-1$, we get $A\cap R\cap S \ne \emptyset$ and hence $\sharp(A\cap H\cap S) \ge 3$ even in this case. Now assume $A \ne M$ and $A \nsubseteq L$; since $S \subset M\cup L$, we have $w(A) = w(A\cap M) + m_{A\cap L}$; by step (f) we may assume $w(A\cap M) \le 2d$; since $m_{A\cap L} \le d-1$ (Lemma 3), we conclude, unless $w(A) = 3d-1$ and $m_{A\cap L} = d-1$; in this case the Segre condition is satisfied, because $A\cap L \in S\cap H$.

(i) By Steps (f), (g) and (h) we may assume that M contains no element of B_1 and no $A \in B_2$ with $w(A) = 2d+1$. Assume the existence of $E \in B_2$ with $E \subset M$ and $w(E) = 2d$. Fix $p \in L\cap S$ with maximal multiplicity and set $H := \langle E \cup \{p\}\rangle$. We have $w_{\mathrm{Res}_H(Z)}(\mathbb{P}^4) \le 4d+1-\sharp(S\cap L\cap H)-\sharp(E\cap S)$ and hence to check the Segre conditions in degree $d-1$ for $\mathrm{Res}_H(Z)$ is is sufficient to use that $\sharp(E\cap S) \ge 3$ (Remark 2). Since $q \in L\cap S$, L satisfies the Segre condition in degree $d-1$. By assumption we do not need to test any other element of B_1.

(i1) Fix $A \in B_3$. If $A = M$, then we use that $\sharp(E\cap S) \ge 3$. Now assume $A \ne M$ and $A \not\supseteq L$; we have $w(A) = w(A\cap M) + m_{A\cap L}$; we may assume $m_{A\cap L} < d$ by Lemma 3; $h^1(\mathcal{I}_Z(d)) = 0$ if $w(A\cap M) = 2d+1$ by step (f); hence $h^1(\mathcal{I}_Z(d)) = 0$ in this case (if $m_1 = d-1$, then $m_p = d-1 \in A\cap H\cap S$). Now assume $A \supset L$; since $p \in H\cap A\cap S$, to check the Segre condition we may assume $w(A) \ge 3d$; we get $w(A\cap M) \ge 2d-1$ and hence $E\cap A\cap S \ne \emptyset$; to check the Segre condition we may assume $w(A) = 3d+1$ and in this case we use Lemma 4.

(i2) Fix $A \in B_2$. First assume $A \subset M$. By step (f) we may assume $w(A) = 2d$ and it is sufficient to prove that $A\cap H\cap S \ne \emptyset$; we have $w(A\setminus A\cap E) \le$

$w(M) - w(E) \le d$ and hence $A \cap E \cap S \neq \emptyset$. Now assume $A \nsubseteq M$ and $A \nsupseteq L$; since $S \subset M \cup S$, we get $A \cap L \cap S \neq \emptyset$ and $w(A) = w(A \cap M) + m_{A\cap L}$, contradicting the assumption $w(A \cap M) \le d$, because if $m_{A\cap L} = d$, then $h^1(\mathcal{I}_Z(d)) = 0$ by Lemma 3. Now assume $A \supset L$. Since $p \in H$, to check the Segre condition in degree $d-1$ for A we may assume $w(A) = 2d+1$ and hence $w(A \cap M) = d$. Therefore $B := A \cap M \in B_1'$. In this case $S = (S \cap E) \sqcup (S \cap A \cap M) \sqcup S \cap L$. Let F be the hyperplane spanned by A, and a point $o \in E \cap S$ with maximal multiplicity. We need to check the Segre conditions of $\mathrm{Res}_F(Z)$ in degree $d-1$. Since $\sharp(A \cap S) \ge \sharp(B \cap S) + \sharp(A \cap L \cap S)$, the Segre condition for $\mathbb{P}^4$ is satisfied. Fix $U \in B_3$. If $U = M$, then we use that $w(M) \le 3d$ and $\sharp(A \cap M \cap S) \ge 2$. If $U \supset L$, then we use that $\sharp(L \cap S) \ge 3$, unless ♠ and ⋄♠ and L contains the point with maximal multiplicity; in the latter case we may assume $w(U) = 3d+1$ (and conclude by Lemma 4), unless $U \cap M = E$; in this case we use that $w(U \cap M) = 2d$, $w(M \setminus A) \le d$ and hence $w(U \cap M \cap A) > 0$.

If $U \nsupseteq L$, then $w(U) = w(U \cap E) + w(U \cap B) + m_{U\cap L}$ and hence$U \cap L \in S$; we may assume $w(U) \ge 3d$ and, since $m_{U\cap L} < d$ by Lemma 3, we may assume $w(U) = 3d+1$; use Lemma 4.

Fix $U \in B_2$. If $U = E$, then we use $F \cap E \cap S \neq \emptyset$ and that $w(E) = 2d$. If $U \subset M$ and $U \neq M$, then we use that $w(U) = 2d$ by step (f) and that $U \cap B \cap S \neq \emptyset$, because $w(U) + w(E) > w(M)$. If $U \supset L$, then we use that $\sharp(L \cap S) \ge 2$. If $U \nsubseteq M$ and $U \nsupseteq L$, then $w(U) = w(U \cap M) + w(U \cap L)$ with $w(U \cap M) \le d$; we apply Lemma 3 to the point $U \cap L$.

Fix $U \in B_1$. Since M contains no element of B_1, it is sufficient to observe that $S = (S \cap M) \cup (S \cap L)$ and hence $U \cap L \cap S \neq \emptyset$.

(j) Assume that M contains no element of $B_1 \cup B_2$. We take $p, q \in M \cap S$ and take a hyperplane H containing $L \cup \{p, q\}$. We heck the Segre conditions in degree $d-1$ with respect to $\mathrm{Res}_H(Z)$. To test $\mathbb{P}^4$ it is sufficient to use that $p, q \in H \cap S$ and that $\sharp(L \cap S) \ge 2$. Fix $U \in B_3$. If $U = M$, then we use that $w(M) \le 3d$ and that $p, q \in S$. If $U \neq M$, then we use that $w(U) = w(U \cap M) + w(M \cap L)$ with $w(U \cap M) \le 2d-1$; since $m_1 < d$, we get $M \supset L$; in the case $\sharp(L \cap S) = 2$, to check the Segre conditions we may assume $w(U) = 3d+1$, which gives $w(U \cap M) = 2d$, a contradiction. Fix $U \in B_2$. By assumption $U \nsubseteq M$. If $U \supset L$, then we use that $\sharp(L \cap S) \ge 2$. If $U \nsupseteq L$, then we get $w(U) = w(U \cap M) + m_{U\cap L}$ with $w(U \cap M) \le d$ and so we apply Lemma 3.

Fix $U \in B_1$. Since $U \nsubseteq M$, U is spanned by $U \cap S$ and $S = (S \cap M) \cup (S \cap L)$, then $U \cap L \cap S \neq \emptyset$ and hence $w_{\mathrm{Res}_H(Z)}(U) \le d$.

(k) Assume $S \nsubseteq M \cup L$, i.e. assume $w(\mathbb{P}^4) = 4d+1$, $w(M) = 3d-1$.

We check that $h^1(\mathcal{I}_Z(d)) = 0$, except at most in one case with $\diamond$. Set $S' := S \cap (M \cup L)$. Let o be the only point of $S \setminus S'$. We have $m_o = 1$, because $w(M) + w(L) = 4d$. Let Q be a general element of $|\mathcal{I}_{M\cup L}(2)|$. Since $M \cup L$ is the base locus of $|\mathcal{I}_{M\cup L}(2)|$ and Q is general, then $o \notin Q$. By the inductive assumption we have $h^1(\mathcal{I}_{Z\cap Q}(d)) = 0$ and hence $h^1(Q, \mathcal{I}_{Z\cap Q}(d)) = 0$. Therefore it is sufficient to prove that $\mathrm{Res}_Q(Z)$ satisfies the Segre conditions in degree $d-2$. Write $m_1 \geq \cdots \geq m_s = 1$.

Claim 2: We have $s \geq 9$.

Proof of Claim 2: We have $m_1 + \cdots + m_{s-1} = 4d$. Hence $(s-1)m_1 \geq 4d$. Therefore to prove Claim 2 we may assume $m_1 \geq \lceil (d+1)/2 \rceil$. Since $m_1 + m_i \leq d+1$ for all $i > 1$, we get $4d = m_1 + \cdots + m_{s-1} \leq (s-2)(d+1) - (s-3)\lceil (d+1)/2 \rceil$. Hence $s > 8$ if $d \geq 4$.

By Claim 2 $w_{\mathrm{Res}_Q(Z)}(\mathbb{P}^4) = w(\mathbb{P}^4) - \sharp(S') \geq 4d+1-9 = 4(d-2)+1$ and hence $\mathbb{P}^4$ satisfies the Segre condition in degree $d-2$ with respect to $\mathrm{Res}_Q(Z)$. Fix $A \in B_c \cup B'_c$, $c = 1,2,3$. Since $A \cap S$ spans A, we have $\sharp(A\cap S') \geq \sharp(A \cap S) - 1 \geq c$. Hence if $A \in B'_c$, then it satisfies the Segre condition in degree $d-2$ with respect to $\mathrm{Res}_Q(Z)$. Now assume $A \in B_c$. We have $w_{Z\setminus\{o\}}(A) \geq w(A) - 1$. If $c = 1$ we use that $m_1 \leq d-1$ and hence $\sharp(S' \cap A) \geq 2$. Now assume $c = 2$ and $w(A) = 2d$. We have $\sharp(A \cap S') \geq 3$, because $m_1 + m_2 \leq d+1 < 2d-1$ (if $d > 3$). Now assume $c = 2$ and $w(A) = 2d+1$. We need to check that $\sharp(A \cap S') \geq 4$. Assume $A \cap S' \subseteq \{o_1, o_2, o_3\}$ with $m_{o_1} \geq m_{o_2} \geq m_{o_3}$. Since $m_{o_1} + m_{o_2} \leq d+1$, we get $2d \leq m_{o_1} + m_{o_2} + m_{o_3} \leq 2(d+1) - m_{o_1}$. We get a contradiction if $m_{o_1} \geq 3$. If $m_{o_1} \leq 2$, we get $2d \leq 6$, contradicting the assumption $d > 3$. Now assume $c = 3$ and write $A \cap S' = \{o_1, \ldots, o_h\}$ with $m_{o_1} \geq \cdots \geq m_{o_h}$. We get

$$w(A) - 1 \leq (h-1)(d+1) - (h-2)m_{o_1} \tag{2}$$

First assume $w(A) = 3d-1$. The inequality (2) gives $h > 3$. Now assume $w(A) = 3d$. From (2) we get $h > 4$ if $m_{o_1} \geq 3$. If $m_{o_1} \leq 2$ and $h \leq 4$ we get $3d - 1 \leq 8$, contradicting the assumption $d > 3$. Now assume $w(A) = 3d+1$. Since $m_1 + \cdots + m_h \leq hm_{o_1}$, we have $h \geq 6$ (and hence A satisfies the Segre condition in degree $d-2$ with respect to $\mathrm{Res}_Q(Z)$) if $m_{o_1} \leq d/2$. Assume $m_{o_1} \geq \lceil (d+1)/2 \rceil$ and $h \leq 5$. If $d = 4$, we use Proposition 2. By (2) we have $3d \leq (5d+5)/2$, which is false if $d \geq 7$. If $d = 6$ we use that $m_1 \geq 4$ and (2) to get $h > 5$. Now assume $d = 5$ and $h \leq 5$. From (2) it is sufficient to check the case $m_{o_1} = 3$. We get $h = 5$ and $m_{o_1} = 3$ for all i. We are in case $\diamond$ with $m_5 = (d+1)/2$. We exclude this case taking a non-general Q. We may take as Q any reducible quadric $Q' = H \cup$Since $L \in B_1$, two of the points of S' are the points of $L \cap S$. Hence all other points of S' are contained in M.

4. $B_3 \neq \emptyset$

In this section we assume $B_3 \neq \emptyset$. We assume $m_1 + m_2 \leq d$. By section 3 we may assume $M \cap L \cap S \neq \emptyset$ for all $M \in B_3$ and all $L \in B_1$.

Call H an element of B_3 with maximal weight. It is sufficient to prove that $W := \mathrm{Res}_H(Z)$ satisfies the Segre condition in degree $d-1$. Our assumption gives that W satisfies the Segre condition with respect to all lines. We have $w_W(\mathbb{P}^4) = w(\mathbb{P}^4) - \sharp(H \cap S) \leq 4d+1-4$ by Remark 2. We have $w_W(H) \leq w(H) - 4 \leq 3(d-1)$. Take $M \in B_3$ with $M \neq H$ (if any). Since $w(\mathbb{P}^4) \leq 4d+1$, we have $w(M \cap H) \geq w(M) + w(H) - 4d - 1$. Since $w(M) + w(H) - 4d - 1 \geq 2d-3 > 0$, we get $M \cap H \cap S \neq \emptyset$. Hence to check that M satisfies the Segre condition for W in degree $d-1$ we may assume $w(M) \geq 3d$. Our choice of H gives $w(H) \geq 3d$. Hence $w(M) + w(H) - 4d - 1 \geq 2d-1$. Since $m_1 \leq d-1$, we get $\sharp(M \cap H) \geq 3$.

(a) Fix $N \in B_2$. N satisfies the Segre condition for W in degree $d-1$ if and only if either $w(N) = 2d$ and $N \cap H \cap S \neq \emptyset$ or $w(N) = 2d+1$ and $\sharp(N \cap H \cap S) \geq 2$. Since $w(\mathbb{P}^4) \leq 4d+1$, we have $w(N \cap H) \geq w(M) + w(N) - 4d - 1$. Hence $w(N \cap H) > 0$, i.e. $N \cap H \cap S \neq \emptyset$. Thus to check the Segre condition for W we may assume $w(N) = 2d+1$. We get $w(N \cap H) \geq w(H) - 2d$. We get $\sharp(N \cap H) \geq 2$, unless either $w(H) = 3d-1$, $m_1 = d-1$ and $N \cap H$ contains the only point of S with maximal multiplicity or $w(H) = 3d$, and $m_1 = d$. In the latter case we have $h^1(\mathcal{I}_Z(d)) = 0$ by Lemma 3.

(b) Assume $w(H) = 3d-1$, $w(N) = 2d+1$, $m_1 = d-1$ and that $N \cap H \cap S = \{o\}$ with o the only point with multiplicity $d-1$ of Z. In this case we have $S \subset H \cup N$. Fix $p \in H \cap S \setminus N \cap S$ and let M be the hyperplane spanned by N and p. We check for which choice of p the scheme $\mathrm{Res}_M(Z)$ satisfies the Segre condition in degree $d-1$. Since $\sharp(S \cap N) \geq 3$ (Remark 2) we have $w_{\mathrm{Res}_M(Z)}(\mathbb{P}^4) \leq 4d+1-4$. Since $p \in H \cap S$ and $w(H) = 3d-1$, H satisfies the Segre condition in degree $d-1$ for $\mathrm{Res}_M(Z)$. Fix another $F \in B_3$ (if any). Since $w(H) = 3d-1$, the maximality property of $w(H)$ gives $w(F) = 3d-1$. Since $F \neq H$, $F \cap H$ is a plane and hence $w(F \cap H) \leq 2d+1$. Hence $w(F \setminus F \cap H) \geq d-2 > 0$. Since $S \subset H \cup N$, we get $F \cap N \cap S \neq \emptyset$ and hence $w_{\mathrm{Res}_M(Z)}(F) \leq 3(d-1)+1$. Fix $A \in B_2$. Since $S \subset H \cup N$, we have $A \cap S = (H \cap A \cap S) \cup (N \cap S)$; if $A \nsubseteq H$ we get $w(A \cap H) \leq d+1$ and hence either $A = N$ or $w(A) = 2d$ and $o \in A$; in these cases$w_{\mathrm{Res}_M(Z)}(A) \leq 2d-1$ and hence A satisfies the Segre condition for $\mathrm{Res}_M(Z)$. Fix $L \in B_1$ and assume $L \cap S \cap N = \emptyset$. Since $S \subset H \cup N$ and $\sharp(L \cap S) \geq 2$, we get $L \subset H$; if it exists we need to take $p \in L$. If the line L is unique any $p \in L \cap S$ will do. Take $R \in B_1$ such that $R \cap N \cap S = \emptyset$ and $R \neq L$. Since $w(\mathbb{P}^4) \leq 4d+1$, we get

$w(L\cap R) \ge 2(d+1)+(2d+1)-4d-1$ and hence $L\cap R \in S$. Since $L\cap R \ne o$, we have $m_{L\cap R} \le 2$ and hence $w(L\cup R) \ge 2d$. Since $w(\mathbb{P}^4) \le 4d+1$, we get $S\cup L\cup R\cup N$; in this case we tmay $p = L\cap R$, unless there is $D \in B_1$ with $D\cap N\cap S = \emptyset$ and $L\cap R \notin D$; D is contained in the plane $\langle D\cup R\rangle$ and we have $2d+1 \ge w(\langle D\cup R\rangle) \ge 3(d+1) - m_{D\cap L} - m_{D\cap R} - m_{R\cap L}$ with $m_u \le 2$ if $u \in \{D\cap L, D\cap R, R\cap L\}$, a contradiction.

5. B_2

In this section we assume $B_3 = \emptyset$ and $m_1+m_2 \le d$ with the next lemma as the target of the section.

Lemma 7. *If $d \ge 5$, then $\sharp(B_2) \le 1$.*

Remark 9. Assume $B_3 = \emptyset$. If $E \in B_2$, $w(B) \ge d-1$ and $E\cap B \ne \emptyset$, then $E\cap B\cap S \ne \emptyset$. If $A \in B_2'$, $B \in B_1$ and $A\cap B \ne \emptyset$, then $A\cap B\cap S \ne \emptyset$.

Proof of Lemma 7. Assume $\sharp(B_2) \ge 2$. Call E, F two different elements of B_2.

Observation 1: Since $B_3 = \emptyset$ and $w(L) \le d+1$ for each line $L \subset \mathbb{P}^4$, $E\cap F$ is a single point o.

Observation 2: If $o \notin S$, then $w(\mathbb{P}^4 \setminus E\cup F) \le 1$ and either $S \subset E\cup F$ or $w(E) = w(F) = 2d$ and $S\cap(E\cup F)$ is a unique point with multiplicity 1, because $w(E) \ge 2d$, $w(F) \ge 2d$ and $w(\mathbb{P}^4) \le 4d+1$.

Claim 1: We have $\sharp(B_2) \le 2$.

Proof of Claim 1: Assume $\sharp(B_2) \ge 3$ and call E_1, E_2, E_3 3 distinct elements of B_2. Observation 1 gives that for each $i \ne j$, the set $E_i\cap E_j$ is a single point, o_h, where $\{i,j,h\} = \{1,2,3\}$. With no loss of generality we may assume $m_{o_1} \ge m_{o_2} \ge m_{o_3}$. The points o_1, o_2, o_3 are not necessarily distinct, but in any case we have $w(E_1\cup E_2\cup E_3) \le w(E_1)+w(E_2)+w(E_3) - m_{o_1} - \epsilon_1 m_{o_2} - \epsilon_2 m_{o_3}$, with $\epsilon_1 = 1$ if $o_1 \ne o_2$, $\epsilon_2 = 1$ if o_1, o_2, o_3 are distinct points, $\epsilon_1 = 0$ if $o_1 = o_2$ and $\epsilon_2 = 0$ if $\sharp(\{o_1,o_2,o_3\}) \le 2$. We first get that $\sharp(\{o_1,o_2,o_e\}) = 3$, and then, using $m_{o_1}+m_{o_3} \le m_{o_1}+m_{o_2} \le d+1$ and $d \ge 4$, a contradiction.

(a) In this step we assume the existence of $E_1, E_2 \in B_2$ such that $E_1\cap E_2 \notin S$. We assume $w(E_1) \ge w(E_2)$.

(a1) Assume $S \subset E_1\cup E_2$. Fix general hyperplanes H, H' containing E_1 and general hyperplanes M, M' containing E_2. Set $Q := H\cup M$ and $T := Q\cap(H'\cup H')$. We first check that $\mathrm{Res}_Q(Z)$ satisfies the Segre conditions in degree $d-2$. By [2] we may assume $s \ge 8$ and hence (since $S \subset Q$) $\mathbb{P}^4$ satisfies the Segre condition for $\mathrm{Res}_{H\cup M}(Z)$. For each $A \in B_i'$, $i = 1,2,3$, it is sufficient to use that $A\cap S$ spans A (Remark 2). If $A \in B_2$ we use then $A = E_i$ and we use that $\sharp(A \cap S) \geq 4$ (Remark 2). If $A \in B_1$, then we use that $\sharp(B \cap S) \geq 2$. By the residual sequence of Q it is sufficient to prove that $h^1(Q, \mathcal{I}_{Q\cap Z,Q}(d)) = 0$. The scheme T is an effective Cartier divisor of Q and hence it is defined the residual scheme $\mathrm{Res}_T(Z \cap Q)$. Since $\mathrm{Res}_T(Z \cap Q) \subseteq \mathrm{Res}_Q(Z)$, we have $h^1(Q, \mathcal{I}_{\mathrm{Res}_T(Z\cap Q)}(d-2)) \leq h^1(\mathcal{I}_{\mathrm{Res}_Q(Z)}(d-2)) = 0$. By the residual exact sequence of T in Q it is sufficient to prove that $h^1(T, \mathcal{I}_{Z\cap T,T}(d)) = 0$. We have $Z \cap T = Z \cap (E_1 \cup E_2)$, because $E_1 \cap E_2 \notin S$. Use that $h^1(E_i, \mathcal{I}_{Z\cap E_i}(d)) = 0$ by the Segre conditions and that the restriction map $H^0(\mathcal{O}_{\mathbb{P}^4}(d)) \to H^0(E_1 \cup E_2, \mathcal{O}_{E_1\cup E_2}(d))$ is surjective (use that $h^1(\mathbb{P}^3, \mathcal{I}_{D\cup E}(2)) = 0$ for any two skew lines D, R of $\mathbb{P}^3$).

(a2) Assume $S \nsubseteq E_1 \cup E_2$. We have $w(E_1) = w(E_2) = 2d$, $S \setminus S \cap (E_1 \cup E_2)$ is a single point o and $m_o = 1$. Let H_i be a general hyperplane containing E_i. Set $Q := H_1 \cup E_2$. Since $o \notin Q$, the inductive assumption gives $h^1(Q, \mathcal{I}_{Q\cap Z}(d-2)) = 0$. Hence it is sufficient to prove that $\mathrm{Res}_Q(Z)$ satisfies the Segre conditions in degree $d-2$. Since $\sharp(Q\cap S) = \sharp(S\cap E_1) + \sharp(S\cap E_1) \geq 4+4$, $\mathbb{P}^4$ satisfies the Segre condition. Fix $U \in B_3' \cup B_3$. Since $B_3 = \emptyset$, it is sufficient to prove that $\sharp(U \cap (S \setminus \{o\})) \geq 3$; this is true, because $m_1 + m_2 \leq d + 1$ and $w(U) - 1 \geq 3d - 5 > d + 1$. Fix $A \in B_2' \cup B_2$. First assume $A \in B_2$, i.e. $A = E_i$ for some i. Since $w(E_i) = 2d$ and $m_o = 1$, it is sufficient to use that $w(E_i \cap S) - 1 > m_1 + m_2 + m_3$; this is true if $2d - 2 \leq m_1 + m_2 + m_3$, which is true, because $m_1 + m_2 \leq d$ and hence $m_1 + m_2 + m_3 \leq 2d - m_1$, while $2d - 2 \geq m_1 + m_2 + m_3$ if $m_1 = 1$. Now assume $A \in B_2'$. Use that $w(A) - 1 > 0$ if $w(A) = 2d - 2$ and $\sharp(A \cap (S \setminus \{o\}) \geq 2$ if $w(A) = 2d - 1$ (since $m_1 \leq 2d - 3$ for all $d \geq 4$). Fix $B \in B_1' \cup B_1$. If $w(B) = d$ it is sufficient to use that $d > 1 = m_o$. If $w(B) = d + 1$ it is sufficient to use that $m_1 < d$ (Lemma 3).

(b) In this step and in step (c) $\sharp(B_2) = 2$, say $B_2 = \{E_1, E_2\}$ and that $\{o\} := E_1 \cap E_2 \in S$. In this step we assume $S \subset E_1 \cup E_2$. Since $\sharp(B \cap S) \geq 3$ for all $B \in B_1$, then each element of B_1 is contained either in E_1 or in E_2. Let $\mathcal{T}_i$, $i = 1, 2$, be the set of all $B \in B_1$ such that $B \subset E_1$ and $o \notin B$. Assume for the moment $\sharp(\mathcal{T}_i) \leq 2$ for some i, say $\sharp(\mathcal{T}_2) \leq 2$. Therefore there is $p \in E_2 \cap S \setminus \{o\}$ such that every element of $\mathcal{T}_2$ contains p. Set $H := \langle E_1 \cup \{p\}\rangle$. By construction $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree $d - 1$ with respect to lines and $\mathbb{P}^4$, trivially with respect to hyperplanes (because $B_3 = \emptyset$). Fix $A \in B_2$, i.e. $A = E_1$ or $A = E_2$. If $A = E_1$, then use that $\sharp(A \cap S) \geq 2$. If $A = E_2$, then we use that $o, p \in A \cap H \cap S$. Now assume $\sharp(\mathcal{T}_i) \geq 3$ for all i. Fix $D_1, D_2, D_3 \in \mathcal{T}_1$. By Remark 3 $D_i \cap D_j$, $i < j$, is an element of S. Since $w(E_1) \leq 3(d + 1) - m_{D_i\cap D_j}$ we get that $D_1 \cap D_2$, $D_1 \cap D_3$ and $D_2 \cap D_3$ are distinct points and $3d + 3 - 2d - 1 + m_o \leq m_{D_1\cap D_2} + m_{D_1\cap D_3} + m_{D_2\cap D_3}$, i.e.

$$d + 2 + m_o \leq m_{D_1\cap D_2} + m_{D_1\cap D_3} + m_{D_2\cap D_3} \tag{3}$$

Moreover, no 3 of the points o, $D_1 \cap D_2$, $D_1 \cap D_3$ and $D_2 \cup D_3$ are collinear. Hence $\mathcal{I}_{\{D_1\cap D_2, D_1\cap D_3, D_3\cap D_3\}\cup E_2}(2)$ is spanned. Thus there is a quadric hypersurface $Q \in |\mathcal{I}_{\{D_1\cap D_2, D_1\cap D_3, D_3\cap D_3\}\cup E_2}(2)|$ with $Q \cap S \neq S$. By the inductive assumption it is sufficient to prove that $\mathrm{Res}_Q(Z)$ satisfies the Segre conditions in degree $d-2$.

(b1) These conditions are satisfied by $\mathbb{P}^4$, each element of B_2, each element of $B_1 \cup B_1'$ contained in E_2. They are satisfied by each D_i and each $B \in B_1 \setminus \{D_1, D_2, D_3\}$ contained in E_1, because $B \cap (D_1 \cup D_2 \cup D_3) = S \cap B \cap (D_1 \cup D_2 \cup D_3)$ (Remark 3) and $\sharp(B \cap (D_1 \cup D_2 \cup D_3)) \geq 2$, since $D_1 \cap D_2 \cap D_3 = \emptyset$. They are satisfied by each $B \in B_1'$ contained in E_1, because $w(E_1 \setminus \{o, D_1 \cap D_2, D_1 \cap D_3, D_3 \cap D_3\}) < d$ by (3), and all other elements of B_1' meets $E_2 \cap S$, since $S \subset E_1 \cup E_2$.

(b2) Fix $U \in B_3'$. We have $\sharp(S \cap Q \cap U) \geq w(U) - 3d + 5$, because $w(\mathbb{P}^4 \setminus Q) \leq 4d + 1 - 2d - 3(d+1) + m_{D_1\cap D_2} + m_{D_1\cap D_3} + m_{D_2\cap D_3} = m_{D_1\cap D_2} + m_{D_1\cap D_3} + m_{D_2\cap D_3} - d - 2 \leq d/2$.

(b3) Fix $A \in B_2'$. First assume $o \notin A$. To check the Segre condition for A we may assume $\sharp(A \cap E_1 \cap S) \leq 1$. Since $A \cap S$ spans A we get that A contains a unique point $E_2 \cap S \setminus \{o\}$, while the other points are contained in a line $L \subset E_1$; we get $w(L) \geq w(A) - m_q$; to check the Segre for A we may assume $w(A) = 2d - 1$ and in this case we get $L \in B_1'$ and we checked that $L \cap Q \cap S \neq \emptyset$. Now assume $o \in A$. To check the Segre condition for A we may assume $w(A) = 2d - 1$ and that $A \cap E_2 \cap S = \{o\}$. Since $S \subset E_1 \cup E_2$ and $A \cap S$ spans A (Remark 2), we get $A \subseteq E_1$, a contradiction.

(c) Assume $o \in S$ and $S \nsubseteq E_1 \cup E_2$. We have $w(\mathbb{P}^4 \setminus (E_1 \cup E_2)) \leq 4d + 1 - 2d - 2d + m_o = m_o + 1$ and equality holds only if $w(\mathbb{P}^4) = 4d + 1$ and $w(E_i) = 2d$ for all i.

Since $\mathcal{I}_{E_1\cup E_2}(2)$ is spanned (use reducible quadrics) there is $Q \in |\mathcal{I}_{E_1\cup E_2}(2)|$ such that $Q \cap S \neq S$. The inductive assumption on the weight gives $h^1(\mathcal{I}_{Z\cap Q}(2)) = 0$ and so it is sufficient to prove that $\mathrm{Res}_Q(Z)$ satisfies the Segre conditions in degree $d-2$. We will find some obstructions and take a suitable q to overcome them. Since $Q \supset E_1 \cup E_2$ and $\sharp(S \cap E_i) \geq 4$, $\mathbb{P}^4$, E_1 and E_2 satisfy the Segre condition. Fix $U \in B_3 \cup B_3'$. Since $w(\mathbb{P}^4 \setminus Q) \leq w(\mathbb{P}^4 \setminus (E_1 \cup E_2)) \leq m_o + 1$, we get $S \cap U \cap Q \neq \emptyset$. If $w(U) \geq 3d - 3$, we also get $\sharp(U \cap Q \cap S) \geq 2$, because $1 + 2m_2 \leq 2d - 1 \leq 3d - 4$. Now assume $w(U) = 3d - 2$ and $\sharp(S \cap U \cap Q) = 2$, say $S \cap U \cap Q = \{p, q\}$. We get $m_p + m_q + m_o + 1 \geq 3d - 2$; if $o \in \{p, q\}$, then we use that $2m_1 + m_2 \leq 3d - 4$, since $m_1 \leq d - 1$ (Lemma 3), $m_1 + m_2 \leq d$ and $d \geq 5$; if $o \notin \{p_q\}$ we use that $m_1 + m_2 + m_3 + 1 \leq 2d + 1 - m_1 + m_3$.

Take $B \in B_1 \cup B_1'$. If $B \cap Q \cap S = \emptyset$, the $w(B) \leq 1 + m_o$; we get $w(B) = d$ and $m_o = d - 1$; we also get $w(\mathbb{P}^4) = 4d + 1$ and $w(E_i) = 2$ for all i; call + this case. Take $B \in B_1$ with $\sharp(B \cap Q \cap S) = 1$, say $B \cap Q \cap S = \{q\}$. First assume $q = o$; we also get $w(\mathbb{P}^4) = 4d+1$ and $w(E_i) = 2$ for all i in this case $w_{\mathrm{Res}_Q(Z)}(B) = d-1$, because Q is singular at o. Now assume $q \ne o$. We get $w(\mathbb{P}^4) = 4d+1$ and $m_q + m_o = d$; call ++ this case. Take $A \in B_2'$. Since $w(\mathbb{P}^4 \setminus (E_1 \cup E_2)) \le m_o + 1 \le d < 2d-2$, we have $\sharp(A \cap Q \cap S) \ne \emptyset$. Assume $w(A) = 2d-1$ and $\sharp(A \cap Q \cap S) = 1$, say $A \cap Q \cap S = \{q\}$. We get $m_q + m_o + 1 \le 2d-1$; this is possible only if $q = o$ and in this case $w_{\mathrm{Res}_Q(Z)}(A) \le 2d-4$, because Q is singular at o. In summary, for any Q the Segre conditions are satisfied by all linear spaces of dimension > 1. Hence from now on we assume $w(\mathbb{P}^4) = 4d+1$ and $w(E_i) = 2d$, $i = 1, 2$.

(c1) Assume case +, i.e. $m_o = d-1$. Since $m_1 + m_2 \le d$, we get $m_i = 1$ for all $i > 0$. We saw the existence of $B \in B_1'$ such that $S \subset E_1 \cup E_2 \cup B$ and $B \cap (E_1 \cup E_2) \cap S = \emptyset$ and that for an arbitrary Q B is the only linear space for which the Segre condition in degree $d-2$ may fail. By Remark 9 we have $B \cap E_=\emptyset$. Fix $p \in B \cap S$ and set $M := \langle E_1 \cup \{p\}\rangle$. Let H be any hyperplane with $H \supset E_2$ and $H \cap B \cap S = \emptyset$. The latter condition gives $(M \cup H) \cap S \ne \emptyset$. The former condition gives that $\mathrm{Res}_{H \cup M}(Z)$ satisfies the Segre condition for B and hence for all linear spaces.

(c2) Assume case ++, i.e. the existence of $L \in B_1$ and $q \in (E_1 \cup E_2) \cap S \setminus \{o\}$ with $m_q + m_o = d$ and $L \cap (E_1 \cap E_2) \cap S = \{q\}$. We also have $S \subset L \cup E_1 \cup E_2$ and that L is the only linear space for which (for an arbitrary $Q \supset E_1 \cup E_2$) the Segre condition in degree $d-2$ may fail. With no loss of generality we may assume $q \in E_1$. Set $H := \langle E_2 \cup \{q\}\rangle$ and $D := \langle \{o, q\}\rangle \in B_1'$. $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree $d-1$ with respect to $\mathbb{P}^4$ (because $\sharp(H \cap S) \ge 4$) and with respect to all hyperplanes. Fix $R \in B_1$. If $R \cap E_2 \cap S \ne \emptyset$, then $w_{\mathrm{Res}_H(Z)}(R) \le d$. If $R \cap E_2 \cap S = \emptyset$, then either $R = L$ or $R \subset E_1$, because $S \subset L \cup E_1 \cup E_2$ and $\sharp(R \cap S) \ge 3$. If $R = L$, then we use that $q \in H$. If $R \subset E_1$, then $D \cap R \cap S \ne \emptyset$, because $D \subset E_1$, $R \subset E_1$ and $w(D) + w(R) = 2d+1 > w(E_1)$. □

6. 3 Mutually Disjoint Elements of B_1

In this section we assume $B_3 \ne \emptyset$ and the existence of $L_1, L_2, L_3 \in B_1$ such that $L_i \cap L_j = \emptyset$ for all $i \ne j$. With this assumption we prove that $h^1(\mathcal{I}_Z(d)) = 0$.

By Remark 7 we have $(L_1 \cup L_2 \cup L_3) \cap S \cap R \ne \emptyset$ for all $R \in B_1 \cup B_1'$. We have $w(\mathbb{P}^4 \setminus (L_1 \cup L_2 \cup L_3)) \le 4d+1-3(d+1) = d-2$. Let J be the only line meeting all lines L_1, L_2, L_3 (Remark 8).

We often use the notation introduced in the statement of the following lemma.

Lemma 8. *For any $i, j \in \{1, 2, 3\}$with $i \neq j$ let U_h, $h := \{1, 2, 3\} \setminus \{i, j\}$, be the hyperplane spanned by $L_i \cup L_j$ and F_i the plane spanned by $L_i \cup J$. Take $a \in \mathbb{P}^4 \setminus J \cup L_1 \cup L_2 \cup L_3$. If $a \in F_i$ for some i, then $|\mathcal{I}_{\{a\} \cup J \cup L_1 \cup L_2 \cup L_3}(2)|$ has $F_i \cup L_1 \cup L_2 \cup L_3$ as its base locus. In all other cases, the base locus is $\{a\} \cup J \cup L_1 \cup L_2 \cup L_3$.*

Proof. First assume that $a \in U_h$ for some U_h, say $h = 1$. Take reducible quadrics U_1 as a component and use that $\mathcal{I}_{J \cup L_i \cup L_j \cup \{a\}, U_1}(2)$ has no base points outside $U_1 \cap (J \cup L_1 \cup L_2 \cup L_3 \cup \{a\})$, unless $a \in F_2 \cup F_3$, say $a \in F_3$. Using hyperplanes containing L_3 we see that the same is true for $|\mathcal{I}_{\{a\} \cup J \cup L_1 \cup L_2 \cup L_3}(2)|$. Now assume $a \notin U_h$ for any h. The linear system of hyperplanes containing L_h and a has the plane $E_{h,a} := \langle L_h \cup \{a\} \rangle$ as its base locus. The family of reducible quadrics $U_h \cup M$, M a hyperplane containing $E_{h,a}$, shows that outside $J \cup L_1 \cup L_2 \cup L_3$ the base locus is contained in $E_{h,a}$. Since $E_{1,a} \cap E_{2,a} \cap E_{3,a} = \{a\}$, we get the lemma even in this case. □

Lemma 9. *Fix a plane $A \subset \mathbb{P}^4$ and lines $L_i \subset \mathbb{P}^4$, $1 \leq i \leq 3$, such that $L_i \cap L_j = \emptyset$ for all $i \neq j$, $\sharp(A \cap L_i) = 1$ for all i and A not the intersection of two hyperplanes $\langle L_i \cup L_j \rangle$, i.e. $A \neq F_h$ for any h. Then $h^0(\mathcal{I}_{A \cup L_1 \cup L_2 \cup L_3}(2)) = 3$.*

Proof. By assumption A is contained in at most one hyperplane $\langle L_i \cup L_j \rangle$. Hence may assume that $A \nsubseteq \langle L_1 \cup L_3 \rangle$ and that $A \nsubseteq \langle L_2 \cup L_3 \rangle$. Set $\{o_i\} := A \cap L_i$. Since each L_i meets A, we have $h^0(\mathcal{I}_{A \cup L_1 \cup L_2 \cup L_3}(2)) \geq 3$. We have $h^1(\mathcal{I}_{A \cup L_1 \cup L_2}(2)) = 0$ (all triples (A, L_1, L_2) are projectively equivalent), i.e $h^0(\mathcal{I}_{A \cup L_1 \cup L_2}(2)) = 5$. Since $A \nsubseteq \langle L_h \cup L_3 \rangle$ for any $h \in \{1, 2\}$, the set $\langle A \cup L_h \rangle \cap L_3$, $h = 1, 2$, is a unique point, e_h. Fix $p \in L_3$ with $p \neq o_3$ and $p \neq e_1$. Taking the union of $\langle A \cup L_1 \rangle$ and a general hyperplane M containing L_2 we get $h^0(\mathcal{I}_{A \cup L_1 \cup L_2 \cup \{p\}}(2)) = 4$. Let H be a general hyperplane containing $L_1 \cup \{p\}$. Fix $o \in L_3 \setminus \{p, e_1, e_2\}$. Since $H \cup \langle A \cup L_2 \rangle \in |\mathcal{I}_{A \cup L_1 \cup L_2 \cup \{p\}}(2)|$ and $o \notin H \cup \langle A \cup L_2 \rangle$, we get $h^0(\mathcal{I}_{A \cup L_1 \cup L_2 \cup L_3}(2)) \leq 3$. □

Lemma 10. *Assume $S \subset L_1 \cup L_2 \cup L_3$. Then $h^1(\mathcal{I}_Z(d)) = 0$.*

Proof. Fix $N \in B_2$. Since $\sharp(N \cap S) \geq 4$, it contains at least one of the lines L_i. Since N is unique (Lemma 7), we may change the labels, so that, if N exists, then it contains L_1. Since $\sharp(B \cap S) \geq 3$ for all $B \in B_1$ (Remark 6)), we have $B_1 \subseteq \{L_1, L_2, L_3, J\}$. Fix a hyperplane containing L_1, a point of $L_2 \cap S$ and a point of $L_3 \cap S$. Fix $B \in B_1$. Since $\sharp(B \cap S) \geq 3$, either $B \cap S$ meets each set $L_i \cap S$, $i = 1, 2, 3$, or B contains at least two points of some L_i and so $B = L_i$. In both cases we have $B \cap S \cap H \neq \emptyset$. The scheme $\mathrm{Res}_H(Z)$

satisfies the Segre condition in degree $d-1$ with respect to $\mathbb{P}^4$, all lines, all planes (because $N \supset L_1$) and (since $B_3 = \emptyset$) all hyperplanes. □

Lemma 11. *If $S \subset J \cup L_1 \cup L_2 \cup L_3$, then $h^1(\mathcal{I}_Z(d)) = 0$.*

Proof. By Lemma 10 we may assume $S \nsubseteq L_1 \cup L_2 \cup L_3$. By Remark 6 we have $B_1 \subseteq \{L_1, L_2, L_3, J\}$.

(i) First assume $J \cap S \cap (L_1 \cup L_2 \cup L_3) \neq \emptyset$, say $J \cap L_3 \in S$. Let H be a hyperplane containing F_1 and a point of $L_2 \cap S \setminus J \cap S$ with maximal multiplicity. It is sufficient to prove that $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree $d-1$. It satisfies the Segre condition for $\mathbb{P}^4$, because $\sharp(L_1 \cap S) \geq 3$. It satisfies the Segre condition for lines, because $H \cap S$ meets J and each L_i. Since $B_3 = \emptyset$ by assumption, we only need to check the Segre condition for planes. Fix $N \in B_2$. We have $\sharp(N \cap S) \geq 4$ (and hence either N contains a different point of $J \cap S$, $L_1 \cap S$, $L_2 \cap S$, $L_3 \cap S$ or N contains a line L_i) with equality only if $w(N) = 2d$ (Remark 6). Therefore $N \cap S \cap H \neq \emptyset$ and to check the Segre condition for $\mathrm{Res}_H(Z)$ it is sufficient to check the case $w(N) = 2d+1$ with N containing one among L_2 and L_3 (say L_h) but not the other one. If $N = F_h$, we get $\sharp(S \cap N \cap H) \geq 2$, because $J \subset H$. Now assume$N \neq F_h$ and so $N \cap J = L_h \cap J$. Since $w(N \setminus L_h) = d$, we have $\sharp(N \cap S \setminus L_h \cap S) \geq 2$ and these points must be contained in $L_1 \cup L_{5-h}$, no two of them in the same line (since two of the lines L_1, L_2, L_3 spans a $\mathbb{P}^3$). Hence $\sharp(N \cap S \cap H) \geq 2$.

(ii) Now assume $J \cap S \cap (L_1 \cup L_2 \cup L_3) = \emptyset$. Hence $w(J) \leq d-2$, $B_1 = \{L_1, L_2, L_3\}$, $B_1' = \emptyset$ and $w(F_i) \leq 2d-1$.

Take a hyperplane U containing one of the points with maximal multiplicity of each line J, L_1, L_2, L_3 and spanned by points of S, except that if $B_2 \neq \emptyset$, say $B_2 = \{N\}$ (Lemma 7), we impose that for any line J, L_1, L_2, L_3 intersecting $N \cap S$ we impose that we take this point as one of the prescribed points of $U \cap S$. We need to check the Segre conditions for $\mathrm{Res}_U(Z)$ in degree $d-1$. The one for $\mathbb{P}^4$ is satisfied, because $\sharp(U \cap S) \geq 4$ by its definition. The Segre conditions for lines are satisfied, because $L_i \cap S \cap U \neq \emptyset$. Since $B_3 = \emptyset$, it is sufficient to check the ones for planes $N \in B_2$. By Remark **??** $S \cap (N \setminus J)$ contains at least two points of $L_1 \cup L_2 \cup L_3$, because $w(F_i) \leq 2d-1$ and hence $N \neq F_i$. □

Remark 10. Fix a quadric $Q \supset L_1 \cup L_2 \cup L_3$ and assume $Q \cap S \neq S$, so that the inductive assumption gives $h^1(\mathcal{I}_{Z \cap Q}(d)) = 0$. We check what conditions on Z and Q gives that $\mathrm{Res}_Q(Z)$ satisfies the Segre condition in degree $d-2$. The condition for $\mathbb{P}^4$ is satisfied, because $\sharp(L_i \cap S) \geq 3$. Fix $U \in B_3' \cup B_3 = B_3'$. Since $w(\mathbb{P}^4 \setminus (L_1 \cup L_2 \cup L_3)) \leq d-2$ and $d-2+m_1 \leq 2d-3 < 3d-4$, we have $U \cap Q \cap S \neq \emptyset$. If $w(U) \geq 3d-3$ (resp. $w(U) = 3d-2$), then we have

$\sharp(U \cap Q \cap S) \geq 2$ (resp. $\sharp(U \cap Q \cap S) \geq 3$), because $d - 2 + m_1 < 3d - 3$ (resp. $d - 2 + m_1 + m_2 \leq 2d - 2 < 3d - 2$). Fix $A \in B_2 \cup B_2'$. Since $d - 2 < 2d - 2$ and $d - 2 + m_1 < 2d - 1$, we have $A \cap Q \cap S \neq \emptyset$ and $\sharp(A \cap Q \cap S) \geq 2$ if $w(A) \geq 2d - 1$. If $w(A) \geq 2d$, then $\sharp(A \cap H \cap S) \geq 3$, because $d - 2 + m_1 + m_2 < w(N)$. Thus we only need to test the element of B_2 if $B_2 \neq \emptyset$ and only if it has weight $2d + 1$. Fix $B \in B_1'$. By Remark 5 we have $S \cap B \cap (L_1 \cup L_2 \cup L_2) \neq \emptyset$ and hence $w_{\mathrm{Res}_Q(Z)}(B) \leq d + 1$. Let $\mathcal{B}$ be the set of all $B \in B_1$ such that $\sharp(S \cap B \cap (L_1 \cup L_2 \cup L_3 \cap J)) = 1$. Fix $B \in B_1$. By Remark 5 we have $S \cap B \cap (L_1 \cup L_2 \cup L_2) \neq \emptyset$. Since J is in the base locus of $|\mathcal{I}_{L_1 \cup L_2 \cup L_3}(2)|$, B may fail the test (for some Q) only if $B \in \mathcal{B}$. If $B \in \mathcal{B}$, then $B \cap J \cap S \subset (L_1 \cup L_2 \cup L_3)$. For $i = 1, 2, 3$ set $\mathcal{B}_i := \{L \in \mathcal{B} : L \cap L_i \neq \emptyset\}$.

Remark 11. Fix a plane $N \in B_2 \cup B_2'$ and set $x := w(N)$. If $(y - 1)\lfloor d/2 \rfloor + d - 2 < x$, then $\sharp(N \cap S \cap (L_1 \cup L_2 \cup L_3)) \geq y$. If $x \geq 2d - 1$, we may take $y \geq 3$, while if $x = 2d - 2$ we have $y > 0$. So only the planes N with $w(N) = 2d + 1$ may give troubles. Here we assume $w(N) = 2d + 1$. By Lemma 8 we may assume $\sharp(B_2) = 1$ and we fix $p \in N \cap S \setminus S \cap (L_1 \cup L_2 \cup L_3)$. Assume the existence of a plane $w(N) = 2d + 1$ and $\sharp(N \cap S \cap (L_1 \cup L_2 \cup L_3)) = 3$ (N is unique by Lemma 7). Set $N \cap S \cap (L_1 \cup L_2 \cup L_3) = \{o_1, o_2, o_3\}$ with, say $m_{o_1} \geq m_{o_2} \geq m_{o_3}$. Since $w(\mathbb{P}^4 \setminus (L_1 \cup L_2 \cup L_3)) \leq d - 2$, and $w(N) > d - 2 + d + 1$, the points o_1, o_2, o_3 are not collinear. Hence $\sharp(S \cap L_i) = 1$ for all i and $N \not\supseteq L_i$ for any i. In particular $N \neq F_i$ for any i. We also get that at most one of the points o_1, o_2, o_3 is contained in J. Since $w(\mathbb{P}^4 \setminus (L_1 \cup L_2 \cup L_3)) \leq d - 2$, we get $m_{o_1} + m_{o_2} + m_{o_3} \geq d + 3$. If N contains a point of $J \cap S$ distinct from the points o_1, o_2, o_3, then $w_{\mathrm{Res}_Q(Z)}(N) \leq 2n - 3$, because $Q \supset J$. We say that N *exists* only if $N \cap J \cap S \subset \{o_1, o_2, o_3\}$. Since $m_1 + m_2 \leq d$ by our assumption $\diamond\spadesuit$, we have $m_1 + m_2 + m_3 + m_4 \leq 2d < w(N)$. Thus $\sharp(N \cap S) \geq 5$, Assume $\mathcal{B} \neq \emptyset$. Fix any $D \in \mathcal{B}$ and set $\{u\} := S \cap D \cap (L_1 \cup L_2 \cup L_3) = S \cap D \cap (J \cup L_1 \cup L_2 \cup L_3)$. Since $m_1 + m_2 \leq d$, we have either $m_{o_1} + m_{o_2} + m_{o_3} + m_u \leq \min\{4m_{o_1}, 3d - 2m_{o_1}\} < w(N) + w(D) - d + 2$ (case $m_{o_1} \geq m_u$) or $m_{o_1} + m_{o_2} + m_{o_3} + m_u \leq \min\{4m_u, 3d - 2m_u\} < w(N) + w(D) - d + 2$ (case $m_u > m_{o_1}$) and hence $D \cap S \cap (N \setminus N \cap (L_1 \cup L_2 \cup L_3)) \neq \emptyset$.

Remark 12. Assume $F_1 \in B_2$. Fix $h \in \{2, 3\}$. Since $w(\langle F_1 \cup L_h \rangle) \geq 2d + d + 1 - m_{J \cap L_h}$ and $B_3 = \emptyset$, we have $J \cap L_h \in S$ for all h. By the definition of existence of N given in Remark 11 if N exists, then $N \neq F_i$ for any i.

Remark 13. Assume $\sharp(\mathcal{B}) \geq 2$ and $D_i \in \mathcal{B}$, $i = 1, 2$ with $D_1 \neq D_2$. We have $D_1 \cap D_2 \cap (L_1 \cup L_2 \cup L_3) = \emptyset$, because $\sharp(D_1 \cap D_2) \leq 1$ and $2(d+1) > m_u + d - 2$ for all $u \in \mathbb{P}^4$. Set $q_i := D_i \cap (L_1 \cup L_2 \cup L_3)$. We have $D_1 \cap D_2 \cap S \neq \emptyset$ (call q this point), because $m_{q_1} + m_{q_2} + d - 2 < 2d + 2$. We have $m_{q_1} + m_{q_2} + m_q + d - 2 \geq 2d + 2$

and hence $m_{q_1} + m_{q_2} + m_q \geq d + 4$.

(a) We also get that $q_1 \neq q_2$ and hence that if $\sharp(\mathcal{B}_j) \geq 2$, no two of the lines of $\mathcal{B}_j$ contain the same point of L_j.

Assume the existence of another line $D_3 \in \mathcal{B}$. Since $D_i \cap D_j \neq \emptyset$ for all i, j, either the lines D_1, D_2, D_3 are contained in a plane, A, or they pass through the same point. In the latter case q is their common point.

Assume for the moment that D_1 and D_2 meets the same line L_i. Since $D_1 \cap D_2 \neq \emptyset$, $L_i \subset \langle D_1 \cup D_2 \rangle$. Hence each D with $q \notin D$ meets L_i and hence $\langle D_1 \cup L_i \rangle$ is the plane A in this case. In the other case in which there is a plane A we have $\sharp(\mathcal{B}) = 3$, $\sharp(A \cap L_j) = 1$ for all j and we may rename the lines so that $D_i \cap L_j \neq \emptyset$ if and only if $i = j$. In the former case either D_1, D_2, D_3 meets different lines L_1, L_2, L_3, say $D_i \cap L_j \neq \emptyset$ for all $i \neq j$, or they meet the same line L_i, and $A = \langle L_i \cup D_1 \rangle$. Hence if all lines $D \in B_1$ with $\sharp(D \cap (L_1 \cup L_2 \cup L_3) \cap S) = 1$ are through a common point, there are at most 3 of them. If they are all contained in A, either all intersects L_i and D_i (at a point of S by Remark 2) or there are at most 3 such lines and $\sharp(A \cap L_i) = 1$ for all i such that $L_i \cap D$ for some i; in the first case they meets D_i and L_i at a point $\neq L_i \cap D_1$ and hence $\sharp(D \cap S \cap (L_i \cup D_1) \cap S) \geq 2$ for all $D \in B_1$ with $\sharp(D \cap (L_1 \cup L_2 \cup L_3) \cap S) = 1$.

Lemma 12. *Assume that N exists and set $\{o_1, o_2, o_3\} := N \cap S \cap (L_1 \cup L_2 \cup L_3)$ with $o_i \in L_i$. If $R \in \mathcal{B}_i$ and $R \nsubseteq N$, then $o_i \notin R$.*

Proof. Assume $o_i \in R$. Since $\sharp(R \cap N) \leq 1$, we get $3d+2 = w(N)+w(R) \leq d - 2 + m_{o_1} + m_{o_2} + m_{o_3}$, a contradiction. □

Lemma 13. *Assume that N exists and that it contains $B \in \mathcal{B}$. Then $h^1(\mathcal{I}_Z(d)) = 0$.*

Proof. With no loss of generality we may assume $B \in \mathcal{B}_1$. If $o_1 \notin B$, then $N = \langle B \cup L_1 \rangle$ and hence $N \supset L_1$, a contradiction. Since all elements of $\mathcal{B}_1$ contain o_1, part (a) of Remark 13 gives $\mathcal{B}_1 = \{B\}$. We have $h^0(\mathcal{I}_{B \cup L_1 \cup L_2 \cup L_3}(2)) \geq 4$. Let τ be the base locus of $|\mathcal{I}_{B \cup L_1 \cup L_2 \cup L_3}(2)|$. Fix a general $Q \in |\mathcal{I}_{B \cup L_1 \cup L_2 \cup L_3}(2)|$. Since $Q \cap S \setminus N \cap S \cap (L_1 \cup L_2 \cap L_3) \neq \emptyset$ and each element of $\mathcal{B}$ meets $B \cap S \setminus \{o_1\}$ (Remark 13), Remark 11 gives that $\mathrm{Res}_Q(Z)$ satisfies the Segre conditions in degree $d-2$. Hence $h^1(\mathcal{I}_Z(d)) = 0$, unless $S \subset \tau$. Since $N \neq F_h$ for any h (Remark 12), we have $h^0(\mathcal{I}_{N \cup L_1 \cup L_2 \cup L_3}(2)) = 3$ (Lemma 9). Since every quadric hypersurface containing $B \cup L_1$ and a point of $N \setminus B \cup L_1$ contains N, we get $h^0(\mathcal{I}_{B \cup L_1 \cup L_2 \cup L_3}(2)) = 4$. Hence $\tau \cap N = B \cup L_1$. Hence $N \cap S = (L_1 \cap S) \cup (B \cap S)$. We get $m_{o_1} = w(L_1) + w(B) - w(N) = 1$. Since $m_{o_1} + m_{o_2} + m_{o_3} \geq d + 3$, we get a contradiction. □

Lemma 14. *If $\mathcal{B} = \emptyset$, then $h^1(\mathcal{I}_Z(d)) = 0$.*

Proof. By Remarks 10 and 11 we may assume that N exists with $N \cap L_i = \{o_i\} \in S$, $i = 1, 2, 3$, o_1, o_2, o_3, not collinear, and $N \cap J \cap S \subset \{o_1, o_2, o_3\}$. Fix $o \in N \cap S \setminus \{o_1, o_2, o_3\}$ and let τ be the base locus of $|\mathcal{I}_{\{o\} \cup L_1 \cup L_2 \cup L_3}(2)|$. Fix a general $Q \in |\mathcal{I}_{\{o\} \cup L_1 \cup L_2 \cup L_3}(2)|$. We have $w_{\mathrm{Res}_Q(Z)}(N) \le 2d - 3$. Since $\mathcal{B} = \emptyset$, $\mathrm{Res}_Q(Z)$ satisfies the Segre conditions in degree $d - 2$. Hence $h^1(\mathcal{I}_Z(d)) = 0$, unless $S \subset \tau$. If $o \notin F_i$ for any i, then $\tau = \{o\} \cup J \cup L_1 \cup L_2 \cup L_3$ (Lemma 8). Since $\sharp(N \cap S) \ge 5$ (Remark 11), we get $S \nsubseteq \tau$, a contradiction. If $o \in F_i$ for some i, say for $i = 1$, then $\tau = F_1 \cup L_2 \cup L_3$. Hence $S \subset F_1 \cup L_2 \cup L_3$ and hence $N \cap S = (F_1 \cap N \cap S) \cup (L_2 \cap N \cap S) \cup (L_3 \cap N \cap S) = (F_1 \cap N \cap S) \cup \{o_2, o_3\}$. Since $N \ne F_i$ (Remark 12), $F_1 \cap N$ is a point or a line. Since $w(N) = 2d + 1$, we get that $N \cap F_1$ is a line, $N \cap B_1 \in B_1$ and $m_{o_2} + m_{o_3} = d$. Since $\mathcal{B} = \emptyset$, $F_1 \not\supseteq L_2$, $F_1 \not\supseteq L_3$ and J is the only line meeting all lines L_1, L_2, L_3 either $N \cap F_1 = L_1$ or $N \cap F_1 = J$ or $N \cap F_1 \cap S$ contains a point of $L_1 \cap S$ and a point of $L_h \cap J \cap S$ for some $h = 2, 3$ (say $h = 2$). By the definition of N given in Remark 11 in the latter case we have $o_2 = L_2 \cap J \cap S$, while the second case cannot occur. We exclude the case $N \cap F_1 = L_1$, because $N \cap L_1 \cap S = \{o_1\}$. We get $N \cap F_1 = \langle \{o_1, o_2\} \rangle$. Let H be a hyperplane containing $\{o, o_1, o_2, o_3\}$ (i.e. containing $(N \cap F_1) \cup \{o_2, o_3\}$). The scheme $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree $d - 1$ with respect to $\mathbb{P}^4$, all hyperplanes (because $B_3 = \emptyset$), all planes (there is at most one, N, by Lemma and $w_{\mathrm{Res}_H(Z)}(N) \le 2d - 3$) and with respect to $L_1, L_2, L_3, F_1 \cap N$. Fix another $B \in B_1$. If $B = J$, then we use o_2. If $B \subset F_1$, then we use $N \cap F_1$ and Remark 3. In the other case since $S \subset F_1 \cup L_2 \cup L_3$, then $B \cap S = \{u_1, u_2, u_3\}$ with $u_1 \in F_1$, $u_2 \in L_2$ and $u_3 \in L_3$. □

Lemma 15. *Assume that N exists and that either $\mathcal{B} = \emptyset$ or the elements of $\mathcal{B}$ are not through a common point q and hence $\sharp(\mathcal{B}) \ge 3$ and there is a plane A containing all of them (Remark 13). Then $h^1(\mathcal{I}_Z(d)) = 0$.*

Proof. If $\mathcal{B} = \emptyset$, then we use Lemma 14. Assume $\mathcal{B} \ne \emptyset$ (and hence $\sharp(\mathcal{B}) \ge 3$ and the existence of A) and fix $R_1 \in \mathcal{B}$, say $R_1 \in \mathcal{B}_1$. We have $h^0(\mathcal{I}_{R_1 \cup L_1 \cup L_2 \cup L_3}(2)) \ge 4$. Let η be the base locus of the linear system

$$|\mathcal{I}_{R_1 \cup L_1 \cup L_2 \cup L_3}(2)|.$$

Let Q be a general element $|\mathcal{I}_{R_1 \cup L_1 \cup L_2 \cup L_2 \cup L_3}(2)|$. By Remarks 11 and 13 $\mathrm{Res}_Q(Z)$ satisfies the Segre condition in degree $d - 2$ and so $h^1(\mathcal{I}_Z(d)) = 0$, unless $S \subset \eta$. By Lemma 12 we have $R_1 \nsubseteq N$ and hence $A \ne N$ and $N \cap R_1$ is at most one point u_1. If $N \cap R_1 = \emptyset$, then set $m_{u_1} := 0$. Set

$\{e_1\} := R_1 \cap (L_1 \cup L_2 \cup L_3)$. We get $3d+2 = w(N)+w(R \geq d-2+\alpha$, where α is the sum of the multiplicities among the points u_1, e_1, o_1, o_2, o_3 with the convention that if $N \cap R \neq \emptyset$ and $u_1 \in \{e_1, o_1, o_2, o_3\}$, then $\alpha = m_{e_1}+m_{o_1}+m_{o_2}+m_{o_3}$. We first get that $N \cap R \neq \emptyset$, the points u_1, e_1, o_1, o_2, o_3 are distinct and then $m_{e_1} + m_{u_1} + m_{o_1} + m_{o_2} + m_{o_3} \geq 2d + 4$.

(a) Assume $\mathcal{B} = \{R_1, R_2, R_3\}$ with $R_i \cap L_j \neq \emptyset$ if and only if $i = j$. We get points u_h and e_h, $h = 2, 3$, with $m_{e_h} + m_{u_h} + m_{o_1} + m_{o_2} + m_{o_3} \geq 2d+4$. We have $h^0(\mathcal{I}_{A \cup L_1 \cup L_2 \cup L_3}(2)) \geq 3$. Since $\sharp(B \cap R_i) \geq 3$ for all i, we get $R_i \subset \eta$ for all i. Since A contains the 3 distinct lines R_1, R_2, R_3, then $A \subset \eta$. If $A \neq F_h$ for any h, then Lemma 9 gives $h^0(\mathcal{I}_{A \cup L_1 \cup L_2 \cup L_3}(2)) = 3$, a contradiction. Assume for instance $A \subset M := \langle L_1 \cup L_2 \rangle$. The set $M \cup L_3$ is the base locus of $|\mathcal{I}_{M \cup L_3}(2)|$. Since $R_1 \cup L_1 \cup L_2 \cup L_3 \subset M \cup L_3$, we get $S \subset M \cup L_3$. Since $N \cap L_3$ is a single point, o_3 with $m_{o_3} < d$ (Lemma 3) and $m_{o_3} + d + 1 < w(N)$, we have $N \subset M$. By assumption $L_1 \nsubseteq A$. Since $L_1 \cap N = \{o_1\}$, $L_1 \nsubseteq A \cup N$. Since $L_1 \cup A \cup N \subset \tau$, we get $h^0(\mathcal{I}_{B \cup L_1 \cup L_2 \cup L_3}(2)) = h^0(\mathcal{I}_{M \cup L_3}(2)) = 3$, a contradiction.

(b) Now assume that $\sharp(\mathcal{B}) \geq 3$ and that $A \supset L_h$ for some h, say $h = 1$. We have $\mathcal{B} = \mathcal{B}_1$, because each line of A meets L_1 and $L_1 \cap L_j = \emptyset$ if $j = 2, 3$. Fix $B \in \mathcal{B}$. Since $N \not\supseteq L_1$, we have $A \neq N$. We get that $A = \langle B \cup L_1 \rangle$ and that $N \cap A$ is a line T containing o_1. Call β the base locus of $|\mathcal{I}_{B \cup L_1 \cup L_2 \cup L_3}(2)|$. As in part (a) it is sufficient to do the case $S \subset \beta$. If $L_2 \cap A = L_3 \cap A = \emptyset$, then Lemma 6 gives $h^0(\mathcal{I}_{A \cup L_2 \cup L_3}(2)) = 3 < h^0(\mathcal{I}_{R_1 \cup L_1 \cup L_2 \cup L_3}(2)) \geq 4$, a contradiction. Hence either $A \cap L_2 \neq \emptyset$ or $A \cap L_3 \neq \emptyset$, say $A \cap L_2 \neq \emptyset$. Hence $A \subset M := \langle L_1 \cup L_2 \rangle$.

(b1) First assume $L_3 \cap A \neq \emptyset$. Since $L_2 \nsubseteq A$ and $L_3 \nsubseteq A$ all configurations (A, L_2, L_3) with $\langle A \cup L_2 \rangle \cap L_3 = \emptyset$ are projectively equivalent, then $h^0(\mathcal{I}_{A \cup L_2 \cup L_3}(2)) = 5$ and $A \cup L_2 \cup L_3$ is the base locus of $|\mathcal{I}_{A \cup L_2 \cup L_3}(2)|$ (use reducible quadrics containing A and that $\langle A \cup L_2 \rangle \cap \langle A \cup L_3 \rangle = A$). Assume for the moment $A \cap N \cap S \nsubseteq \{o_1, o_2, o_3\}$. Fix $Q' \in |\mathcal{I}_{A \cup L_2 \cup L_3}(2)|$. Since $L_1 \subset A$ and each element of $\mathcal{B}$ is contained in A, our assumption gives that $\mathrm{Res}_{Q'}(Z)$ satisfies the Segre conditions in degree $d - 2$. Hence $h^1(\mathcal{I}_Z(d)) > 0$, unless $Q' \cap S = S$ for all Q', i.e. unless $S \subset A \cup L_2 \cup L_3$. Hence $N \cap S \subset A \cap N \cup \{o_2, o_3\}$. We get that $A \cap N$ is a line $m_{o_2} + m_{o_3} = d$, $(A \cap N) \cap \{o_2, o_3\} = \emptyset$ and that $A \cap N \in B_1$. Since o_2 and o_3 are the only points of $N \cap (L_2 \cup L_3)$, we get that either $A \cap N = L_1$ (not true, because $L_1 \cap N = \{o_1\}$) or $B \in \mathcal{B}_1$ (and we use Lemma 13). Now assume $A \cap N \cap S \subseteq \{o_1, o_2, o_3\}$. Since $N \cap B \cap S \neq \emptyset$, each $B \in \mathcal{B}_1$ contains o_1, contradicting part (a) of Remark 13.

(b2) Now assume $L_3 \cap A = \emptyset$. Set $G := \langle A \cup L_2 \rangle$ and $\{u\} := L_3 \cap G$. In this case all configurations (A, L_2, L_3) with $L_2 \cap L_3 = \emptyset$ are projectively equivalent, $h^0(\mathcal{I}_{A \cup L_2 \cup L_3}(2)) = 4$ and $A \cup \langle \{u\} \cup L_2 \rangle \cup L_3$ is the base locus of $|\mathcal{I}_{A \cup L_2 \cup L_3}(2)|$. Since $h^0(\mathcal{I}_{B \cup L_1 \cup L_2 \cup L_3}(2)) \geq 4$ and $B \cup L_1 \subset A$, we get $\beta = A \cup \langle \{u\} \cup L_3 \rangle \cup L_3$.

Hence each point of $N \cap S$ is contained in one of the linear spaces A, $\langle \{u\} \cup L_2 \rangle$, and L_3. We have $o_3 = N \cap L_3$. Since $m_{o_3} + d + 1 < w(N)$, N contains at least 3 non collinear points of M and hence $N \subset M$. Since $\{u\}$ is the only point of $L_3 \cap M$, we get $u = o_3$ and $N \cap S \subset (A \cap S) \cup (\langle \{o_3\} \cup L_2 \rangle \cap S)$. Since $\sharp(N \cap S) \geq 5$ (Remark 11) and $N \cap S$ spans N, we get that either $A = N$ (excluded by Lemma 13) or $A = \langle \{o_3\} \cup L_2 \rangle$ (excluded because $L_1 \subset A$ and $L_1 \cap L_2 = \emptyset$). □

Lemma 16. *Assume that N exists and that $\mathcal{B} \neq \emptyset$. Then $h^1(\mathcal{I}_Z(d)) = 0$.*

Proof. Fix $B \in \mathcal{B}$, say $B \in \mathcal{B}_1$. We have $h^0(\mathcal{I}_{B \cup L_1 \cup L_2 \cup L_3}(2)) \geq 4$. Let η be the base locus of the linear system $|\mathcal{I}_{B \cup L_1 \cup L_2 \cup L_3}(2)|$. Let Q be a general element $|\mathcal{I}_{B \cup L_1 \cup L_2 \cup L_2 \cup L_3}(2)|$. By Remarks 11 and 13 $\mathrm{Res}_Q(Z)$ satisfies the Segre condition in degree $d-2$ and so $h^1(\mathcal{I}_Z(d)) = 0$, unless $S \subset \eta$. Hence we may assume $S \subset \eta$. Set $E := \langle L_1 \cup B \rangle$. Since $L_h \cap L_1 = \emptyset$, $h = 1, 2$, each $\langle E \cup L_h \rangle$, is a hyperplane. Assume $\langle E \cup L_2 \rangle = \langle E \cup L_3 \rangle$. Since $\langle L_2 \cup L_3 \rangle$ is a hyperplane, we get $L_1 \subset \langle E \cup L_2 \rangle \subset \langle L_2 \cup L_3 \rangle$. Hence $w(\langle L_2 \cup L_3 \rangle) \geq 3d+3$, a contradiction. Therefore there is an index $h \in \{2, 3\}$, say $h = 3$, such that $L_3 \nsubseteq \langle E \cup L_2 \rangle$. Taking reducible quadrics $\langle E \cup L_2 \rangle \cup M$ with M a hyperplane containing L_3 we get that the base locus of $|\mathcal{I}_{E \cup L_2 \cup L_3}(2)|$ is contained in $\langle E \cup L_2 \rangle \cup L_3$. Since $B \cup L_1 \subset E$, we get $\tau \subset \langle E \cup L_2 \rangle \cup L_3$. Hence $N \cap S \subseteq (N \cap \langle E \cup L_2 \rangle \cap S) \cup \{o_3\}$. Since $m_{o_3} \leq d - 1$ (Lemma 3), $w(N) = 2d + 1$ and $w(L) \leq d + 1$ for every line L, we get $N \subset \langle E \cup L_2 \rangle$. We have $E \neq N$, because $N \cap L_1 = \{o_1\}$. If $L_2 \nsubseteq \langle E \cup L_3 \rangle$, we also get $N \subseteq \langle E \cup L_2 \rangle \cap \langle E \cup L_3 \rangle$. Since $L_1 \subset E$ and $\langle L_1 \cup L_2 \cup L_3 \rangle = \mathbb{P}^4$, we first get $\langle E \cup L_2 \rangle \neq \langle E \cup L_3 \rangle$ and then $N \subseteq E$, a contradiction. □

Lemma 17. *If $\mathcal{B} \neq \emptyset$, then $h^1(\mathcal{I}_Z(d)) = 0$.*

Proof. By Lemma 16 we may assume that N does not exists. Fix $B \in \mathcal{B}$, say $B \in \mathcal{B}_1$. We have $h^0(\mathcal{I}_{B \cup L_1 \cup L_2 \cup L_3}(2)) \geq 4$. Let η be the base locus of the linear system $|\mathcal{I}_{B \cup L_1 \cup L_2 \cup L_3}(2)|$. Let Q be a general element $|\mathcal{I}_{B \cup L_1 \cup L_2 \cup L_2 \cup L_3}(2)|$. By Remarks 11 and 13 $\mathrm{Res}_Q(Z)$ satisfies the Segre condition in degree $d-2$ and so $h^1(\mathcal{I}_Z(d)) = 0$, unless $S \subset \eta$. Hence we may assume $S \subset \eta$.

(a) Assume the existence of a point $q \in S$ such that each $R \in \mathcal{B}$ contains q; if $\sharp(\mathcal{B}) = 1$ we also assume $q \notin J \cup L_1 \cup L_2 \cup L_3$; we may do this by Lemma 11, because $J \notin \mathcal{B}$. Let β be the base locus of $|\mathcal{I}_{\{q\} \cup L_1 \cup L_2 \cup L_3}(2)|$. Take $Q' \in |\mathcal{I}_{\{q\} \cup L_1 \cup L_2 \cup L_3}(2)|$. Since N does not exists, $\mathrm{Res}_{Q'}(Z)$ satisfies the Segre conditions in degree $d-2$ (Remark 10). Hence $h^1(\mathcal{I}_Z(d)) = 0$, unless $S \subset \beta$. If $q \in J \cup L_1 \cup L_2 \cup L_3$, then we use Lemma 11. If $q \notin F_h$ for all h, then

$\beta = \{q\} \cup L_1 \cup L_2 \cup L_3$; hence $\sharp(R \cap S) \le 2$ for all $R \in \mathcal{B}$, a contradiction. Now assume $q \in F_h \setminus (J \cup L_1 \cup L_2 \cup L_3)$ for some h, say $h = 1$. Since $\mathcal{I}_{F_1 \cup L_2 \cup L_3}(2)$ is spanned (use reducible quadrics $\langle F_1 \cup L_2 \rangle \cup M_3$ and $\langle F_1 \cup L_3 \rangle \cup M_2$ with M_i a hyperplane containing L_i), we get $\beta \subset F_1 \cup L_2 \cup L_3$. Hence for each $B \in \mathcal{B}$ we get $B \cap S = (B \cap F_1 \cap S) \cup (B \cap L_2 \cap S) \cup (B \cap L_3 \cap S)$; assume for the moment $R \nsubseteq F_1$ and hence $\sharp(R \cap F_1 \cap S) \le 1$; since $\sharp(B \cap L_2) + \sharp(B \cap L_3) \le 1$, we get $\sharp(R \cap S) \le 2$, a contradiction. Thus each element of $\mathcal{B}$ is contained in F_1. Hence it meets L_1. By Remark 3 we get $\mathcal{B} = \mathcal{B}_1$.

Let Ψ be the set of all $R \in B_1$ such that $R \neq J$, $R \cap L_2 \neq \emptyset$ and $R \cap L_3 \neq \emptyset$. By Remark 3 for each $R \in \Psi$ we have $R \cap L_2 \in S$ and $R \cap L_3 \in S$.

Claim : We have $\sharp(\Psi) \le 2$.

Proof of the Claim: Assume $\sharp(\Psi) \ge 3$ and fix 3 distinct elements $R_1, R_2, R_3 \in \Psi$. If $R, T \in \Psi$ with $R \neq T$ and $R \cap T \neq \emptyset$, then the point $R \cap T$ is contained in $L_2 \cup L_3$, because the plane $\langle R \cup T \rangle$ cannot contain $L_2 \cup L_3$. Since $w(\mathbb{P}^4 \setminus (L_1 \cup L_2 \cup L_3)) \le d - 2$, we get $3d + 3 = w(R_1) + w(R_2) + w(R_3) \ge d - 2 + w(L_2) + w(L_3)$, a contradiction.

Since $\sharp(\Psi) \le 2$ and each $R \in \Psi$ contains a point of $L_2 \cap S$ and a point of $L_3 \cap S$, we may find $u \in L_2 \cap S$ and $v \in L_3 \cap S$ such that each element of Ψ meets $\{u, v\}$. Let H be a hyperplane containing $L_1 \cup \{u, v\}$. It is sufficient to prove that $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree $d - 1$. It satisfies it with respect to $\mathbb{P}^4$ and, since $B_3 = \emptyset$, all hyperplanes. To check the condition for lines, it is sufficient to check it for all $R \in B_1$. This condition is satisfied by all $R \in \mathcal{B}$, because $\mathcal{B} = \mathcal{B}_1$. It is satisfied by L_1, L_2, L_3 and by all $R \in B_1$ meeting L_1. By the definition of Ψ, u, v it is satisfied by all $R \in B_1$, with $R \notin \{L_2, L_3\}$ and $R \cap L_1 = \emptyset$. Assume $B_2 \neq \emptyset$, say $B_2 = \{F\}$. Since N does not exists, we have $\sharp(S \cap F \cap (L_1 \cup L_2 \cup L_3 \cup J)) \ge 4$.

(a1) Assume for the moment $\sharp(S \cap F \cap (L_1 \cup L_2 \cup L_3)) \ge 4$. Hence $L_j \subset F$ for some j. If $j = 1$, then $\sharp(F \cap H \cap S) \ge 2$. Assume $j \neq 1$, say $j = 2$. Since $u \in F \cap H \cap S$, to check the Segre condition for F we may assume $w(F) = 2d + 1$ and $F \cap L_1 \cap S = \emptyset$. Since $w(F) + w(L_1) > 3d + 1$, we get $F \cap L_1 = \emptyset$. Hence $F \cap F_1$ is a unique point, e. Since $\langle L_2 \cup L_3 \rangle \not\supseteq F$, $F \cap L_3$, is at most one point, o; set $m_o = 0$ if $F \cap L_3 = \emptyset$. Since $S \subset F_1 \cup L_2 \cup L_3$, we get $d = w(F) - w(L_2) = m_e + m_o$. By Lemma 3 we get $o \in S$, $F \cap L_3 \neq \emptyset$, $F \cap L_3 \in S$ and $m_o + m_e = d$. If $e = v$, then $w_{\mathrm{Res}_H(Z)}(F) \le 2d - 1$. Hence we may assume that we cannot take $v = e$. In particular (since $\sharp(B_2) = 1$) we may assume $\sharp(\Psi) = 2$, say $\Psi = \{R_1, R_2\}$. Since $o \notin L_1 \cup L_2 \cup L_3$ and $w(\mathbb{P}^4 \setminus (L_1 \cup L_2 \cup L_3)) \le d - 2$, we get $d - 2 - m_o \ge w(R_1) + w(R_2) - w(L_2 \cap (R_1 \cup R_2)) - w(L_3 \cap (R_1 \cup R_2))$. Since $\sharp(L_2 \cap S) \ge 3$, we have $w(L_2 \cap (R_1 \cup R_2)) \le d$. Since $e \notin (R_1 \cup R_2)$, we get $w(L_3 \cap (R_1 \cup R_2)) \le d + 1 - m_e$. Hence $d - 2 - m_o \ge 2d + 2 - d - d - 1 + m_e$, contradicting the equality $m_e + m_o = d$.

(a2) Now assume $\sharp(S \cap F \cap (L_1 \cup L_2 \cup L_3)) \le 3$. First assume $J \subset F$. Since $w(F) - w(J) \ge d-1 > w(\mathbb{P}^4 \setminus (L_1 \cup L_2 \cup L-3))$ there is $i \in \{1,2,3\}$ and $o_i \in L_i \cap S \cap F$ with $o_i \in F_i$. We get $F = F_i$. By Remark 12 we have $F \cap L_i \in S$ for all i. Since $\sharp(S \cap F \cap (L_1 \cup L_2 \cup L_3)) \le 3$, we get $S \cap F \cap (J \cup L_1 \cup L_2 \cup L_3)) = S \cap J$ and hence $w(F) \le d+1$, a contradiction. Now assume $J \nsubseteq F$. Since J is the only line containing 3 points of $L_1 \cup L_2 \cup L_3$, we get that $\sharp(S \cap F \cap (L_1 \cup L_2 \cup L_3)) = 3$, $F = \langle S \cap (L_1 \cup L_2 \cup L_3)\rangle$ and $F \cap J$ is a point of $S \cap J \setminus S \cap J \cap (L_1 \cup L_2 \cup L_3)$. Hence each $F \cap F_i$ is a line not containing the point $J \cap L_i$. We get that $F_1 \cup F_2 \cup F_3$ is contained in the base locus of $|\mathcal{I}_{F \cup L_1 \cup L_2 \cup L_3}(2)|$. Since $|\mathcal{I}_{J \cup L_1 \cup L_2 \cup L_3}(2)| = |\mathcal{I}_{L_1 \cup L_2 \cup L_3}(2)|$, $\sharp(F \cap S \cap (J \cup L_1 \cup L_2 \cup L_3)) \ge 4$ and $F \cap (J \cup L_1 \cup L_2 \cup L_3)$ is not formed by 4 collinear points, we have $h^0(\mathcal{I}_{F \cup L_1 \cup L_2 \cup L_3}(2)) \ge 4 > h^0(\mathcal{I}_{F_1 \cup F_2 \cup F_3}(2))$, a contradiction.

(b) Assume that there is no point q contained in all $R \in \mathcal{B}$. By Remark 13 we have $\sharp(\mathcal{B}) \ge 3$ and there is a plane A containing all elements of $\mathcal{B}$. Since A contains 3 distinct elements of B_1 and $\sharp(R \cap S) \ge 3$ for all $R \in B_1$, we have $A \subset \tau$. Since $B \subset A$, we get $h^0(\mathcal{I}_{A \cup L_1 \cup L_2 \cup L_3}(2)) = h^0(\mathcal{I}_{B \cup L_1 \cup L_2 \cup L_3}(2)) \ge 4$. If $A \ne F_i$ for all i and $L_i \nsubseteq A_i$ for all i, then we get $\sharp(L_i \cap A) = 1$ for all i and and hence $h^0(\mathcal{I}_{A \cup L_1 \cup L_2 \cup L_3}(2)) = 3$ (Lemma 9). Since $B \cup L_1 \subset A$, and $h^0(\mathcal{I}_{B \cup L_1 \cup L_2 \cup L_3}(2)) \ge 4$, we get a contradiction. Now assume $L_j \subset A$ for some j. Since $\mathcal{B}_1 \ne \emptyset$, we get $j = 1$ and $\mathcal{B}_2 = \mathcal{B}_3 = \emptyset$. If $A \cap L_2 = A \cap L_3 = \emptyset$, then $h^0(\mathcal{I}_{A \cup L_2 \cup L_3}(2)) = 3$ (Lemma 6) and again we get a contradiction. Hence we may assume $A \cap L_2 \ne \emptyset$, i.e. $A \subset \langle L_1 \cup L_2\rangle$.

(b1) First assume $L_3 \cap A = \emptyset$, i.e. $A \ne F_1$. For a general $v \in A$ we have $h^0(\mathcal{I}_{\{v\} \cup L_1 \cup L_2 \cup L_3}(2)) = 5$ and $\{v\} \cup J \cup L_1 \cup L_2 \cup L_3$ is the base locus of $|\mathcal{I}_{\{v\} \cup L_1 \cup L_2 \cup L_3}(2)|$ (Lemma 11). Hence $h^0(\mathcal{I}_{A \cup L_2 \cup L_3}(2)) \le 4$. We get that equality holds; write $w := \langle A \cup L_2\rangle \cap L_3$ and $A' := \langle L_2 \cup \{w\}\rangle$. The set $A \cup A' \cup L_3$ is the base locus of $|\mathcal{I}_{A \cup L_2 \cup L_3}(2)|$. Hence $S = (A \cap S) \cup (A' \cap S) \cup (L_3 \cap S)$ with $A \cup A' \subset \langle L_1 \cup L_2\rangle$. Take $u \in L_2 \cap S$ and $v \in L_3 \cap S$ and call H a hyperplane containing $L_1 \cup \{u, v\}$. It is sufficient to check that $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree $d-1$ and this is done as in step (a), except for the planes. Hence we may assume $B_2 \ne \emptyset$ and call E the only element of B_2. Since N does not exists, we get $\sharp(E \cap (L_1 \cup L_2 \cup L_3)) \ge 4$ and hence E contains one of the lines L_i (and exactly one such line). Hence $E \cap H \cap S \ne \emptyset$ for any choice of u, v and so we may assume $w(E) = 2d+1$. If $L_1 \subset E$, then $\sharp(E \cap H \cap S) \ge 3$ and hence $w_{\mathrm{Res}_H(Z)}(E) \le 2d-2$.

(b1.1) Now assume $L_2 \subset E$. Since $L_1 \cup \{u\} \subset H \cap S$, to prove the inequality $w_{\mathrm{Res}_H(Z)}(E) \le 2d-1$, we may assume $E \cap L_1 \cap S = \emptyset$. Since

$w(E) + w(L_1) > 3d + 1$, the Segre conditions of Z give $E \cap L_1 = \emptyset$. Since $L_2 \cap L_1 = \emptyset$, we get $E \neq A$. Assume $E = A'$. Since $A' \subset \langle L_1 \cup L_2 \rangle$, we have $E \cap L_1 \neq \emptyset$. Since $w(E) + w(L_1) > 3d + 1$, we get $E \cap L_1 \in S$ and hence $w_{\mathrm{Res}_H(Z)}(E) \leq 2d - 1$. Thus we may assume $E \neq A'$.Since $L_2 \cap L_3 = \emptyset$, the set $E \cap L_3$ is either empty or a single point e; in the latter case we set $m_e = 0$.

(b1.1.1) Assume $m_e = 0$. Since $E \cap S$ spans S (Remark 2), we get $E \subset \langle L_1 \cup L_2 \rangle$. Hence $E \cap L_1 \neq \emptyset$, a contradiction.

(b1.1.2) Assume $m_e > 0$. If $e \neq w$, then $\langle L_1 \cup L_2 \rangle \cap E = L_2$. Since $S \subset \langle L_1 \cup L_2 \rangle \cup L_3$, we get $S \cap E = \{e\} \cup (L_2 \cap S)$ and hence $m_e = d$; we conclude by Lemma 3.

(b1.2) Now assume $L_3 \subset E$. We get $w(E \setminus L_3) = d$. The set $D := E \cap \langle L_1 \cup L_2 \rangle$ is a line containing w and $w(D \setminus \{w\}) = d$. Since $A \cap L_3 = \emptyset$, we have $e \notin A$ and hence $D \cap A$ is a single point, α. If $D \nsubseteq A'$ we get $D \cap A' = \{w\}$ and hence $m_\alpha = d$; we conclude by Lemma 3. Now assume $D \subset A'$. Call β the only point of $D \cap L_3$. Fix $R \in \mathcal{B}_1$ and call γ the only point of $R \cap L_1$. Since $w(\mathbb{P}^4 \setminus (L_1 \cup L_2 \cup L_3)) \leq d - 2$, $w(R) = d + 1$ and $w(D \setminus \{w\}) = d$, we get $d - 2 \geq d + 1 - m_\gamma + d - m_\beta$ and hence $m_\beta + m_\gamma \geq d + 3$, a contradiction.

(b2) Now assume $A = F_1$. Since $F_1 \cup L_2 \cup L_3$ is the base locus of $|\mathcal{I}_{F_1 \cup L_2 \cup L_3}(2)|$, we get $S \subset F_1 \cup L_2 \cup L_3$. We get a contradiction as in step (a). □

7. Proof of Lemma 1

In this section we assume $B_3 = \emptyset$ and conclude the proof of Lemma 1.

(a) Assume the existence of $L, D \in B_1$ such that $L \cap D = \emptyset$. By section 6 and Remark 3 we may assume $S \cap R \cap (L \cup D) \neq \emptyset$ for any $R \in B_1$. Set $H := \langle D \cup L \rangle$. By assumption $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree $d-1$ with respect to lines. It satisfies it with respect to $\mathbb{P}^4$ (since $\sharp(H \cap S) \geq 4$) and with respect to the hyperplanes (since $B_3 = \emptyset$). If $B_2 = \emptyset$, we are done. Hence we may assume the existence of $N \in B_2$. By Lemma 7 N is unique. Since $w(L)+w(D)+w(N) > 4d+1$, we have $N \cap S \cap (L \cup D) \neq \emptyset$. Hence $h^1(\mathcal{I}_Z(d)) = 0$, unless $w(N) = 2d+1$ and $\sharp(N \cap S \cap (L \cup D)) = 1$, say $N \cap S \cap L$ is a single point, o, and $N \cap S \cap D = \emptyset$. Let Δ be the set of all line $B \in B_1$ with $B \cap D = \emptyset$. Fix $q \in N \cap S \setminus \{o\}$ and set $H_q := \langle \{o, q\} \cup D \rangle$. The scheme $\mathrm{Res}_{H_q}(Z)$ satisfies the Segre conditions in degree $d-1$ with respect to $\mathbb{P}^4$, all hyperplanes, N and hence with respect to all planes since $\{N\} = B_2$, and with respect to all lines, except the elements of Δ not containing either o or q. Hence $h^1(\mathcal{I}_Z(d)) = 0$, unless $\sharp(\Delta) \geq 3$ and not all elements of Δ are through the same point. Since any two elements of B_1 disjoint from D meet, we get the existence of a plane A containing all elements of Δ. Since $N \cap D = \emptyset$, we saw that each element of Δ meets $N \cap S$. Since we assumed that not all elements of Δ pass through the same point, we get that either $A = N$ (excluded, because $L \subset A$) or $A \cap N$ is a line. Note that each $R \in \Delta$ meets $(A \cap N) \cap S$ and that $\sharp(A \cap N \cap S) \geq 2$ by our assumptions on Δ (not all through a common point). Set $U := \langle (A \cap N) \cup D \rangle$. The scheme $\mathrm{Res}_U(Z)$ satisfies the Segre conditions in degree $d-1$ with respect to $\mathbb{P}^4$, all hyperplanes, N (because $\sharp(A \cap N \cap S) \geq 2$) and hence with respect to all planes since $\{N\} = B_2$, with respect to all elements of Δ (they contain of $A \cap N \cap S$) and of $B_1 \setminus \Delta$ (they intersects $D \cap S$), concluding this case.

(b) Assume $B_1 \neq \emptyset$ and that any two elements of B_1 meets. Fix $L \in B_1$. If there is $N \in B_2$ take two points of $p, q \in N \cap S$. Otherwise take any two elements of p, q of $S \setminus \{p, q\}$. Let H be a hyperplane containing $L \cup \{p, q\}$. The scheme $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree $d-1$ with respect to $\mathbb{P}^4$ (since $\sharp(H \cap S) \geq 4$) all hyperplanes (since $B_3 = \emptyset$), all planes (since either $B_2 = \emptyset$ or the only element of B_2 has two points of $H \cap S$) and all lines (since each element of B_1 meets L).

(c) Assume $B_3 = B_1 = \emptyset$. Let H be a hyperplane containing 4 points of S with the only restriction that if $B_2 \neq \emptyset$, say $B_2 = \{N\}$, then H contains two points of $N \cap S$. The scheme $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree $d-1$ with respect to $\mathbb{P}^4$ (since $\sharp(H \cap S) \geq 4$) all hyperplanes (since $B_3 = \emptyset$), all planes (since either $B_2 = \emptyset$ or the only element of B_2 has two points of $H \cap S$) and all lines (since $B_1 = \emptyset$).

Parts (a), (b) and (c) conclude the proof of Lemma 1.

8. $d = 4, 5$

Proposition 2. *If Z satisfies the Segre conditions in degree 4, then $h^1(\mathcal{I}_Z(4)) = 0$.*

Proof. By Lemma 3 we may assume $m_1 = 3$. Set $S_i := \{p \in S : m_p = i\}$, $i = 1, 2, 3$. Set $\{o\} := S_3$, $h := \sharp(S_2)$ and $c := \sharp(S_1)$. Since $w(\mathbb{P}^4) \leq 17$, we have

$$2h + c \leq 14 \tag{4}$$

The elements of B_1 are the lines through o containing a point of S_2 (there are h of them, they meet only at o and, since they have weight 5, no 4 of them are in a plane and they are disjoint from S_1), the ones (call B_- the set of these lines) containing o and two points of S_1 (they exists only if $c \geq 2$, they are disjoint from S_2), the ones (call B_+ the set of these lines) formed by 5 collinear points of S_3 (they exists only if $c \geq 5$ and they are disjoint from $S_2 \cup \{o\}$), the ones (call them $B_=$) containing 2 points of S_2 and one point of S_1, and the ones (call $B_\vee$ the set of these lines) containing a point of S_2 and 3 points of S_1. We call *starred* (resp. *bistarred*) a point of $S_1 \cup S_2$ contained in a line $L \in B_-$ (resp. $L \in B_\vee$).

We assume that Proposition 2 fails and we take a counterexample S, Z with minimal weight. By [2] we may assume $h + c \geq 7$. Let $\ell : \mathbb{P}^4 \setminus \{o\} \to \mathbb{P}^3$ be the linear projection from o. Set $S'_i := \ell(S_i)$, $i = 1, 2$, and $S' := S'_1 \cup S'_2$. We have $S'_1 \cap S'_i = \emptyset$ and $\sharp(S') = h + c - \sharp(B_-)$. We say that a point of S'_i has multiplicity i. We say that $p \in S'_1$ is a *marked point* if $\ell^{-1} \cap (S_2 \cup S_1)$ is not a point, i.e. if it is the image of a line in B_-; in this case the fiber contains no element of S_2 and two elements of S_1.

Claim 1: If $h^1(\mathbb{P}^3, \mathcal{I}_{S'}(2)) = 0$, then $h^1(\mathcal{I}_Z(4)) = 0$.

Proof of Claim 1: First assume $c > 0$ and that not all points of S'_1 are marked points. Fix $q' \in S'_1$, q' not marked, and call q the point of S_1 with $\ell(q) = q'$. Since $h^1(\mathbb{P}^3, \mathcal{I}_{S'}(2)) = 0$, there is quadric surface $Q' \subset \mathbb{P}^3$ with $Q' \cap S' = S' \setminus \{q\}$. Let $Q \subset \mathbb{P}^4$ be the quadric cone with vertex o and Q' as a basis. We have $Q \cap S \neq S$ and $\mathrm{Res}_Q(Z) \subseteq \{o\} \cup S_2 \cup \{q\}$. Hence it is sufficient to prove that $h^1(\mathcal{I}_{\{o\} \cup S_2 \cup \{q\}}(2)) = 0$. The Segre conditions for Z gives that no plane through o contains 4 points of $S_2 \cup \{q\}$, that no plane contains 6 points of $S_2 \cup \{q\}$ and that no line contains 3 points of $\{o\} \cup S_2 \cup \{q\}$. Hence $h^1(\mathcal{I}_{\{o\} \cup S_2 \cup \{q\}}(2)) = 0$. Now assume $c > 0$ (and hence $h \leq 6$), but that each point of S'_1 is marked. Fix $q'' \in S'_1$ and call q, q' the point of S_1 with q'' as their image. Take a quadric surface $Q' \subset \mathbb{P}^3$ with $Q' \cap S' = S' \setminus \{q''\}$. Let $Q \subset \mathbb{P}^4$ be the quadric cone with vertex o and Q' as a basis. We have $Q \cap S \neq S$ and $\mathrm{Res}_Q(Z) \subseteq \{o\} \cup S_2 \cup \{q, q'\}$. Hence it is sufficient to prove that $h^1(\mathcal{I}_{\{o\} \cup S_2 \cup \{q,q'\}}(2)) = 0$. Since q'' is marked, there is a line R with $R \cap S = \{o, q, q'\}$. The Segre conditions give that $\{o\} \cup S_2 \cup \{q, q'\}$ has no 4-secant line, R is its unique 3-secant line and that if A is a plane containing at least 5 points of $\{o\} \cup S_2 \cup \{q, q'\}$, then either $A \supset R$ and A contains $\{o, q, q'\}$ and two points of S_2 or it contains 4 points of S_2 and only one among the points q, q'. Hence $h^1(\mathcal{I}_{\{o\} \cup S_2 \cup \{q,q'\}}(2)) = 0$. Now assume $c = 0$. Fix $q \in S_2$ and set $E := S_2 \setminus \{q\}$. Taking a quadric surface Q' with $Q' \cap S' = S' \setminus \{\ell(q)\}$ we reduce to prove that $h^1(\mathcal{I}_{\{o\} \cup E \cup 2q}(2)) = 0$. Let $\mu : \mathbb{P}^4 \setminus \{q\} \to \mathbb{P}^3$ be the linear projection from q. Note that $\mu|S \setminus \{q\}$ is injective. Hence it is sufficient to prove that $h^1(\mathbb{P}^3, \mathcal{I}_{\mu(\{o\} \cup E)}(2)) = 0$. We have $\sharp(\mu(\{o\} \cup E)) = h \leq 7$. By the Segre condition no line contains 4 points of $\mu(\{o\} \cup E)$ and (since $w(U) \leq 13$ for every hyperplane U) no plane contains ≥ 6 points of $\mu(\{o\} \cup E)$. Hence

$h^1(\mathbb{P}^3, \mathcal{I}_{\mu(\{o\}\cup E)}(2)) = 0$.

(a) By Claim 1, to check the case $h = 7$ it is sufficient to observe that in the case $c = 0$ for each plane $A \subset \mathbb{P}^3$ we have $\sharp(A \cap S_2) \le 5$ by the Segre conditions for Z. In a similar way we do the case $(h, c) = (6, 1)$. Now assume $(h, c) = (6, 2)$. If S' has a marked point, then $\sharp(S') = 6$ and again $\sharp(A \cap S') \le 5$ even for the planes $A \subset \mathbb{P}^3$ containing the marked point. Now assume that S' has no marked point. Assume the existence of a plane $A \subset \mathbb{P}^3$ such that $\sharp(A \cap S') \ge 6$. Let $H \subset \mathbb{P}^4$ be the hyperplane containing o and with $\ell(H \setminus \{o\}) = A$. We have $\mathrm{Res}_H(Z) = \{o\} \cup S_2$ and $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(3)) = 0$, because $\{o\} \cup S_2$ has cardinality 8, no 4-secant line, and it is not in a plane.

(b) Assume $h = 5$ and hence $c \le 4$. Therefore $B_+ = \emptyset$. The cases with $c \le 2$ are done as in the case $h = 6$. We assume $c = 4$, because the proof for $c = 3$ is very similar, but easier.

(b1) Assume $h^1(\mathcal{I}_S(2)) = 1$. Since $h^1(\mathcal{I}_{\{o\}\cup S_2}(2)) = 0$ (by the Segre condition for Z) we get the existence of a quadric Q such that $\{o\} \cup S_2 \subset Q$ and $2 \le \sharp(Q \cap S_3) \le 3$. The scheme $\mathrm{Res}_Q(Z)$ is contained in the scheme W union of $2o$, S_2 and 2 points of S_3. We have $h^1(\mathcal{I}_Z(4)) \le h^1(\mathcal{I}_W(2))$. Set $E := W \setminus 2o$. We have $h^1(\mathcal{I}_W(2)) = h^1(\mathbb{P}^3, \mathcal{I}_{\ell(E)}(2))$. Assume the existence of a plane containing all points of $\ell(E)$. We get the existence of a hyperplane U containing o, S_2 and 2 points of S_1. The scheme $\mathrm{Res}_U(Z)$ satisfies the Segre condition in degree 3 with respect to $\mathbb{P}^4$, because $\sharp(U \cap S) \ge 4$, and with respect to all lines, because $B_+ = \emptyset$ and hence every element of B_1 meets $S_2 \cup S_3 \subseteq U \cap S$. Fix $B \in B_2$. Since $w(B) \ge 8$, B contains at least 2 points of $U \cap S$ and hence $w_{\mathrm{Res}_U(Z)}(B) \le 7$. Fix $B \in B_3$. Since $w(B) \ge 11$, B contains at least 2 points of $U \cap S$ and hence $w_{\mathrm{Res}_U(Z)}(B) \le 9$.

Hence $h^1(\mathcal{I}_W(2)) = 0$, unless there a line containing 4 points of $\ell(E)$. If this line exists, then we get a plane A with $w(A) \ge 3+2+2+1+1$ and hence $A \cap S$ contains exactly 2 points of S_1, 2 points of S_2 and $\{o\}$. Fix $p \in S_2 \setminus S_2 \cap A$ and set $H := \langle \{p\} \cup A \rangle$. The scheme $\mathrm{Res}_H(Z)$ satisfies the Segre condition in degree 3 with respect to $\mathbb{P}^4$, because $\sharp(H \cap S) \ge 4$. It satisfies the Segre conditions with respect to all lines, because $S_2 \cup S_3 \subset H$ and $B_+ = \emptyset$. Since $w(\mathbb{P}^4 \setminus H) \le 6$, each $B \in B_3$ contains at least 2 points of $H \cap S$ and hence to check the Segre condition for U in degree 3 we may quote Lemma 4. Fix $B \in B_2$. Since $w(\mathbb{P}^4 \setminus H) \le 6$, we have $B \cap H \cap S \neq \emptyset$ and hence to check the Segre in degree 3 we may assume $w(B) = 9$; since $w(\mathbb{P}^4 \setminus H) \le 6$, we get $\sharp(B \cap H \cap S) \ge 2$, unless $w(\mathbb{P}^4 \setminus H) = 6$ (and hence $H \cap S = \{p\} \cup (A \cap S)$) and $\{o\} = A \cap B$. For each $q \in S_3 \cap B$ set $H_q := \langle \{q\} \cup A \rangle$. We get $h^1(\mathcal{I}_{\mathrm{Res}_{H_q}(Z)}(2)) = 0$ and hence $h^1(\mathcal{I}_Z(2)) = 0$, unless there is a plane $B_q \supset \{p\} \cup (B \cap S \setminus \{q\})$. If B_q exists, then $B_q \cap B$ is a line containing o, at least another element of S_2 and 2 points of S_3, contradicting the Segre conditions for Z.

(b2) Assume $h^1(\mathcal{I}_S(2)) = 0$. As in step (b1) we get the existence of a quadric Q such that $S \setminus Q \cap S$ is a point of S_1. Working as in step (b1) we quickly get a contradiction.

(b3) Assume $h^1(\mathcal{I}_S(2)) \geq 2$.

(b3.1) Assume the existence of a line L with $\sharp(S \cap L) \geq 4$. Since $B_+ = \emptyset$, we get $\sharp(S \cap L) = 4$ and that either $S_1 \subset L$ or L contains one point of S_2 and 3 points of S_1. In all case L is spanned by 3 collinear points of S_1 and hence it is unique. Take $q \in L \cap S$ and set $T := S \setminus \{q\}$. Let H be a plane containing L and with $a := \sharp(A \cap H)$ maximal. By Lemma 3 we may assume $a \leq 8$.

If $a = 6$, then $S \setminus S \cap L$ has no 3-secant line and hence $h^1(H, \mathcal{I}_{H \cap S}(2)) = 1$. The residual exact sequence of H gives $h^1(\mathcal{I}_{S \setminus S \cap H}(1)) > 0$ and hence the 4 points of $S \setminus S \cap H$ span a plan A. Since $a = 6$, we have $A \cap L = \emptyset$. Since $\sharp(T) = 9$, T satisfies the Segre condition in degree 2. Fix a plane E. If $E \supset L$, then $\sharp(E \cap T) = -1 + \sharp(E \cap S) \leq 4$. If $E \not\supseteq L$, we get that either $E = A$ and hence $\sharp(E \cap T) = 3$ or $\sharp(T \cap A) \leq 5$, because $\sharp(S \cap A \cap E) \leq 2$. Fix a hyperplane U. If $U \supset L$, then we get $\sharp(U \cap T) \leq 5$ by the definition of a. Take $U \not\supseteq L$ and assume $\sharp(U \cap T) \geq 8$. Since U contains at most one point of L and all but one of the points of $S \setminus S \cap L$ have multiplicity > 1, we get $w(U) \geq 8 + 6$, contradicting the Segre condition for Z. Hence $h^1(\mathcal{I}_{S \setminus \{q\}}(2)) = 0$ and hence $h^1(\mathcal{I}_S(2)) \leq 1$, a contradiction.

Now assume $a = 8$. Since $\sharp(S \setminus S \cap H) = 2$, we have $h^1(\mathcal{I}_{S \setminus S \cap H}(1)) = 0$. Hence $h^1(H, \mathcal{I}_{T \cap H}(2)) > 0$. Since $\sharp(T \cap H) = 9$ and T has no 4-secant line we get the existence of a plane $G \subset H$ with $\sharp(G \cap T) \geq 6$ and hence $\sharp(G \cap S) \geq 7$. Since $c = 4$, we get $w(G) \geq 10$, a contradiction.

Now assume $a = 7$. First assume $h^1(\mathcal{I}_{S \setminus S \cap H}(1)) = 0$. As in the case $a = 8$ we first get $h^1(H, \mathcal{I}_{T \cap H}(2)) = 0$ and then a contradiction. Now assume $h^1(\mathcal{I}_{S \setminus S \cap H}(1)) > 0$, i.e. assume the existence of a line $R \supset S \setminus S \cap H$. Taking $\langle L \cup R \rangle$ instead of H we conclude, unless there is a line D containing the 3 points of $S \cap H \setminus S \cap L$. Since $a = 7$, we have $D \cap L = \emptyset = D \cap R$. Since $h^1(\mathcal{I}_{D \cup L \cup R}(2)) = 0$, we get $h^1(\mathcal{I}_S(2)) = h^1(D \cup R \cup L, \mathcal{I}_S(2)) = 1$, a contradiction.

(b3.2) Now assume the non-existence of a quadrisecant line of S. Let E be a plane such that $b := \sharp(E \cap S)$ is maximal. Let M be a hyperplane with $a := \sharp(M \cap S)$ maximal. By Lemma 3 we may assume $a \leq 8$.

First assume $b \geq 7$. Since $c = 4$, we get $w(E) \geq 10$, a contradiction.

Now assume $b = 6$. Since $w(E) \leq 9$, we get that E contains 3 points of S_1 and 3 points of S_2 or 4 points of S_1 and two points of $S_2 \cup S_3$. If $o \notin E$, let H be a hyperplane containing $\{o\} \cup E$. If $o \in E$ let H be a hyperplane containing E and another point of S_2. Since $w(\mathbb{P}^4 \setminus H) \le 6$, $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 3 with respect to $\mathbb{P}^4$, all $U \in B_3$ and all $A \in B_2$ with $w(A) = 8$. Take a plane A with $w(A) = 9$. We get $w_W(A) \le 7$, unless $A \cap H \cap S = \{o\}$ and $w(\mathbb{P}^4 \setminus H) = 6$. In this case $E \cap S$ contains $\{o\} \cup S_1$ and exactly one point of S_2 $S \setminus S \cap (E \cup A)$ is a point p with multiplicity 2 and A is spanned by o and 4 points of S_2. Let p' be one of the points of $A \cap S_2$. Using $H' := \langle E \cup \{p'\}\rangle$ instead of H we get $h^1(\mathcal{I}_{\mathrm{Res}_{H'}(Z)}(3)) = 0$.

Now assume $b = 5$. We get $h^1(A, \mathcal{I}_{S\cap A}(2)) = 0$ for every plane A. If $a \le 7$, then $h^1(M, \mathcal{I}_{S\cap M}(2)) = 0$ by the Segre conditions; if $a = 7$, we get $h^1(\mathcal{I}_S(2)) \le 1$, because $h^1(\mathcal{I}_{S\setminus S\cap M}(1)) \le 1$; if $a = 6$ we get $h^1(\mathcal{I}_{S\setminus S\cap M}(1)) \le 1$, because S has no quadrisecant line. Assume $a = 8$. Since $h^1(\mathcal{I}_{S\setminus S\cap M}(1)) = 0$, it is sufficient to observe that for each $q \in S\cap M$, $S\cap M \setminus \{q\}$ satisfies the Segre conditions in degree 2 and hence $h^1(M, \mathcal{I}_{S\cap M}(2)) \le 1 + h^1(M, \mathcal{I}_{S\cap M\setminus\{q\}}(2)) = 1$.

Now assume $b = 4$. If $a \ge 6$, then we conclude as in the case $b = 5$. Now assume $a = 5$. We conclude, because the 5 points of $S \setminus S\cap M$ are not contained in a plane.

Now assume $b = 3$, then S has no trisecant line. If $a \ge 5$, we conclude as in the case $b = 4, 5$. If $a = 4$, then S is in linearly general position and hence the inequality $h^1(\mathcal{I}_S(2)) \le \sharp(S) - 9 = 1$ is classical.

(c) Assume $h = 4$. We have $c \le 6$. By [2] we may assume $c \ge 4$ (anyway, in the case $c \le 3$ we have $h^1(\mathcal{I}_{S'}(2)) = 0$).

(c1) Assume $B_+ = B_\vee = \emptyset$. Let H be a hyperplane containing o and at least 3 points of S_2. It is sufficient to prove that $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 3. The condition for $\mathbb{P}^4$ is satisfied, because $\sharp(H \cap S) \ge 4$. The condition for lines is satisfied, because $B_+ = B_\vee = \emptyset$, $o \in H$ and $\sharp(S_2 \setminus S_2 \cap H) \le 1$, so that $H \cap S$ meets each element of $B_=$. We have $w(\mathbb{P}^4 \setminus H) \le c+2 \le 8$. Fix $U \in B_x$, $x = 2, 3$. We get $\sharp(S\cap U\cap H) \ge w(U) - c - 3$. Hence each hyperplane satisfies the Segre condition for $\mathrm{Res}_H(Z)$. A plane A may not satisfy the Segre condition for $\mathrm{Res}_H(Z)$ only if it contains 6 points of S_1 and either o or a point of $S_2 \setminus S_2 \cap H$. Hence we may assume that $c = 6$ and that S_1 is contained in a plane A. Since $w(S_1) = 6$, we get $A = \langle S_1\rangle$. Therefore A is unique. Hence if it contains a point $q \in S_2$, then we may take this point q as a point of $S_2 \cap H$, so that $A \cap S \cap H \ne \emptyset$; since $w(A) = 8$, in this case A satisfies the Segre condition for $\mathrm{Res}_H(Z)$. Now assume $o \in A$. Let M be a hyperplane containing A and at least one point of S_2. We have $\mathrm{Res}_M(Z) = 2o \cup E$ with $E \subset S_2$ and $\sharp(E) \le 3$ and hence $h^1(\mathcal{I}_{2o\cup E}(3)) = 0$.

(c2) Assume $B_+ \ne \emptyset$. Since $c \le 6$, B_+ has a unique element, R. Fix $p \in S_2$ and let H be a hyperplane containing $\{o, p\} \cup R$. It is sufficient to prove that $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 3. The condition for $\mathbb{P}^4$

is satisfied, because $\sharp(H \cap S) \geq 4$. Since $\mathbb{P}^4 \setminus H$ contains at most $c - 5 \leq 1$ points of S_1 and 3 points of S_2, the conditions for hyperplanes is satisfied. Fix $A \in B_2$. Since $w(A) \geq 8 > w(\mathbb{P}^4 \setminus H)$, we have $A \cap H \cap S \neq \emptyset$. Hence to check the Segre condition we may assume $w(A) = 9$ and $\sharp(A \cap H \cap S) = 1$. Since $c \leq 6$, we get $c = 6$, that $A = \langle S_1 \rangle$ and that $o \in A$. Let M be a hyperplane containing A and at least one point of S_2. We have $\mathrm{Res}_M(Z) = 2o \cup E$ with $E \subset S_2$ and $\sharp(S_2) \leq 3$ and hence $h^1(\mathcal{I}_{2o \cup E}(3)) = 0$. The Segre condition in degree 3 for $\mathrm{Res}_H(Z)$ is satisfied by lines $\notin B_=$, because $R \subset H$ and $o \in H$. Since we may choose any $p \in S_2$ we may satisfies the Segre condition for lines, unless there are $D_1, D_2 \in B_=$ such that $S_2 \cap D_1 \cap D_2 = \emptyset$. If $D_1 \cap S_1 \in R$ (resp. $D_2 \cap S_1 \in R$), it is sufficient to take $p \in S_2 \cap D_2$ (resp. $p \in S_2 \cap D_1$). Assume $R \cap D_1 = R \cap D_2 = \emptyset$. We get $c = 6$ and that $D_1 \cap D_2$ is the only point of $S_1 \setminus S_1 \cap R$. Fix $q \in R \cap S_1$ and let H' be a hyperplane containing $\langle D_1 \cup D_2 \rangle \cup \{q\}$. We have $\mathrm{Res}_{H'}(Z) = 3o \cup (S_1 \setminus \{q\})$. Since $h^1(\mathcal{I}_{\ell(S_1 \setminus \{q\}}(2)) = 0$, we have $h^1(\mathcal{I}_{3o \cup (S_1 \setminus \{q\}}(3)) = 0$ and hence $h^1(\mathcal{I}_Z(4)) = 0$.

(c3) Assume $B_+ = \emptyset$ and $B_\vee \neq \emptyset$. If all points of S_2 are bistarred, then, $c \leq 6$, there is a plane B containing at least 3 points of S_2 and 5 points of S_1 and hence $w(B) \geq 11$, a contradiction. Hence there is a hyperplane H containing o and 3 points of S_2 and such that $H \cap S_2$ contains all bistarred points of S_2. The scheme $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 3 with respect to $\mathbb{P}^4$ (because $\sharp(H \cap S) \geq 4$) and with respect to all lines (because $B_+ = \emptyset$, $o \in H$ and all bistarred points of S_2 are in H). Fix $U \in B_3$. Since $w(\mathbb{P}^4 \setminus H) \leq 8$, we have $U \cap H \cap S \neq \emptyset$ and $\sharp(U \cap H \cap S) \geq 2$ if $w(U) \geq 12$. If $w(U) = 13$ we may quote Lemma 4. Fix $A \in B_2$. Since $w(\mathbb{P}^4 \setminus H) \leq 8$, we have $A \cap H \cap S = \emptyset$ only if $c = 6$, $w(A) = 8$ and A contains S_1 and the point of $S_2 \setminus S_2 \cap H$; let M be a hyperplane containing $A \cup \{o\}$; since $w(\mathbb{P}^4 \setminus M) \leq 6$, $\mathrm{Res}_M(Z)$ satisfies the Segre conditions in degree 3 with respect to all linear spaces of dimension > 1; it satisfies the condition for all lines, because $\{o\} \cup S_1 \subset M$ and each element of B_1 meets $\{o\} \cup S_1$. Now assume the existence of a plane A with $w(A) = 9$ and $\sharp(A \cap H \cap S) = 1$. Since $c \leq 6$, we get $c = 6$ and $A \supset \{o\} \cup S_1$; take a hyperplane M containing A and an element of S_2 and proceed as above.

(d) Assume $h = 3$. We have $c \leq 8$.

(d1) Assume $c = 8$ and that $E := \langle S_1 \rangle$ is a plane. Let M be a hyperplane containing $E \cup \{o\}$. It is sufficient to prove that $W := \mathrm{Res}_M(Z)$ satisfies the Segre conditions in degree 3. These conditions are satisfied by $\mathbb{P}^4$ (because $\sharp(M \cap S) \geq 4$) and by all lines (because $E \cup \{o\}$ meets each element of B_1). Fix $U \in B_3$. Since $S \setminus S \cap M \subseteq S_2$, we have $\sharp(U \cap M \cap S) \geq 3$. Fix $A \in B_2$. Since $S \setminus S \cap M \subseteq S_2$, we get $A \cap M \cap S \neq \emptyset$. Assume $w(A) = 9$ and $\sharp(A \cap S) = 1$. We get $S \cap H = S_1 \cup \{o\}$, $A \cap S = \{o\} \cup S_2$ and hence that $A = \langle \{o\} \cup S_2 \rangle$. We get $S \subset A \cup E$. Fix $q \in E \cap S$ and set $H_q := \langle A \cup \{q\}\rangle$ and $W_q := \mathrm{Res}_{H_q}(Z)$; if $B_+ \neq \emptyset$, then assume that q is one of the 5 points of the only elements of B_+. The last condition and the inclusion of $S_2 \cup S_3$ in H_q implies that W_q satisfies the Segre condition in degree 3 with respect to all lines. It satisfies it with respect to $\mathbb{P}^4$ and with respect to all hyperplanes, because $W_q = 2o \cup S_2 \cup (S_1 \setminus \{q\})$ and $w(\mathbb{P}^4 \setminus H_q) = 7$. Each $B \in B_2$ meets $H_q \cap S$. Assume $w(B) = 9$ and $\sharp(B \cap S) = 1$. We get that $\sharp(B \cap S_1) \geq 6$. Hence $B = \langle B \cap S_1\rangle$. Thus B contains q and at least one point of $S_2 \cup S_3$, a contradiction.

(d2) Assume $B_+ = \emptyset$; if $c = 8$ assume that $\langle S_1\rangle$ is not a plane. Let H be a hyperplane containing $S_3 \cup S_2$ and spanned by points of S. We have $w(\mathbb{P}^4 \setminus H) \leq 8$ with strict inequality if $\langle S_3 \cup S_2\rangle$ is a plane. The scheme $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 3 with respect to $\mathbb{P}^4$ and all lines. Fix $U \in B_3$. We have $\sharp(U \cap H \cap S) \geq 2$, because $c = 8$. If $w(U) = 13$, then we use Lemma 4. Fix $A \in B_2$. Since $w(\mathbb{P}^4 \setminus H) \leq 8$, we have $A \cap H \cap S = \emptyset$ if and only if $c = 8$ and $A = \langle S_1\rangle$; we excluded this case. Now assume $w(A) = 9$ and $\sharp(A \cap H \cap S) = 1$. We get that either A contains a point of S_2 and 7 points of S_1 or it contains o and 6 points of S_1. Since $c \leq 8$ and we assumed $B_+ = \emptyset$, there are at most two planes A, A', spanned by 6 points of S_1 and in this case we have $\sharp(A \cap A' \cap S') = 4$, $\sharp(A \cap S_1) = \sharp(S \cap A') = 6$ and $A \cap A'$ contains o and 4 points of S_1, so that $w(A \cap A') \geq 7$, a contradiction. Hence we may assume that A is the only plane spanned by 6 points of S_1. Let M be a hyperplane containing A and a point of $S_3 \cup S_2 \setminus (S_3 \cup S_2) \cap A$ with maximal multiplicity. Since $w(\mathbb{P}^4 \setminus M) \leq 6$, $\mathrm{Res}_M(Z)$ satisfies the Segre conditions in degree 3 with respect to all linear spaces of dimension ≥ 3 and with respect to all planes with weight ≤ 8. Fix a plane B with $w(B) = 9$ and assume $\sharp(B \cap M \cap S) = 1$. It means that $B \cap M \cap S = \{o\}$ and that B contains 2 points of S_2 not in in M and 2 points of S_1. We get $w(\mathbb{P}^4 \setminus H) \leq 6$ and in this case the plane A does not arise.

(d3) Assume $B_+ \neq \emptyset$; if $c = 8$ assume that $\langle S_1\rangle$ is not a plane. Since $c \leq 8$, B_+ has a unique element, R. Since $\sharp(S_1 \setminus S_1 \cap R) = 3$ and $\langle S_1\rangle$ is not a plane, we get $\sharp(B_\vee) \leq 1$. Fix $q \in R \cap S$. Let M be a hyperplane containing q, o and two points of S_2; if $B_\vee \neq \emptyset$ we take inside M at least one bistarred point of S_2. The scheme $\mathrm{Res}_M(Z)$ satisfies the Segre conditions in degree 3 with respect to $\mathbb{P}^4$. Now we check the Segre condition in degree 3 for lines for the scheme $\mathrm{Res}_M(W)$; since $\{q, o\} \in M$, it is satisfied, except at most by the elements of $B_=$ and $B_\vee$. Since $\sharp(S_2 \cap M) = 2$, every element of $B_=$ satisfies the Segre condition. By our choice of the points of $S_2 \cap M$ every element of $B_\vee$ satisfies the Segre condition in degree 3.

Fix $U \in B_3$. Since $w(\mathbb{P}^4 \setminus M) \leq 9$, we have $\sharp(M \cap U \cap S) > 0$. Now assume

$w(U) = 12$ and $\sharp(M \cap U \cap S) = 1$. Since $w(\mathbb{P}^4 \setminus M) \le 9$, we get $o \in U$, $c = 8$ and that U contains the point of $S_2 \setminus S_2 \cap M$ and the seven points of $S_1 \setminus \{q\}$; since R is spanned by two points of $R \cap S_1$, we get $q \in U$ and hence $\sharp(S \cap U \cap M) \ge 2$. If $w(U) = 13$, then we use Lemma 4.

Fix $B \in \mathcal{B}_2$. If $B \cap M \cap S = \emptyset$, then B must contain at least two points of $R \cap (S_1 \setminus \{q\})$, because $w(\mathbb{P}^4 \setminus M) \le 9$ and hence $q \in B$, a contradiction. Now assume that B is a plane with $w(B) = 9$ and $\sharp(B \cap M \cap S) = 1$.

(d3.1) Assume $B \cap M \cap S \in S_3 \cup S_2$. Since $q \notin B$, we get that $c = 8$, $o \in B \cap M \cap S$, B contains a point of $S_2 \setminus S_2 \cap M$, the 3 points of $S_1 \setminus S_1 \cap R$ and one point p of $R \cap S_1$ with $p \ne q$. By the Segre condition for Z we have $B = \langle \{o\} \cup (S_1 \setminus S_1 \cap R) \rangle$. Hence we may avoid the case taking inside M the only point of $S_2 \cap B$, because every $T \in \mathcal{B}_\vee$ contains a point of $S_1 \setminus S_1 \cap R$.

(d3.2) Now assume $B \cap M \cap S \in S_1$. Hence $o \notin B$ and $\sharp(S_2 \cap B) \le 1$. Hence $\sharp(B \cap S_1) \ge 7$. Since $c \le 8$, we get $\sharp(S_2 \cap B) = 1$, $R \subset B$ and $B = \langle B \cap S_1 \rangle$. Since $c \le 8$, B is unique. Since $\sharp(S_1 \setminus S_1 \cap B) = 1$, all elements of $\mathcal{B}_\vee$ (if any) are contained in B. Hence we may take the point of $S_2 \cap B$ as one of the points of $S_2 \cap M$ and avoid this case (note that (d3.1) and (d3.2) cannot occur for different planes).

(f) Assume $h = 2$. We have $c \le 10$. Note that either $\mathcal{B}_= \emptyset$ or $\mathcal{B}_= \{\langle S_2 \rangle\}$.

(f1) Assume $\sharp(\mathcal{B}_+) \ge 2$. Since $c \le 10$, we have $\sharp(\mathcal{B}_+) = 2$, say $\mathcal{B}_+ = \{D_1, D_2\}$ and either $D_1 \cap D_2 = \emptyset$, $c = 10$ and $S_1 \subset D_1 \cup D_2$ or $D_1 \cap D_2 \ne \emptyset$ and the plane $\langle D_1 \cup D_2 \rangle$ contains at least 9 points of S_1 (by the Segre condition of Z it contains 9 points of S_1 and no point of $S_2 \cup S_3$). First assume $D_1 \cap D_2 \ne \emptyset$ and let H be a hyperplane containing $D_1 \cup D_2 \cup \{o\}$. Since $w(\mathbb{P}^4 \setminus H) \le 5$, $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 3 with respect to all linear spaces of dimension > 1. It satisfies the Segre condition for all lines, unless $c = 10$, $\sharp(S_1 \cap \langle D_1 \cup D_2 \rangle) = 9$ and the line $\langle S_2 \rangle$ contains the point of $S_1 \setminus \langle D_1 \cup D_2 \rangle$. Set $F := \langle D_1 \cup S_2 \rangle$. F is a hyperplane. $\mathrm{Res}_F(Z)$ satisfies the Segre conditions in degree 3 with respect to $\mathbb{P}^4$ and all lines, because it contains S_2 and no line though o may contain 2 points of $S_1 \setminus S_1 \cap F = D_2 \cap S_1 \setminus D_1 \cap D_2$. Fix $U \in \mathcal{B}_3$. Since $w(\mathbb{P}^4 \setminus F) \le 7$ and each point of $S \cap F$ has multiplicity ≤ 2, we get $\sharp(F \cap U \cap S) \ge 2$, with strict inequality if $w(U) = 13$. Fix $A \in \mathcal{B}_2$. Since $w(\mathbb{P}^4 \setminus F) \le 7$, we get $A \cap F \cap S \ne \emptyset$. Now assume $w(A) = 9$ and $\sharp(A \cap F \cap S) = 1$. We get that $A \cap F \cap S \in S_2$ and that A contains o and the 4 elements of $S_1 \setminus D_1 \cap D_2$. Hence $D_1 \cap D_2 \in A \cap F \cap S$ and so $\sharp(A \cap F \cap S) > 1$.

Now assume $D_1 \cap D_2 = \emptyset$. Fix $p_i \in D_i \cap S_1$, $i = 1, 2$, and let G be a hyperplane containing $\{o, o_2, o_3\}$ and one point of S_2. By construction $\mathrm{Res}_G(Z)$ satisfies the Segre conditions with respect to $\mathbb{P}^4$ and all lines. Fix $U \in \mathcal{B}_3$. Since $w(\mathbb{P}^4 \setminus G) \le 10$, we have $U \cap G \cap S \ne \emptyset$. Now assume $w(U) = 12$

and $\sharp(U \cap G \cap S) = 1$; since $w(U) - w(S_2 \cup S_3) \geq 5$, U must contain at least 4 points of $S \cap (D_1 \cup D_2)$ and hence at least one of the points p_i; we also get that either $U \cap G \cap (S_2 \cup S_3) \neq \emptyset$ or that $G \supset \{p_1, p_2\}$ and hence $\sharp(U \cap G \cap S) > 1$. Now assume $w(U) = 13$ and $\sharp(U \cap G \cap S) = 2$; we exclude both the cases $\sharp(U \cap G \cap (S_2 \cup S_3)) = 2$ and $\sharp(U \cap D_i \cap S) \leq 1$ for all i and the case $\sharp(U \cap G \cap (S_2 \cup S_3)) = 1$ and $\sharp(U \cap D_i \cap S) = 1$ for at least one i (because $3 + 5 + 1 < 13$) and the case $U \cap G \cap (S_2 \cup S_3)) = \emptyset$ (because $c = 10$, $o \notin U$ and U has at most one element of S_2). Fix $A \in B_2$. Since $w(\mathbb{P}^4 \setminus G) \leq 10$ and if A contains 3 of the 8 points of $S_1 \setminus \{o_1, o_2\}$, then it also intersects $\{o_1, o_2\}$, we get $A \cap G \cap S \neq \emptyset$. Now assume $w(A) = 9$. If $A \cap G \cap (S_3 \cup S_2) = \emptyset$, then $A \cap S$ is at most one point of multiplicity 2 and 7 points of multiplicity 1 and hence it contains $\{o_1, o_2\}$. If $\sharp(A \cap G \cap (S_3 \cup S_2)) = 1$, then it contains at least 6 points of $S \cap (D_1 \cup D_2)$ and hence at least one of the points o_1, o_2.

(f2) Assume $\sharp(B_+) = 1$, say $B_+ = \{R\}$. Let H be a hyperplane containing $S_3 \cup S_2$ and $p \in R \cap S_1$. The scheme $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 3 with respect to $\mathbb{P}^4$, all lines and (since $w(\mathbb{P}^4 \setminus H) \leq 9$) and all hyperplanes with weight ≤ 11. By Lemma 4 we do not need to test the hyperplanes with $w(U) = 13$. Let U be a hyperplane with $w(U) = 12$. Since $c \leq 10$, U contains a point of $S_2 \cup S_3$. If it contains a unique point of $S_3 \cup S_2$, then it contains at least 9 points of S_1 and hence it contains two points of $R \cap S_1$ and hence R and so another point p of $H \cap S$. Take a plane A with $w(A) \geq 8$ and assume $A \cap (S_3 \cup S_2) = \emptyset$. Since $c \leq 10$, $A \cap R \cap S_1) \geq 2$ and so $A \supset R$ and hence $A \cap H \cap S \neq \emptyset$. Now assume $w(A) = 9$, $\sharp(A \cap (S_3 \cup S_2)) = 1$ and $p \notin A$. Since $c \leq 10$, we get $c = 10$, $o \in A$, that A contains all points of $S_1 \setminus S_1 \cap R$ and a unique point $q \in R \cap S_1$. If A is unique, to avoid this case it is sufficient to take q as the point p. Since $w(T) \leq 5$ for any line, $\langle \{o\} \cup (S_1 \setminus S_1 \cap R) \langle$ is a plane and so $A = \langle \{o\} \cup (S_1 \setminus S_1 \cap R) \langle$ is unique.

(f3) Assume $B_+ = \emptyset$. Let H be a hyperplane containing $S_3 \cup S_2$ and $p \in S_1$. The scheme $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 3 with respect to $\mathbb{P}^4$, all lines and (since $w(\mathbb{P}^4 \setminus H) \leq 9$), all hyperplanes with weight ≤ 11. Let U be a hyperplane with $w(U) = 12$. Since $c \leq 10$, we have $(S_2 \cap S_3) \cap U \neq \emptyset$. If $\sharp(S_2 \cup S_3) \geq 2$, then U satisfies the Segre condition in degree 3 for $\mathrm{Res}_H(Z)$. If $(S_2 \cap S_3) \cap U = \{o'\}$ with $o' \in S_2 \cup S_3$, then either $o' \in S_2$, $c = 10$ and $U = \langle S_1 \rangle$, or $o' = o$ and $\sharp(U \cap S_1) = 9$. In the first case $p \in U$ and hence U satisfies the Segre condition in degree 3 for $\mathrm{Res}_H(Z)$. In the second case we may take $p \in U \cap S_1$ if U is unique. If there is another U' containing o and 9 points of S_1, we get that $U \cap U'$ contain o and at least 8 points of S_1 and so $w(U \cap U') \geq 11$ and so $U \cap U'$ is not a plane, i.e. $U = U'$. By Lemma 4 we do not need to test the hyperplanes with weight 13.

Take a plane $A \in B_2$ with $\sharp(A \cap H \cap S) \le 9 - w(A)$.

(f3.1) First assume $A \cap (S_3 \cup S_2) = \emptyset$. A is the only plane containing at least 8 points of S_1; hence if $w(A) = 8$ it is sufficient to take $p \in S_1 \cap A$; note that in case all the hyperplanes U, U_1 contain A and hence we may choose p also for those hyperplanes. Thus we may assume $w(A) = 9$. Set $M := \langle \{o\} \cup A \rangle$. The scheme $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 3 with respect to $\mathbb{P}^4$ and (since $w(\mathbb{P}^4 \setminus M) \le 5$ and $m_1 < 4$) all linear spaces of dimension > 1. It also satisfies the Segre conditions for lines, unless $c = 10$ and the line $\langle S_2 \rangle$ contains the point of $S_1 \setminus S_1 \cap A$; we assume that we are in this case. Let M be a hyperplane containing o, one point of S_2 and two points $p_1, p_2 \in A \cap S_1$. The scheme $\mathrm{Res}_M(Z)$ satisfies the Segre conditions in degree 3 with respect to $\mathbb{P}^4$ and all lines. Fix a hyperplane U with $w(U) \ge 11$. By Lemma 4 we may assume $w(U) \le 12$. If $U \cap M \cap (S_3 \cup S_2) = \emptyset$, then U contains at least 9 points of S_1 and hence at least 8 points of $S_1 \cap A$; since these points span A by the Segre condition for Z; we get $p_1, p_2 \in U$. Now assume $\sharp(U \cap M \cap (S_3 \cup S_2)) = 1$; we get that U has at least 5 points of $S_1 \cap A$ and (since $B_+ = \emptyset$), these points span A, so that $\sharp(U \cap M \cap S) > 1$. Let E be a plane with $w(E) \ge 8$ and $E \cap M \cap S = \emptyset$. We get that $\sharp(E \cap S_1) \ge 6$ and hence $\sharp(A \cap S_1) \ge 5$ and hence $E = A$ (since $B_+ = \emptyset$) and hence $E \cap M \cap S \ne \emptyset$. Now assume $w(E) = 9$ and $\sharp(A \cap M \cap S)) = 1$. As above we see that $A \cap M \cap (S_3 \cup S_2))$ is a single point o' and that $E \cap S_1$ contains at least 3 point of A (case $o' = o$) or 4 points of A (case $o' \in S_2$); we may also assume $\langle E \cap A \cap S_1 \rangle \ne A$, i.e. that all points of $A \cap S_1 \cap E$ are collinear. In the case $o' \in S_2$ there are at most 3 4-secant lines of $A \cap S_1$ and all of them pass through 3 points of $A \cap S_1$; for suitable p_1, p_2 we find $E \cap M \cap S_1 \ne \emptyset$ and hence E satisfies the Segre condition in degree 3 for $\mathrm{Res}_M(Z)$. Now assume $o' = o$. Since E contains $S_1 \setminus S_1 \cap A$ and one point of S_2, it contains the other point of S_2, because $\langle S_2 \rangle$ contains the point of $S_1 \setminus S_1 \cap A$.

(f3.2) Now assume that $A \cap (S_3 \cup S_2)$ is a point q with $m_q = 2$. We get that A contains at least 4 points of $A \cap S_1$. Since $A \ne E$, the points of $E \cap A \cap S_1$ are collinear. There are at most 3 4-secant lines of $A \cap S_1$ and all of them pass through 3 points of $A \cap S_1$; for suitable p_1, p_2 we find $E \cap M \cap S_1 \ne \emptyset$ and hence E satisfies the Segre condition in degree 3 for $\mathrm{Res}_M(Z)$.

(f3.3) Now assume $\{o\} = A \cap (S_3 \cup S_2)$. We get that E contains at least 3 points of $A \cap S_1$. If E contains 4 points of $A \cap S_1$, then we conclude as in step (f3.2). Thus we may assume that E contains exactly 3 points of $A \cap S_1$ and hence it contains the point of $S_1 \setminus S_1 \cap A$. Since E also contain a point of S_2 and the line $\langle S_2 \rangle$ contains the point of $S_1 \setminus S_1 \cap A$, we get $S_2 \subset E$ and hence $\sharp(E \cap A \cap (S_3 \cup S_2)) = 2$, a contradiction.

(g) Assume $h = 1$. We have $c \le 12$. Since $h = 1$, $B_- = \emptyset$. Set

$L := \langle S_3 \cup S_2 \rangle$. Every $D \in B_1 \setminus B_+$ meets L.

(g1) Assume the existence of $L_1, L_2 \in B_+$ such that $L_1 \cap L_2 = \emptyset$. Since $L \cap L_i \cap S = \emptyset$, Remark 3 gives $L \cap L_i = \emptyset$. Since $\sharp(S_1) - \sharp(S_1 \cap (L_1 \cup L_2)) \leq 2$, we get $B_+ = \{L_1, L_2\}$. Fix $p_i \in L_i \cap S$, $i = 1, 2$, and let H be a hyperplane containing $L \cup \{p_1, p_2\}$. $\mathrm{Res}_H(Z)$ satisfies the Segre condition in degree 3 with respect to $\mathbb{P}^4$ and all lines. Fix $U \in B_3$. Since $w(\mathbb{P}^4 \setminus H) \leq 10$, we have $U \cap H \cap S \neq \emptyset$. Assume $w(U) = 12$. Since $c \leq 12$ and $S_2 \cup S_3 \subset H$, we get $S_1 = U \cap S$ and hence $\{p_1, p_2\} \in U$. Lemma 4 takes care of the case $w(U) = 13$. Fix $A \in B_2$. We have $A \cap S \cap H \neq \emptyset$, because $A \cap S$ cannot be the $c - 10$ points of $S_1 \setminus S_1 \cap (L_1 \cup L_2)$ and one point for each L_i. Now assume $w(A) = 9$ and $\sharp(A \cap H \cap S) = 1$; $A \cap H \cap S$ cannot be the $c - 10$ points of $S_1 \setminus S_1 \cap (L_1 \cup L_2)$, at most 5 points of $L_i \cap S_1$ and one point f $L_{3-i} \cap S_1$; $A \cap H \cap S$ cannot be a point of $S_3 \cup S_2$, the $c - 10$ points of $S_1 \setminus S_1 \cap (L_1 \cup L_2)$ and one point for each L_i.

(g2) Assume $\sharp(B_+) \geq 2$ and take $D_1, D_2 \in B_+$ with $D_1 \neq D_2$. By step (g1) we may assume $D_1 \cap D_2 \neq \emptyset$. Hence $D_1 \cap D_2 \in S_1$, the plane $\langle D_1 \cup D_2 \rangle$ contains 9 points of S_1 and no other point of S. First assume that either $c \leq 11$ or the 3 points of $S_1 \setminus S_1 \cap \langle D_1 \cup D_2 \rangle$ are not contained in an element of $B_\vee$. Let H be a hyperplane containing $D_1 \cup D_2 \cup \{o\}$. Our assumption gives that $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 3 with respect to $\mathbb{P}^4$ and all lines. Since $w(\mathbb{P}^4 \setminus H) \leq 5$, $m_1 < 4$ and $m_1 + m_2 + 5 < 13$, $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 3 with respect to all linear spaces of dimension > 1. Now assume $c = 12$ and the existence of $R \in B_\vee$ containing the 3 points of $S_1 \setminus S_1 \cap \langle D_1 \cup D_2 \rangle$. Let M be a hyperplane containing $L \cup D_1$. Assume for the moment $B_- \emptyset$ and take $T \in B_-$; since $S_1 \subset \langle D_1 \cup D_2 \rangle \cup R$, and $o \notin \langle D_1 \cup D_2 \rangle$, we get $T \cap R \neq \emptyset$ and hence $T \cap R \cap S \neq \emptyset$ (Remark 3). Thus $\mathrm{Res}_M(Z)$ satisfies the Segre conditions in degree 3 with respect to $\mathbb{P}^4$ and all lines. Fix $U \in B_3$. Since $w(\mathbb{P}^4 \setminus M) \leq 7$, $\mathrm{Res}_M(Z)$ satisfies the Segre conditions in degree 3 with respect to all hyperplanes (if $w(U) = 13$ use that $m_1 + m_2 + w(\mathbb{P}^4 \setminus M) < 13$) and all planes with weight ≤ 8. Fix a plane A with $w(A) = 9$. If A contains at least two points of $D_2 \cap S_1$, then it contains $D_2 \cap D_1 \in M \cap S_1$. Since $S \setminus S \cap M$ is the union of o and 4 points of D_2, then $\sharp(A \cap M \cap S) > 1$.

(g3) Assume $\sharp(B_+) = 1$, say $B_+ = \{R\}$. By [2] we may assume $c \geq 6$. Fix $p \in R \cap S_1$, $q \in S_1 \setminus S_1 \cap R$ and take a hyperplane $H \supset L \cup \{p, q\}$. $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 3 with respect to $\mathbb{P}^4$ and all lines. Fix $U \in B_3$. Since $w(\mathbb{P}^4 \setminus H) \leq 10$, we have $S \cap H \cap U \neq \emptyset$. Assume $w(U) = 12$ and $\sharp(U \cap S \cap H) = 1$. Either $U \supset R$ and $U \cap (S_2 \cup S_3) = \emptyset$ (and hence $c = 12$ and $U = \langle S_1 \rangle$) or U contains one element of $S_2 \cup S_3$, $c - 5$ elements $S_1 \setminus S_1 \cap R$ and one element of R (this case give $w(U) \leq 11$). Assume $c = 12$ and $U = \langle S_1 \rangle$; we get $\sharp(H \cap U \cap S) = 2$, because $p, q \in H \cap U \cap S$. If $w(U) = 13$, then we quote Lemma 4.

Fix $A \in B_2$. First assume $A \cap H \cap S = \emptyset$. Since $c - \sharp(S_1 \cap R) \le 7$ and $q \in H \cap S_1$, we get that A contains at least two points of $R \cap S_1$ and hence it contains p, a contradiction. Now assume $w(A) = 9$ and $\sharp(A \cap H \cap S) = 1$. First assume $A \cap H \cap S \in \{p, q\}$. We get that A contains 8 other points of S_1; call G a hyperplane containing $A \cup \{o\}$; since $w(\mathbb{P}^4 \setminus G) \le 4$, $\mathrm{Res}_G(Z)$ satisfies the Segre conditions in degree 3. Now assume $(S_3 \cap S_2) \cap A \neq \emptyset$ and hence that A contains 6 or 7 points of S_1 (6 if and only $o \in A$), but neither p nor q. Since $p \notin A$, A contains at most one point of R. We have $A = \langle A \cap S_1 \rangle$ and hence to avoid this case would be sufficient to take $q \in A$ if A is unique. Take another A' containing one point of $S_3 \cup S_2$ and 6 or 7 points of S_1 (6 if and only if $o \in A$), but with at most one point of $R \cap S_1$. We get $\sharp(A \cap A' \cap S_1) \ge 3$ and hence $A \cap A'$ is a line. Since $w(\langle A \cup A' \rangle) \ge 18 - w(A \cap A') \le 13$, we get $w(\langle A \cup A' \rangle) = 13$ and we use Lemma 4.

(g4) Assume $B_+ = \emptyset$ and the existence of a plane E such that $\sharp(E \cap S_1) \ge 7$. Let F be a hyperplane containing L and two points p, q of $S_1 \cap E$ and spanned by S. The scheme $\mathrm{Res}_F(Z)$ satisfies the Segre conditions in degree 3 with respect to $\mathbb{P}^4$, all lines and, since $w(\mathbb{P}^4 \setminus F) \le 7$, all planes with weight ≤ 8 and (since $m_1 < 4$) all hyperplanes with weight ≤ 12. For the hyperplanes with weight 13 we quote Lemma 4. Take a hyperplane U with $w(U) = 13$; if $U \supset L$, then it meets each element of B_1, because $B_+ = \emptyset$; since $c < w(U)$, U contains at least one point, o', of $S_2 \cup S_3$; if $U \cap (S_2 \cup S_3) = \{o'\}$, then $\sharp(U \cap S_1) \ge 10$ and hence $\sharp(U \cap E \cap S_1) \ge 5$; since $B_+ = \emptyset$, we get $U \supset E$ and hence $\sharp(U \cap H \cap S) \ge 3$. Take a plane B with $\sharp(B \cap F \cap S) = 1$. If $B \cap (S_3 \cup S_2) = 9$, then B has 9 points of S_1; since $B_+ = \emptyset$, we get $B = E$ and hence $\sharp(B \cap F \cap S) > 1$, a contradiction. Hence we may assume that B contains one point of $S_2 \cup S_3$ and at least 6 points of S_1, none of them on R. We have $B \cap E \cap S_1 \neq \emptyset$. The planes E, B are the only ones containing at least 6 points of S_1 and with weight 9, the other one with at least 7 points of S_1 (because $B_+ = \emptyset$ and hence $\sharp(S_1 \cap A \cap E) \le 4$). Hence we may take as p a point of $B \cap S_1$.

(g5) Assume $B_+ = \emptyset$ and the existence of a plane A containing one point of $S_3 \cup S_2$ and at least 6 points of S_1. Let H be a hyperplane containing A and the points of $(S_3 \cup S_2) \setminus (S_3 \cup S_2) \cap A$. The scheme $\mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 3 with respect to $\mathbb{P}^4$, all lines, all hyperplanes and all planes with weight ≤ 8 (because $w(\mathbb{P}^4 \setminus H) \le 6$ and $m_1 + m_2 + 6 < 13$). Fix a plane B with $w(B) = 9$ and $\sharp(A \cap B \cap S) = 1$. We get that B contains o and 6 points of $S_1 \setminus S_1 \cap H$. First assume $o \in A$. We get $S_2 \nsubseteq A \cup B$ and hence

$Q \cap S \neq S$, where Q is the union of a general hyperplane through A and a general hyperplane through B. The inductive assumption gives $h^1(\mathcal{I}_{Z\cap Q}(4)) = 0$. Since o is a singular point of Q, we have $\mathrm{Res}_Q(Z) = \{o\} \cup 2q$ (with $\{q\} = S_2$) and hence $h^1(\mathcal{I}_{\mathrm{Res}_Q(Z)}(2)) = 0$. Now assume $o \notin A$. Since $o \in B$ we may reverse the role of B and A taking a hyperplane containing B and S_2 and conclude as above.

(g6) Assume $B_+ = \emptyset$. Let H be a hyperplane containing L and containing the maximal number $a \geq 2$ of points of S_1 among the hyperplanes containing L. The scheme $\mathrm{Res}_H(Z)$ satisfies the Segre conditions with respect to $\mathbb{P}^4$, all lines and (since $w(\mathbb{P}^4 \setminus H) \leq 10$) all hyperplanes with weight ≤ 11. Lemma 4 handles the hyperplanes with $w(U) = 13$. Part (g4) handles the case of a plane A with $w(A) \geq 8$ and $A \cap (S_3 \cup S_2) = \emptyset$. Part (g5) handles the case of a plane A with $w(A) = 9$ and $\sharp(A \cap (S_3 \cup S_2)) = 1$. Fix a hyperplane U with $w(U) = 12$ and $\sharp(U \cap H \cap S) = 1$. First assume $U \cap (S_3 \cup S_2) = \emptyset$. We get $c = 12$ and $U \supset S_1$; since $p, q \in H$, we get $\sharp(U \cap H \cap S) > 1$. Now assume $U \cap (S_3 \cup S_2) \neq \emptyset$. We get that U contains 9 points of S_1 and o or 10 points of S_1 and the point, o', of S_2. First assume $o' \in S_2 \cap U$; we get $a = 2$ and in particular $B_\vee = B_- = \emptyset$; we get that $\mathrm{Res}_U(Z)$ satisfies the Segre conditions in degree 3 with respect to all lines and (since $w(\mathbb{P}^4 \setminus U) \leq 5$ and $9 - 5 > m_1$) with respect to all linear spaces of dimension ≥ 2. Now assume $o \in U$; since $a \leq 3$, we get $B_\vee = \emptyset$; hence even in this case $\mathrm{Res}_U(Z)$ satisfies the Segre conditions in degree 3 with respect to all lines. □

Proposition 3. *If Z satisfies the Segre conditions in degree 5, $m_1 \leq 3$ and Z has h points with multiplicity 3 with $h \in \{0, 5, 6, 7\}$, then $h^1(\mathcal{I}_Z(5)) = 0$.*

Proof. Set $S_i := \{p \in S : m_p = i\}$. Set $h := \sharp(S_3)$, $c := \sharp(S_2)$ and $e := \sharp(S_1)$. Since $w(\mathbb{P}^4) \leq 21$, we have

$$3h + 2c + e \leq 21 \tag{5}$$

By Lemma 1 we may assume $h \geq 2$. If $h = 7$, then $\sharp(S) = 7$ and this case is covered by [2]. For the same reason from now on we assume $h + c + e \geq 8$. For any $o \in S_3$ let $\ell_o : \mathbb{P}^4 \setminus \{o\} \to \mathbb{P}^3$ be the linear projection from o. Set $S'_o := \ell_o(S \setminus \{o\})$. The Segre conditions for Z gives $w(A) \leq 11$ for all planes A and $w(U) \leq 16$ for all hyperplanes U. Every line spanned by two points of S_3 is in S_1 and it contains no other point of S.

Observation 1: Assume the existence of $o \in S_3$ such that $h^1(\mathbb{P}^3, \mathcal{I}_{S'_o}(2)) = 0$. Fix $q \in S'_o$. By assumption there is a quadric surface Q' such that $Q' \cap S'_o = S'_o \setminus \{q\}$. Let $Q = \{o\} \cup \ell^{-1}(Q')$ be the quadric cone with vertex containing

o. Let Z'' be the union of the connected components of Z whose support is a point of $S' \setminus \{q\}$ whose image is q and let Z' be the union of the connected components of Z whose support is a point of $S' \setminus \{q\}$ whose image is not q. Let Z_- be the closed subscheme of Z' obtained from Z' decreasing by one all the non-zero multiplicities of Z'. We have $\mathrm{Res}_Q(Z) \subseteq \{o\} \cup Z_- \cup Z''$. Therefore to get $h^1(\mathcal{I}_Z(3)) = 0$ it is sufficient to prove that $\{o\} \cup Z_- \cup Z''$ satisfies the Segre conditions in degree 3.

(a) Assume $h = 6$.

(a1) First assume $c = e = 1$. Fix $o \in S_3$. The set S'_o has at most cardinality 7. Therefore $h^1(\mathbb{P}^3, \mathcal{I}_{S'_o}(2)) = 0$, unless either it has 4 collinear points or it is contained in a plane. In the second case, S is contained in a hyperplane and hence $h^1(\mathcal{I}_Z(5)) = 0$. Assume that S'_o has a 4-secant line. We get the existence of a plane $A \subset \mathbb{P}^4$ containing o and at least 4 other points of S. Hence $w(A) \geq 12$, a contradiction.

Assume for the moment $\sharp(S'_o) = 7$. Let $q \in S'_o$ be the image of the point with multiplicity 1. The scheme $W := \{o\} \cup Z_- \cup Z''$ has 5 points with multiplicity 2 (the points of $S_3 \setminus \{o\}$) and 3 points with multiplicity one ($\{o\} \cup S_2 \cup S_1$). Since each line spanned by two points of S_3 contains no other point of S, $\{o\} \cup Z_- \cup Z''$ satisfies the condition for lines. We have $w_W(\mathbb{P}^4) = 13$. Assume the existence of a line L with $w_W(L) \geq 5$. Since L must contain at least two points of $S_3 \cup S_2$, we get $w(L) \geq 7$, a contradiction. Assume the existence of a plane A such that $w(A) \geq 8$. Since A must contain at least 4 points of $S_3 \cup S_2$, we get $w(A) \geq 12$, a contradiction. Assume the existence of a hyperplane U with $w(U) \geq 11$. Since $e = 1$, U contains either at least 6 points of $S_3 \cup S_2$ or o and at least 4 points of $(S_3 \setminus \{o\}) \cup S_2$ and hence $w(U) \geq 15$, contradicting the Segre condition for Z.

Now assume $\sharp(S'_o) < 7$, i.e. assume the existence of a line L containing o and the two points of $S_1 \cup S_2$, i.e. assume that $\langle S_1 \cup S_2 \rangle \in B_1$. To avoid this case it is sufficient to take as o any of the five points of S_3 not in $\langle S_1 \cup S_2 \rangle$.

(a2) Now assume $e = 2$. In this case S has no 4-secant line and no plane contains 7 points of S (by the Segre conditions for Z) and $\sharp(S) = 8$. Hence $h^1(\mathcal{I}_S(2)) = 0$ and hence for each $q \in S_1$ there is a quadric hypersurface Q with $Q \cap S = S \setminus \{q\}$. The inductive assumption gives $h^1(Q, \mathcal{I}_{Z \cap Q}(5)) = 0$. We have $W := \mathrm{Res}_Q(Z) = \sum_{p \in S_3} 2p \cup \{u\}$ with $S_1 = \{q, u\}$. We have $w_W(\mathbb{P}^4) = 13$. Assume the existence of a line L with $w_W(L) \geq 5$. Since it contains at least two points of S_3, we get $w(L) \geq 7$, a contradiction. Fix a hyperplane. By Lemma 3 we may assume $\sharp(S) - \sharp(S \cap U) \geq 2$ and hence $w_W(U) \leq 10$. Assume the existence of a plane A with $w_W(A) \geq 8$. If A contains at least 4 points of S_3, then we get $w(A) \geq 12$, a contradiction. Hence $S_1 \subset A$ and $\sharp(A \cap S_3) = 4$. Fix a hyperplane H containing A and another point of S_3. The scheme $\mathrm{Res}_H(Z)$

has S_3 as its support, one point of multiplicity 3 and 5 points of multiplicity 2. It satisfies the Segre conditions in degree 3, because no line contains 3 points of S_3, no plane contains 4 points of S_3 and no hyperplane contains S_3.

(b) Assume $h = 5$ and hence $2c + e \le 6$ and $c + e \ge 3$.

(b1) Assume $c = 3$ and hence $e = 0$.

(b1.1) First assume that $\langle S_2\rangle$ is a line. Note that $S_3 \cap \langle S_2\rangle = \emptyset$. Fix $o \in S_3$. By Observation 1 it is sufficient to prove that $h^1(\mathbb{P}^3, \mathcal{I}_{S'_o}(2)) = 0$. S'_o is gives by 4 points images of $\ell_o(S_3 \setminus \{o\})$ and 3 points on the line $\ell_o(\langle S_2\rangle)$. It is sufficient to prove that S'_o is contained in no plane (true, because $\langle S\rangle$) and no line contains 4 points of S'_o. The latter condition is equivalent to the non-existence of a plane E containing $\langle S_2\rangle$, o and at least another point of S_3; this is true for any $o \in S_3$, because $w(E) \le 11$.

(b1.2) Now assume that $\langle S_2\rangle$ is not a line. In this case every element of B_1 is spanned by two points of S_3. Let H be a hyperplane containing 4 points of S_3. We check when $Y := \mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 4. First assume $S_3 \subset H$. In this case the condition for lines is satisfied, because each element of B_1 meets S_3. Assume the existence of a hyperplane U with $w_Y(U) \ge 14$. We have $U \ne H$, because $\sharp(H \cap S) \ge 4$. Since $w(\mathbb{P}^4 \setminus H) \le 6$, we get $\sharp(U \cap S_3) \ge 4$ and hence $w(U) \ge 8$, a contradiction. Fix a plane A such that $w_Y(A) \ge 10$. We get $\sharp(A \cap S_3) \ge 2$ and hence $w(A) \ge 12$, a contradiction.

Now assume $S_3 \nsubseteq H$, i.e. S_3 is linearly independent. Assume the existence of a plane A such that $w_Y(A) \ge 10$. Since $w(S_2) = 6$, we get $\sharp(S_3 \cap A) \ge 2$. If $\sharp(A \cap S_3) \ge 3$, then $\sharp(S \cap A \cap H) \ge 2$ and hence $w(A) \ge 12$, a contradiction. Assume $\sharp(S_3 \cap A) = 2$. We get that A contains S_2. Since $\langle S_2\rangle$ is not a line, we get $A = \langle S_2\rangle$ and it is sufficient to take inside $S_3 \cap H$ both points of $S_3 \cap \langle S_3\rangle$.

Assume the existence of a hyperplane U with $w_Y(U) \ge 14$. We have $U \ne H$, because $\sharp(H \cap S) \ge 4$. Since $w(\mathbb{P}^4 \setminus H) \le 9$, we get $\sharp(S_3 \cap U \cap H) \ge 2$ and hence $w(U) \ge 16$. By the Segre condition for Z we have $w(U) = 16$, i.e. $\sharp(S_2 \cap U) = 2$ and $\sharp(S_3 \cap U) = 4$. Hence $\sharp(U \cap H \cap S_3) \ge 3$ and so $w(U) \ge w_Y(U) + 3 = 17$, a contradiction.

(b2) Assume $c = 2$ and $e = 2$. Fix $o \in S_3$. First assume $h^1(\mathbb{P}^3, \mathcal{I}_{S'_o}(2)) = 0$ and set $W := \{o\} \cup Z_{-} \cup Z''$ with q a point with multiplicity 1. Therefore W has 4 points with multiplicity 2 and 4 points with multiplicity 1. It is sufficient to prove that W satisfies the Segre conditions in degree 3. We have $w_W(\mathbb{P}^4) = 12$. By Lemma 3 we may assume $\sharp(S \cap U) \le \sharp(S) - 2$ for each hyperplane and hence $w_W(U) \le 10$ for each hyperplane U. Assume the existence of a line L with $w_W(L) \ge 5$. If $o \in L$, then $w(L) \ge 7$, a contradiction. If $o \notin L$, then L contains at least two points of $S \setminus \{q\}$ and hence Now assume $h^1(\mathbb{P}^3, \mathcal{I}_{S'_o}(2)) > 0$. By Lemma 3 we may assume $\sharp(A \cap S'_o) \le 6$ for each plane A. Hence there is a line

$L \subset \mathbb{P}^3$ with $\sharp(S'_o) \ge 5$, i.e. there is a plane A containing o and 5 further points $S \setminus \{o\}$. We get $w(A) \ge 12$, a contradiction.

(b3) The cases $(c, e) \in \{(2,1),(1,2)\}$ are done as in step (b2). Assume $c = 1$ and $e = 4$. First assume that $L := \langle S_2 \cup S_1\rangle$ is a line. Fix $q, q' \in S_1$, $q \ne q'$. Since $\sharp(S_3 \cup S_2 \cup \{q,q'\}) = 8$, $\langle S_3 \cup S_2 \cup \{q,q'\}\rangle = \langle L \cup S_3\rangle = \mathbb{P}^4$ and every secant line of S_3 does not contain other points of S, we have $h^1(\mathcal{I}_{S_3 \cup S_2 \cup \{q,q'\}}(2)) = 0$. Hence there is a quadric Q with $Q \cap S = S \setminus \{p_1, p_2\}$ with $p_1, p_2 \in S_1$ and hence it is sufficient to prove that $W := \mathrm{Res}_Q(Z)$ satisfies the Segre conditions in degree 3. W is the union of S_2, $\{p_1, p_2\}$ and $\sum_{p \in S_3} 2p$. We have $w_W(\mathbb{P}^4) = 13$. Assume the existence of a line R such that $w_W(R) \ge 5$. Since R contains at least two points of $S_3 \cup S_2$, we get $w(R) \ge 7$, a contradiction. Assume the existence of a plane A with $w_W(A) \ge 8$. We get $\sharp(A \cap (S_2 \cup S_3)) \ge 4$ and hence $w(A) \ge 12$, unless A contains $\{p_1, p_2\}$. In this case A contains L and hence $w(A) \ge w_W(A) + 3$, a contradiction. In the same way we get that if U is a hyperplane with $w_W(U) \ge 11$, then $w(U) \ge 17$. Hence $\langle S_2 \cup S_1\rangle$ is not a line, i.e. every element of B_1 meets S_3. Let H be a hyperplane containing 4 points of S_3; if there is a line containing a point of S_3 and 3 points of S_1, then there is at most one such line and we impose that H contains the point of S_3 in this line. We check when $Y := \mathrm{Res}_H(Z)$ satisfies the Segre conditions in degree 4. First assume $S_3 \subset H$. In this case the condition for lines is satisfied, because each element of B_1 meets S_3. Assume the existence of a hyperplane U with $w_Y(U) \ge 14$. We have $U \ne H$, because $\sharp(H \cap S) \ge 4$. Since $w(\mathbb{P}^4 \setminus H) \le 6$, we get $\sharp(U \cap S_3) \ge 4$ and hence $w(U) \ge 18$, a contradiction. Fix a plane A such that $w_Y(A) \ge 10$. We get $\sharp(A \cap S_3) \ge 2$ and hence $w(A) \ge 12$, a contradiction.

Now assume $S_3 \nsubseteq H$, i.e. S_3 is linearly independent. In this case the condition for lines is satisfied by our choice of $S_3 \cap H$. Assume the existence of a plane A such that $w_Y(A) \ge 10$. Since $w(S_1 \cup S_2) = 6$, we get $\sharp(S_3 \cap A) \ge 2$. If $\sharp(A \cap S_3) \ge 3$, then $\sharp(S \cap A \cap H) \ge 2$ and hence $w(A) \ge 12$, a contradiction. Assume $\sharp(S_3 \cap A) = 2$. We get that A contains S_2 and at least 3 points of S_1. Since $\langle S_2 \cup S_1\rangle$ is not a line, there is at most one plane containing S_2 and at least 3 points of S_1 and it is sufficient to take inside $S_3 \cap H$ both points of S_3.

Assume the existence of a hyperplane U with $w_Y(U) \ge 14$. We have $U \ne H$, because $\sharp(H \cap S) \ge 4$. Since $w(\mathbb{P}^4 \setminus (S_1 \cup S_2)) = 15$, we get $\sharp(S_3 \cap U \cap H) \ge 2$ and $\sharp(S_3 \cap U) = 3$. We get $S_2 \in U$ and that U contains at least 3 points of S_1. Assume for the moment the existence of a plane $A \subset U$ containing S_2 and at least 3 points of S_1; A is spanned by S_2 and two points of S_1; we took A containing 2 points of $S_3 \cap H$; since $w(A) \le 11$, we have $\sharp(S_1 \cap A) = 3$; we conclude taking U instead of H, unless there is $B \in B_1$ with $B_1 \cap U \cap S = \emptyset$; since $\langle S_2 \cup S_1\rangle$ is not a line, the elements of B_1 are either the lines spanned by two points of S_3 or they contain 3 points of S_1 and in the latter case they are contained in A.

Now assume the non-existence of A; in this case U is spanned by $S_2 \cup (U \cap S_1)$ and it contains 3 points of S_3. We may take U instead of H, because in this case no A exists and every $B \in B_1$ mets a point of $U \cap S_3$.

In the same way we do the case $c = 1$ and $e = 3$.

(b4) Assume $c = 0$, $e = 6$ (the proof works if $c = 0$ and $e < 6$).

(b4.1) First assume the existence of a line L with $\sharp(L \cap S_1) = 6$, i.e. assume that $\langle S_1 \rangle$ is a line. We claim that we may find $o \in S_3$ such that $\ell_o(S_3 \setminus \{o\}) \cap \ell_o(L) = \emptyset$ and $h^1(\mathbb{P}^3, \mathcal{I}_{\ell_o(S_3 \setminus \{o\}}(2)) = 0$. This is true, unless either a plane contains both $L \cup \ell_o(S_3 \setminus \{o\})$ (not possible, because $\langle S_2 \rangle$) or a point of $\ell_o(S_3 \setminus \{o\})$ is contained in $\ell_o(L)$ (not possible, because it would give a plane E containing S_1, o and another point of S_3 and hence with $w(E) > 11$) or $\ell_o(S_3 \setminus \{o\})$ is contained in a line (not possible, because no plane contains 4 points of S_3) or if $\ell_o(S_3 \setminus \{o\})$ contains 3 points in a line D intersecting $\ell_o(L)$. In the latter case there is a hyperplane containing $\sharp(S) - 1$ points of S and hence $h^1(\mathcal{I}_Z(5)) = 0$ by Lemma 3. By the claim there is a quadric Q containing S_3 and two points of S_1, with vertex containing o and with $L \nsubseteq Q$, i.e. with $\sharp(Q \cap S_1) = 4$. Since $Q \cap S \neq S$, the inductive assumption gives $h^1(\mathcal{I}_{Z \cap Q}(5)) = 0$. The scheme $W := \mathrm{Res}_Q(Z)$ is the union of $\{o\}$, 4 points of $S_1 \cap L$ and the schemes $2p$, $p \in S_3 \setminus \{o\}$. It is sufficient to prove that it satisfies the Segre conditions in degree 3. We have $w_W(\mathbb{P}^4) = 13$. W satisfies the Segre conditions for lines, because $L \cap S_3 = \emptyset$ and no line spanned by two points of S_3 meets L (it would give a plane E with $w(E) > 11$. Fix $A \in B_2$. First assume $L \subset A$. We have $w(A \setminus L) = w(A) - 6$ and hence A contains at least $w(A) - 10$ points of S_3; therefore $w_W(A) \leq 7$. Now assume $A \nsubseteq L$. Hence A contains at most one point of $L \cap S_1 = S_1$. If $S_1 \cap A = \emptyset$, then $\sharp(A \cap S) \geq 4$ and hence $w_W(A) \leq 7$. Now assume $\sharp(S_1 \cap A) = 1$. If $w(A) = 10$, then $\sharp(A \cap S_3) \geq 3$, while if $w(A) = 11$ we have $\sharp(A \cap S_3) > 3$, contradicting the inequality $w(A) \leq 11$. Fix $A \in B_2'$, i.e. $8 \leq w(A) \leq 9$. If $A \supset L$, then we get $w_W(A) \leq w(A) - 2 \leq 7$; if $A \not\supseteq L$ we first get $\sharp(S_1 \cap A) \leq 1$, then $S_3 \cap A \neq \emptyset$ and then, if $w(A) = 9$, $\sharp(S_3 \cap A) \geq 2$; in all cases we get $w_W(A) \leq 7$. Fix $U \in B_3$. First assume $U \supset L$. We have $w(U \setminus L) = w(U) - 6$ and hence $\sharp(S_3 \cap U) \geq 3$ with strict inequality if $w(U) = 16$. Hence $w_W(U) \leq 10$. Now assume $U \not\supseteq L$ and hence $\sharp(U \cap A_1) \leq 1$. We get $\sharp(S_3 \cap U) \geq 5$ and hence $S_3 \subset U$; since o appears with multiplicity one in W, we get $w(U) \leq 10$.

Fix $U \in B_3'$, i.e. $11 \leq w(A) \leq 13$. Since $w(U) - w(L) \geq 7$, we always have $\sharp(S_3 \cap U) \geq 3$ and hence $w_W(U) \leq 10$.

(b4.2) Now assume that $\langle S_1 \rangle$ is not a line. Let H be a hyperplane containing 4 points of S_3; there are at most two points of S_3 for which there are lines containing 3 points of S_1 and we take these points as some of the points of $S_3 \cap H$. If $S_3 \subset H$ we conclude as in step (b3). Thus we assume that S_3 is linearly independent. In this case the condition for lines is satisfied by our choice of $S_3 \cap H$. Assume the existence of a plane A such that $w_Y(A) \geq 10$. Since $w(S_1) = 6$, we get $\sharp(S_3 \cap A) \geq 2$. If $\sharp(A \cap S_3) \geq 3$, then $\sharp(S \cap A \cap H) \geq 2$ and hence $w(A) \geq 12$, a contradiction. Assume $\sharp(S_3 \cap A) = 2$. We get that A contains at least 5 points of S_1. Since $\langle S_1 \rangle$ is not a line, there is at most one plane containing at least 5 points of S_1 and it is sufficient to take inside $S_3 \cap H$ both points of S_3.

Assume the existence of a hyperplane U with $w_Y(U) \geq 14$. We have $U \neq H$, because $\sharp(H \cap S) \geq 4$. We have $\sharp(S_3 \cap U \cap H) \geq 2$ and $\sharp(S_3 \cap U) = 3$. We get that U contains at least 5 points of S_1. Assume for the moment the existence of a plane $A \subset U$ containing at least 5 points of S_1; we took A containing 2 points of $S_3 \cap H$; since $w(A) \leq 11$, we have $\sharp(S_1 \cap A) = 3$; we conclude taking U instead of H, unless there is $B \in B_1$ with $B_1 \cap U \cap S = \emptyset$; since $\langle S_1 \rangle$ is not a line, the elements of B_1 are either the lines spanned by two points of S_3 or they contain 3 points of S_1 and in the latter case they are contained in $A \subset U$.

Now assume the non-existence of A; in this case U is spanned by $U \cap S_1$ and it contains 3 points of S_3. We may take U instead of H, because in this case no A exists and every $B \in B_1$ mets a point of $U \cap S_3$. □

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