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COMMON FIXED POINTS FOR COMPATIBLE MAPPINGS AND ITS VARIANTS IN MULTIPLICATIVE METRIC SPACES

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Abstract: In this paper, we introduce the notions of compatible mappings and its variants in multiplicative metric spaces. Next, we prove common fixed point theorems for these mappings.

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1. Introduction and Preliminaries

It is well know that the set of positive real numbers $\mathbb{R}^+$ is not complete according to the usual metric. To overcome this problem, in 2008, Bashirov et al. [1] introduced the concept of multiplicative metric spaces as follows:

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Definition 1.1. Let X be a nonempty set. A multiplicative metric is a mapping $d : X \times X \to \mathbb{R}^+$ satisfying the following conditions:

(i) $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) \cdot d(z, y)$ for all $x, y, z \in X$ (multiplicative triangle inequality).

Example 1.2. ([4]) Let $\mathbb{R}^n_+$ be the collection of all n-tuples of positive real numbers. Let $d : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ be defind as follows:

$$d(x, y) = \left(\left|\frac{x_1}{y_1}\right| \cdot \left|\frac{x_2}{y_2}\right| \cdots \left|\frac{x_n}{y_n}\right|\right),$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$ and $|\cdot| : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$|a| = \begin{cases} a & \text{if } a \geq 1; \\ \frac{1}{a} & \text{if } a < 1. \end{cases}$$

Then it is obvious that all conditions of multiplicative metric are satisfied.

Example 1.3. ([5]) Let $d : \mathbb{R} \times \mathbb{R} \to [1, \infty)$ be defined as $d(x, y) = a^{|x-y|}$ for all $x, y \in X$, where $a > 1$. Then d is a multiplicative metric and hence (X, d) is a multiplicative metric space.

One can refer to [2] and [4] for detailed the multiplicative metric topology.

Definition 1.4. Let (X, d) be a multiplicative metric space. Then a sequence $\{x_n\}$ in X said to be

(1) a *multiplicative convergent* to x if for every multiplicative open ball $B_\epsilon(x) = \{y \mid d(x, y) < \epsilon\}$, $\epsilon > 1$, there exists a natural number N such that $n \geq N$, then $x_n \in B_\epsilon(x)$, that is, $d(x_n, x) \to 1$ as $n \to \infty$.

(2) a *multiplicative Cauchy sequence* if for all $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $m, n > N$, that is, $d(x_n, x_m) \to 1$ as $n \to \infty$.

(3) We call a multiplicative metric space *complete* if every multiplicative Cauchy sequence in it is multiplicative convergent to $x \in X$.

In 2012, Özavsar [4] gave the concept of multiplicative contraction mappings and proved some fixed point theorem of such mappings in a multiplicative metric space.

Definition 1.5. Let f be a mapping of a multiplicative metric space (X, d) into itself. Then f is called a *multiplicative contraction* if there exists a real constant $\lambda \in [0, 1)$ such that

$$d(fx, fy) \leq d^{\lambda}(x, y) \quad \text{for all } x, y \in X.$$

Gu et. al. [3] introduced the notion of commuting and weakly commuting mappings in a multiplicative metric space and proved some fixed point theorems for these mappings.

Definition 1.6. Let f and g be two mappings of a multiplicative metric space (X, d) into itself. Then f and g are called
(1) *commuting mappings* if $fgx = gfx$ for all $x \in X$.
(2) *weakly commuting mappings* if $d(fgx, gfx) \leq d(fx, gx)$ for all $x \in X$.

Remark 1.7. Notice that commuting mappings are obviously weakly commuting. However, the converse need not be true.

Example 1.8. Let $X = [0, 1]$ be usual multiplicative metric d on X. Define constant mappings f and $g : X \to X$ by $fx = a$ and $gx = b$ for all $x \in X$, where $a, b \in [0, 1]$ with $a \neq b$. Then f and g are weakly commuting but not commuting since

$$d(fgx, gfx) = \left|\frac{a}{b}\right| \leq d(fx, gx)$$

for all $x \in X$.

Example 1.9. Let $X = [0, 1]$ be a multiplicative metric d on X defined by $d(x, y) = a^{|x-y|}$ for all $x, y \in X$, where $a > 1$. Define mappings f and $g : X \to X$ by $fx = \frac{x}{3-x}$ and $gx = \frac{x}{3}$ for all $x \in X$. For any $x \in X$,

$$d(fgx, gfx) = a^{\left|\frac{2x^2}{(9-x)(9-3x)}\right|} \leq a^{\left|\frac{x^2}{9-3x}\right|} = d(fx, gx).$$

Then f and g are weakly commuting but f and g are not commuting since

$$fgx = \frac{x}{9-x} < \frac{x}{9-3x} = gfx$$

for any non-zero $x \in X$.

2. Relationships and Properties of Compatible Mappings and Its Variants

Now we introduce the notions of compatible mappings and its variants in the setting of multiplicative metric spaces as follows:

Definition 2.1. Let f and g be two mappings of a multiplicative metric space (X, d) into itself. Then f and g are called

(1) *compatible* if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

(2) *compatible of type* (A) if

$$\lim_{n\to\infty} d(fgx_n, ggx_n) = 1 \quad \text{and} \quad \lim_{n\to\infty} d(gfx_n, ffx_n) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

(3) *compatible of type* (B) if

$$\lim_{n\to\infty} d(fgx_n, ggx_n) \leq \left[\lim_{n\to\infty} d(fgx_n, ft) \cdot \lim_{n\to\infty} d(ft, ffx_n)\right]^{1/2}$$

and

$$\lim_{n\to\infty} d(gfx_n, ffx_n) \leq \left[\lim_{n\to\infty} d(gfx_n, gt) \cdot \lim_{n\to\infty} d(gt, ggx_n)\right]^{1/2},$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

(4) *compatible of type* (C) if

$$\lim_{n\to\infty} d(fgx_n, ggx_n) \leq \left[\lim_{n\to\infty} d(fgx_n, ft) \cdot \lim_{n\to\infty} d(ft, ffx_n) \cdot \lim_{n\to\infty} d(ft, ggx_n)\right]^{1/3}$$

and

$$\lim_{n\to\infty} d(gfx_n, ffx_n) \leq \left[\lim_{n\to\infty} d(gfx_n, gt) \cdot \lim_{n\to\infty} d(gt, ggx_n) \cdot \lim_{n\to\infty} d(gt, ffx_n)\right]^{1/3},$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

(5) *compatible of type* (P) if $\lim_{n\to\infty} d(ffx_n, ggx_n) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Next we give relationships and properties of compatible mappings and its variants.

Proposition 2.2. *Let f and g be compatible mappings of type (A). If one of f and g is continuous, then f and g are compatible.*

Proof. Since f and g be compatible of type (A), so $1 = \lim_{n\to\infty} d(fgx_n, ggx_n)$ and $1 = \lim_{n\to\infty} d(gfx_n, ffx_n)$, whenever $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Suppose that f is continuous. Then $\lim_{n\to\infty} ff = \lim_{n\to\infty} fgx_n = ft$ for some $t \in X$. Now we get $\lim_{n\to\infty} d(fgx_n, gfx_n) = 1$, i.e., f and g be compatible mappings.

Similarly, if g is continuous, then f and g be compatible mappings. □

Proposition 2.3. *Every pair of compatible mappings of type (A) is compatible of type (B).*

Proof. Suppose that f and g are compatible mappings of type (A). Then we have

$$\begin{aligned} 1 &= \lim_{n\to\infty} d(fgx_n, ggx_n) \\ &\leq \Big[\lim_{n\to\infty} d(fgx_n, ft) \cdot \lim_{n\to\infty} d(ft, ffx_n)\Big]^{1/2} \end{aligned}$$

and

$$\begin{aligned} 1 &= \lim_{n\to\infty} d(gfx_n, ffx_n) \\ &\leq \Big[\lim_{n\to\infty} d(gfx_n, gt) \cdot \lim_{n\to\infty} d(gt, ggx_n)\Big]^{1/2}, \end{aligned}$$

as derived. □

Proposition 2.4. *Let f and g be continuous mappings of a multiplicative metric space (X, d) into itself. If f and g are compatible of type (B), then they are compatible of type (A).*

Proof. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. Since f and g are continuous, we have

$$\begin{aligned} \lim_{n\to\infty} d(fgx_n, ggx_n) &\leq \Big[\lim_{n\to\infty} d(fgx_n, ft) \cdot \lim_{n\to\infty} d(ft, ffx_n)\Big]^{1/2} \\ &= 1 \end{aligned}$$

and

$$\lim_{n\to\infty} d(gfx_n, ffx_n) \le \Big[\lim_{n\to\infty} d(gfx_n, gt) \cdot \lim_{n\to\infty} d(gt, ggx_n)\Big]^{1/2} = 1.$$

Therefore, f and g compatible of type (A). This completes the proof. □

Proposition 2.5. *Let f and g be continuous mappings of a multiplicative metric space (X, d) into itself. If f and g are compatible of type (B), then they are compatible.*

Proof. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. Since f and g are continuous, we have

$$\lim_{n\to\infty} ffx_n = ft = \lim_{n\to\infty} fgx_n$$

and

$$\lim_{n\to\infty} gfx_n = gt = \lim_{n\to\infty} ggx_n$$

By multiplicative triangle inequality, we have

$$d(fgx_n, gfx_n) \le d(fgx_n, ggx_n) \cdot d(ggx_n, gfx_n).$$

Letting $n \to \infty$ and taking into account that f and g are compatible of type (B), we have

$$\begin{aligned}&\lim_{n\to\infty} d(fgx_n, gfx_n)\\ &\le \Big[\lim_{n\to\infty} d(fgx_n, ft) \cdot \lim_{n\to\infty} d(ft, ffx_n)\Big]^{1/2} \cdot \lim_{n\to\infty} d(ggx_n, gfx_n)\\ &= 1.\end{aligned}$$

Therefore, f and g are compatible. This completes the proof. □

Proposition 2.6. *Let f and g be continuous mappings of a multiplicative metric space (X, d) into itself. If f and g are compatible, then they are compatible of type (B).*

Proof. Since f and g are compatible so there exists $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$ for which $\lim_{n\to\infty} d(fgx_n, gfx_n) = 1$. Since f and g are continuous, we have

$$\lim_{n\to\infty} ffx_n = ft = \lim_{n\to\infty} fgx_n$$

and

$$\lim_{n\to\infty} gfx_n = gt = \lim_{n\to\infty} ggx_n,$$

so

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n.$$

Now

$$\lim_{n\to\infty} d(fgx_n, ggx_n) \le \left[\lim_{n\to\infty} d(fgx_n, ft) \cdot \lim_{n\to\infty} d(ft, ffx_n)\right]^{1/2}$$

and

$$\lim_{n\to\infty} d(gfx_n, ffx_n) \le \left[\lim_{n\to\infty} d(gfx_n, gt) \cdot \lim_{n\to\infty} d(gt, ggx_n)\right]^{1/2},$$

which implies that f and g be compatible of type (B). □

Proposition 2.7. *Let f and g be continuous mappings of a multiplicative metric space (X, d) into itself. Then*

(1) f and g are compatible if and only if they are compatible of type (B);

(2) f and g are compatible of type (A) if and only if they are compatible of type (B).

Proof. (1) One can easily prove it using Propositions 2.5 and 2.6.

(2) One can easily prove it using Propositions 2.3 and 2.4. □

Proposition 2.8. *Let f and g be compatible mappings of a multiplicative metric space (X, d) into itself. If $ft = gt$ for some $t \in X$, then $fgt = fft = ggt = gft$.*

Proof. Suppose that $\{x_n\}$ is a sequence in X defined by $x_n = t$, $n = 1, 2, \ldots$ for some $t \in X$ and $ft = gt$. Then we have $fx_n, gx_n \to ft$ as $n \to \infty$. Since f and g are compatible, we have

$$d(fgt, gft) = \lim_{n\to\infty} d(fgx_n, gfx_n) = 1.$$

Hence we have $fgt = ggt$. Therefore, since $ft = gt$, we have $fgt = fft = ggt = gft$. This completes the proof. □

From Proposition 2.8 we have

Proposition 2.9. *Let f and g be compatible mappings of a multiplicative metric space (X, d) into itself. Suppose that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. Then*

(a) $\lim_{n\to\infty} gfx_n = ft$ *if f is continuous at t.*

(b) $\lim_{n\to\infty} fgx_n = gt$ *if g is continuous at t.*

(c) $fgt = gft$ *and* $ft = gt$ *if f and g are continuous at t.*

Proof. (a) Suppose that f is continuous at t. Since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$, we have $fgx_n \to ft$ as $n \to \infty$. Since f and g are compatible, we have

$$\begin{aligned}\lim_{n\to\infty} d(gfx_n, ft) &= \lim_{n\to\infty} d(gfx_n, fgx_n) \cdot \lim_{n\to\infty} d(fgx_n, ft) \\ &= 1.\end{aligned}$$

Therefore, $\lim_{n\to\infty} gfx_n = ft$. This completes the proof.

(b) The proof of $\lim_{n\to\infty} fgx_n = gt$ follows by similar arguments as in (a).

(c) Suppose that f and g are continuous at t. Since $gx_n \to t$ as $n \to \infty$ and f is continuous at t, by (a), $gfx_n \to ft$ as $n \to \infty$. On the other hand, g is also continuous at t, $gfx_n \to gt$. Thus, we have $ft = gt$ by the uniqueness of limit and so by Proposition 2.8, $fgt = gft$. This completes the proof. □

Proposition 2.10. *Let f and g be compatible mappings of type (B) of a multiplicative metric space (X, d) into itself. If $ft = gt$ for some $t \in X$, then $fgt = fft = ggt = gft$.*

Proof. Suppose that $\{x_n\}$ is a sequence in X defined by $x_n = t$, $n = 1, 2, \ldots$ for some $t \in X$ and $ft = gt$. Then we have $fx_n, gx_n \to ft$ as $n \to \infty$. Since f and g are compatible of type (B), we have

$$\begin{aligned}d(fgt, ggt) &= \lim_{n\to\infty} d(fgx_n, ggx_n) \\ &\le \left[\lim_{n\to\infty} d(fgx_n, fft) \cdot \lim_{n\to\infty} d(fft, ffx_n)\right]^{1/2} \\ &= 1.\end{aligned}$$

Hence we have $fgt = ggt$. Therefore, since $ft = gt$, we have $fgt = fft = ggt = gft$. This completes the proof. □

From Proposition 2.10 we have

Proposition 2.11. *Let f and g be compatible mappings of type (B) of a multiplicative metric space (X, d) into itself. Suppose that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. Then*

(a) $\lim_{n\to\infty} ggx_n = ft$ if f is continuous at t.

(b) $\lim_{n\to\infty} ffx_n = gt$ if g is continuous at t.

(c) $fgt = gft$ and $ft = gt$ if f and g are continuous at t.

Proof. (a) Suppose that f is continuous at t. Since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$, we have $ffx_n, fgx_n \to ft$ as $n \to \infty$. Since f and g are compatible of type (B), we have

$$\begin{aligned}\lim_{n\to\infty} d(ft, ggx_n) &= \lim_{n\to\infty} d(fgx_n, ggx_n)\\ &\le \Big[\lim_{n\to\infty} d(fgx_n, ft) \cdot \lim_{n\to\infty} d(ft, ffx_n)\Big]^{1/2}\\ &= d(ft, ft) = 1.\end{aligned}$$

Therefore, $\lim_{n\to\infty} ggx_n = ft$. This completes the proof.

(b) The proof of $\lim_{n\to\infty} ffx_n = gt$ follows by similar arguments as in (a).

(c) Suppose that f and g are continuous at t. Since $gx_n \to t$ as $n \to \infty$ and f is continuous at t, by (a), $ggx_n \to ft$ as $n \to \infty$. On the other hand, g is also continuous at t, $ggx_n \to gt$. Thus, we have $ft = gt$ by the uniqueness of limit and so by Proposition 2.10, $fgt = gft$. This completes the proof. □

Remark 2.12. In Proposition 2.10, assume that f and g be compatible mappings of type (C) or of type (P) instead of of type (B). the conclusion of Proposition 2.10 still holds.

Remark 2.13. In Proposition 2.11, assume that f and g be compatible mappings of type (C) or of type (P) instead of of type (B). the conclusion of Proposition 2.11 still holds.

Remark 2.14. Every weakly commuting mapping is compatible but converse need not be true.

Indeed, since f and g be weakly commuting mappings, therefore, $d(fgx, gfx) \le d(fx, gx)$ for all $x \in X$. Let $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. Then

$d(fgx_n, gfx_n) \leq 1$, which implies that $\lim_{n\to\infty} d(fgx_n, gfx_n) = 1$, i.e., f and g be compatible mappings.

Example 2.15. Consider $X = [0, \infty)$ with multiplicative metric $d(x, y) = e^{|x-y|}$ on X. Define mappings f and $g : X \to X$ by $fx = x^3$ and $gx = 2x^3$. Then f and g are compatible but not weakly commuting.

Remark 2.16. Notions of compatible mappings and its variants are independent to each other as follows.

Example 2.17. Let $X = \mathbb{R}$, the set of all real numbers, with the usual multiplicative metric d, define f and $g : X \to X$ by

$$fx = \begin{cases} \frac{1}{x^4} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0 \end{cases} \quad \text{and} \quad gx = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Then f and g are not continuous at $t = 0$. Consider a sequence $\{x_n\}$ in X defined by $x_n = n$, $n = 1, 2, \ldots$. Then for $n \to \infty$ we have $fx_n = \frac{1}{n^4} \to t = 0$, $\quad gx_n = \frac{1}{n^2} \to t = 0$ and

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = \lim_{n\to\infty} \left| \frac{n^8}{n^8} \right| = 1.$$

However, the following limits do not exist:

$$\lim_{n\to\infty} d(fgx_n, ggx_n) = \lim_{n\to\infty} \left| \frac{n^8}{n^4} \right| = \infty,$$

$$\begin{aligned} &\left[\lim_{n\to\infty} d(fgx_n, f0) \cdot \lim_{n\to\infty} d(f0, ffx_n) \right]^{1/2} \\ &= \left[\lim_{n\to\infty} \left| \frac{n^8}{1} \right| \cdot \lim_{n\to\infty} \left| \frac{n^{16}}{1} \right| \right]^{1/2} \\ &= \infty \end{aligned}$$

and

$$\lim_{n\to\infty} d(gfx_n, ffx_n) = \lim_{n\to\infty} \left| \frac{n^{16}}{n^8} \right| = \infty,$$

$$\begin{aligned} &\left[\lim_{n\to\infty} d(gfx_n, g0) \cdot \lim_{n\to\infty} d(g0, ggx_n) \right]^{1/2} \\ &= \left[\lim_{n\to\infty} \left| \frac{n^8}{2} \right| \cdot \lim_{n\to\infty} \left| \frac{n^4}{2} \right| \right]^{1/2} \\ &= \infty. \end{aligned}$$

Also $\lim_{n\to\infty} d(fgx_n, ggx_n) = \infty$ and $\lim_{n\to\infty} d(gfx_n, ffx_n) = \infty$ and we get

$$\begin{aligned}&\left[\lim_{n\to\infty} d(fgx_n, f0)\cdot \lim_{n\to\infty} d(f0, ffx_n)\cdot \lim_{n\to\infty} d(f0, ggx_n)\right]^{1/3}\\ &= \left[\lim_{n\to\infty}\left|\frac{n^8}{1}\right|\cdot \lim_{n\to\infty}\left|\frac{n^{16}}{1}\right|\cdot \lim_{n\to\infty}\left|\frac{n^4}{1}\right|\right]^{1/3}\\ &= \infty\end{aligned}$$

and

$$\begin{aligned}&\left[\lim_{n\to\infty} d(gfx_n, g0)\cdot \lim_{n\to\infty} d(g0, ggx_n)\cdot \lim_{n\to\infty} d(g0, ffx_n)\right]^{1/3}\\ &= \left[\lim_{n\to\infty}\left|\frac{n^8}{2}\right|\cdot \lim_{n\to\infty}\left|\frac{n^{4}}{2}\right|\cdot \lim_{n\to\infty}\left|\frac{n^{16}}{2}\right|\right]^{1/3}\\ &= \infty\end{aligned}$$

Also

$$\lim_{n\to\infty} d(ffx_n, ggx_n) = \left|\frac{n^{12}}{1}\right| = \infty.$$

Therefore, f and g are compatible but they are not compatible of type (A), compatible of type (B), type (C) and of type (P).

Example 2.18. Let $X = [0, 6]$ with the usual multiplicative metric d, define f and $g : X \to X$ by

$$fx = \begin{cases} x & \text{if } x \in [0,3),\\ 6 & \text{if } x \in [3,6]\end{cases} \quad\text{and}\quad gx = \begin{cases} 6-x & \text{if } x \in [0,3),\\ 6 & \text{if } x \in [3,6].\end{cases}$$

Then f and g are not continuous at $t = 3$. Now, we assert that f and g are not compatible but they are compatible of type (A), of type (B), of type (C) and of type (P). To see this, suppose that $\{x_n\} \subset [0, 6]$ and that $fx_n, gx_n \to t$. By definition of f and g, $t \in [3, 6]$. Since f and g agree on $[3, 6]$, we have only to consider $t = 3$. So we can suppose that $x_n \to 3$ and that $x_n < 3$ for all n. Then $gx_n = 6 - x_n \to 3$ from the right and $fx_n = x_n \to 3$ from the left. Thus, since $x_n < 3$ and $6 - x_n > 3$ for all n,

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = \left|\frac{6}{6-x_n}\right| \to 2.$$

Further, we have

$$\lim_{n\to\infty} d(fgx_n, ggx_n) = \left|\frac{6}{6}\right| = 1,$$

$$\begin{aligned}&\left[\lim_{n\to\infty} d(fgx_n, f3) \cdot \lim_{n\to\infty} d(f3, ffx_n)\right]^{1/2}\\ &= \left[\lim_{n\to\infty}\left|\frac{6}{6}\right| \cdot \lim_{n\to\infty}\left|\frac{6}{x_n}\right|\right]^{1/2}\\ &= \sqrt{2}\end{aligned}$$

and

$$\lim_{n\to\infty} d(gfx_n, ffx_n) = \left|\frac{6-x_n}{x_n}\right| = 1,$$

$$\begin{aligned}&\left[\lim_{n\to\infty} d(gfx_n, g3) \cdot \lim_{n\to\infty} d(g3, ggx_n)\right]^{1/2}\\ &= \left[\lim_{n\to\infty}\left|\frac{6}{6-x_n}\right| \cdot \lim_{n\to\infty}\left|\frac{6}{6}\right|\right]^{1/2}\\ &= \sqrt{2}.\end{aligned}$$

Also $\lim_{n\to\infty} d(fgx_n, ggx_n) = 1$ and $\lim_{n\to\infty} d(gfx_n, ffx_n) = 1$ and we get

$$\left[\lim_{n\to\infty} d(fgx_n, f3) \cdot \lim_{n\to\infty} d(f3, ffx_n) \cdot \lim_{n\to\infty} d(f3, ggx_n)\right]^{1/3} = \sqrt[3]{2}$$

and

$$\left[\lim_{n\to\infty} d(gfx_n, g3) \cdot \lim_{n\to\infty} d(g3, ggx_n) \cdot \lim_{n\to\infty} d(g3, ffx_n)\right]^{1/3} = \sqrt[3]{2}$$

as $x_n \to 3$ and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 3$. Also we have

$$\lim_{n\to\infty} d(ffx_n, ggx_n) = \left|\frac{6}{6}\right| = 1.$$

Therefore, f and g are compatible mappings of type (A), of type (B), of type (C) and of type (P) but they are not compatible.

3. Fixed Points for Compatible Mappings and Its Variants

In 2014, He et. al. [2] proved the common fixed points for pair of weak commutative mappings on a complete multiplicative metric spaces as follow:

Theorem 3.1. *Let S, T, A and B be mappings of a complete multiplicative metric space (X, d) into itself satisfying the following conditions:*
$(C1)$ $SX \subset BX, \quad TX \subset AX$;

$$(C2) \qquad \begin{aligned} d(Sx, Ty) \leq [\max\{&d(Ax, By), d(Ax, Sx), \\ &d(By, Ty), d(Sx, By), d(Ax, Ty)\}]^{\lambda} \end{aligned}$$

for all $x, y \in X$, where $\lambda \in \left(0, \frac{1}{2}\right)$.
$(C3)$ *one of the mappings S, T, A and B is continuous.*
Assume that the pairs (A, S) and (B, T) are weakly commuting. Then S, T, A and B have a unique common fixed point.

Now we give the following theorem for compatible mappings.

Theorem 3.2. *Let S, T, A and B be mappings of a complete multiplicative metric space (X, d) into itself satisfying $(C1)$-$(C3)$.*
Assume that the pairs (A, S) and (B, T) are compatible. Then S, T, A and B have a unique common fixed point.

Proof. Since $SX \subset BX$. Now consider a point $x_0 \in X$, there exists $x_1 \in X$ such that $Sx_0 = Bx_1 = y_0$ for this point x_1 there exists $x_2 \in X$ such that $Tx_1 = Ax_2 = y_1$. Continuing in this way, one can construct sequences such that

$$y_{2n} = Sx_{2n} = Bx_{2n+1}, \quad y_{2n+1} = Tx_{2n+1} = Ax_{2n+2}.$$

From the proof of [2, Theorem 3.1], $\{y_n\}$ is multiplicative Cauchy sequence in X. Consequently, the subsequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ of the sequence $\{y_n\}$ also converge to z.

Now suppose that A is continuous. Then AAx_{2n}, ASx_{2n} converge to Az as $n \to \infty$. Since (A, S) are compatible on X, it follows from Proposition 2.9 SAx_{2n} converges to Az as $n \to \infty$.

We claim that $z = Az$. Consider

$$\begin{aligned} d(SAx_{2n}, Tx_{2n+1}) \leq [\max\{&d(AAx_{2n}, Bx_{2n+1}), d(AAx_{2n}, SAx_{2n}), \\ &d(Bx_{2n+1}, Tx_{2n+1}), d(SAx_{2n}, Bx_{2n+1}), \\ &d(AAx_{2n}, Tx_{2n+1})\}]^{\lambda}. \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(Az, z) &\leq [\max\{d(Az, z), d(Az, Az), d(z, z), d(Az, z), d(Az, z)\}]^{\lambda} \\ &= d^{\lambda}(Az, z). \end{aligned}$$

This implies that $d(Az, z) = 1$ implies $Az = z$.

Next we claim that $Sz = z$. Consider

$$\begin{aligned} d(Sz, Tx_{2n+1}) \leq [\max\{&d(Az, Bx_{2n+1}), d(Az, Sz), \\ &d(Bx_{2n+1}, Tx_{2n+1}), d(Sz, Bx_{2n+1}), \\ &d(Az, Tx_{2n+1})\}]^{\lambda}. \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(Sz, z) &\leq [\max\{d(z, z), d(z, Sz), d(z, z), d(Sz, z), d(Sz, z)\}]^{\lambda} \\ &= d^{\lambda}(Sz, z). \end{aligned}$$

This implies that $Sz = z$. Since $SX \subset BX$ and hence there exists a point $u \in X$ such that $z = Sz = Bu$.

We claim that $z = Tu$.

$$\begin{aligned} d(z, Tu) &= d(Sz, Tu) \\ &\leq [\max\{d(Az, Bu), d(Az, Sz), \\ &\qquad d(Bu, Tu), d(Sz, Bu), d(Az, Tu)\}]^{\lambda} \\ &= [\max\{d(z, z), d(z, z), d(z, Tu), d(z, z), d(z, Tu)\}]^{\lambda}. \end{aligned}$$

This implies that $z = Tu$. Since (B, T) is compatible in X and $Bu = Tu = z$, by Proposition 2.8, we have $BTu = TBu$ and hence $Bz = BTu = TBu = Tz$. Also, we have

$$\begin{aligned} d(z, Bz) &= d(Sz, Tz) \\ &\leq [\max\{d(Az, Bz), d(Az, Sz), \\ &\qquad d(Bz, Tz), d(Sz, Bz), d(Az, Tz)\}]^{\lambda} \\ &= [\max\{d(z, Bz), d(z, z), d(Bz, Tz), d(z, Bz), d(z, Bz)\}]^{\lambda}. \end{aligned}$$

This implies that $z = Bz$. Hence, $z = Bz = Tz = Az = Sz$. Therefore, z is common fixed point of S, T, A and B.

Similarly, we can also complete the proof when B is continuous.

Next suppose that S is continuous. Then SSx_{2n}, SAx_{2n} converge to Az as $n \to \infty$. Since A and S are compatible on X, it follows from Proposition 2.9 that ASx_{2n} converges to Az as $n \to \infty$.

Consider

$$\begin{aligned} d(SSx_{2n}, Tx_{2n+1}) \leq [\max\{&d(ASx_{2n}, Bx_{2n+1}), d(ASx_{2n}, SSx_{2n}), \\ &d(Bx_{2n+1}, Tx_{2n+1}), d(SSx_{2n}, Bx_{2n+1}), \\ &d(ASx_{2n}, Tx_{2n+1})\}]^{\lambda}. \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} d(Sz, z) &\leq [\max\{d(Sz, z), d(Sz, Sz), d(z, z), d(Sz, z), d(Sz, z)\}]^\lambda \\ &= d^\lambda(Sz, z), \end{aligned}$$

which implies that $Sz = z$. Since $SX \subset BX$ and hence there exists a point $v \in X$ such that $z = Sz = Bw$. Consider

$$\begin{aligned} d(SSx_{2n}, Tv) \leq [\max\{&d(ASx_{2n}, Bv), d(ASx_{2n}, SSx_{2n}), \\ &d(Bv, Tv), d(SSx_{2n}, Bv), d(ASx_{2n}, Tv)\}]^\lambda. \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(z, Tv) &\leq [\max\{d(z, z), d(z, z), d(z, Tv), d(z, z), d(z, Tv)\}]^\lambda \\ &= d^\lambda(z, Tv). \end{aligned}$$

This implies that $z = Tv$. Since B and T are compatible on X and $Bv = Tv = z$, by Proposition 2.8, we have $BTv = TBv$ and hence $Bz = BTv = TBv = Tz$. Consider

$$\begin{aligned} d(Sx_{2n}, Tz) \leq [\max\{&d(Ax_{2n}, Bz), d(Ax_{2n}, Sx_{2n}), \\ &d(Bz, Tz), d(Sx_{2n}, Bz), d(Ax_{2n}, Tz)\}]^\lambda. \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} d(z, Tz) &\leq [\max\{d(z, Tz), d(z, z), d(Tz, Tz), d(z, Tz), d(z, Tz)\}]^\lambda \\ &= d^\lambda(z, Tz). \end{aligned}$$

This implies that $Tz = z$. Since $TX \subset AX$, so there exists a point $w \in X$ such that $z = Tz = Aw$. Consider

$$\begin{aligned} d(Sw, z) &= d(Sw, Tz) \\ &\leq [\max\{d(Aw, Bz), d(Aw, Sw), \\ &\qquad d(Bz, Tz), d(Sw, Bz), d(Aw, Tz)\}]^\lambda \\ &= [\max\{d(z, z), d(z, Sw), d(Tz, Tz), d(Sw, z), d(z, z)\}]^\lambda. \end{aligned}$$

This implies that $Sw = z$. Since S and A are compatible on X, $Sw = Aw = z$, by Proposition 2.8, we have $ASw = SAw$ and hence $Az = ASw = SAw = Sz$. That is, $z = Az = Sz = Bz = Tz$. Therefore, z is common fixed point of S, T, A and B.

Similarly, we can complete the proof when T is continuous.

Finally, suppose that z and w $(z \neq w)$ are two common fixed points of S, T, A and B. Consider

$$\begin{aligned} d(z,w) &= d(Sz,Tw) \\ &\leq [\max\{d(Az,Bw), d(Az,Sz), \\ &\qquad d(Bw,Tw), d(Sz,Bw), d(Az,Tw)\}]^{\lambda} \\ &= [\max\{d(z,w), d(z,z), d(w,w), d(z,w), d(z,w)\}]^{\lambda} \\ &= d^{\lambda}d(z,w). \end{aligned}$$

This implies that $z = w$. Therefore, z is a unique common fixed point of S, T, A and B. This completes the proofs. □

Next we give the following theorem for compatible mappings of type (A).

Theorem 3.3. *Let S, T, A and B be mappings of a complete multiplicative metric space (X, d) into itself satisfying $(C1)$-$(C3)$.*

Assume that the pairs (A, S) and (B, T) are compatible of type (A). Then S, T, A and B have a unique common fixed point.

Proof. Suppose that A is continuous. Since (A, S) are compatible of type (A), by Proposition 2.2, the pair (A, S) is compatible, so result easily follows from Theorem 3.2.

Similarly if B is continuous and (B, T) is compatible of type (A), then (B, T) is compatible so result easily follows from Theorem 3.2.

Similarly, we can complete the proof when S or T is continuous. This completes the proofs. □

Also we give the following theorem for compatible mappings of type (B).

Theorem 3.4. *Let S, T, A and B be mappings of a complete multiplicative metric space (X, d) into itself satisfying $(C1)$-$(C3)$.*

Assume that the pairs (A, S) and (B, T) are compatible of type (B). Then S, T, A and B have a unique common fixed point.

Proof. From the proof of Theorem 3.2, $\{y_n\}$ is multiplicative Cauchy sequence in X. Consequently, the subsequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ of $\{y_n\}$ converge to z.

Suppose that S is continuous. Then SSx_{2n}, SAx_{2n} converge to Sz as $n \to \infty$. Since the pair (A, S) is compatible of type (B), it follows from Proposition 2.11 that AAx_{2n} converges to Sz as $n \to \infty$.

Consider

$$d(SAx_{2n}, Tx_{2n+1}) \leq [\max\{d(AAx_{2n}, Bx_{2n+1}), d(AAx_{2n}, SAx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), d(SAx_{2n}, Bx_{2n+1}), d(Aax_{2n}, Tx_{2n+1})\}]^{\lambda}.$$

Letting $n \to \infty$, we get

$$d(Sz, z) \leq [\max\{d(Sz, z), d(Sz, Sz), d(z, z), d(Sz, z), d(Sz, z)\}]^{\lambda}.$$

This implies that $Sz = z$. Since $SX \subset BX$ there exists a point $u \in X$ such that $z = Sz = Bu$. Consider

$$d(SAx_{2n}, Tu) \leq [\max\{d(AAx_{2n}, Bu), d(AAx_{2n}, SAx_{2n}), d(Bu, Tu), d(SAx_{2n}, Bu), d(AAx_{2n}, Tu)\}]^{\lambda}.$$

Letting $n \to \infty$, we get

$$d(Sz, Tu) \leq d^{\lambda}(Sz, Tu).$$

This implies that $Tu = Sz$ $(z = Tu)$. Since the pair (B, T) is compatible of type (B) and $Bu = z = Tu$. By Proposition 2.10 we have $TBu = BTu$ and so $Bz = BTu = TBu = Tz$. Consider

$$d(Sx_{2n}, Tz) \leq [\max\{d(Ax_{2n}, Bz), d(Ax_{2n}, Sx_{2n}), d(Bz, Tz), d(Sx_{2n}, Bz), d(Ax_{2n}, Tz)\}]^{\lambda}.$$

Letting $n \to \infty$, we get

$$d(z, Tz) \leq d^{\lambda}(z, Tz),$$

which implies that $Tz = z$. Since $TX \subset AX$, there exists a point $v \in X$ such that $z = Tz = Av$. Consider

$$d(Sv, Tz) \leq [\max\{d(Av, Bz), d(Av, Sv), d(Bz, Tz), d(Sv, Bz), d(Av, Tz)\}]^{\lambda},$$

which implies that

$$d(Sv, z) \leq d^{\lambda}(Sv, z).$$

This implies $Sv = z$. Since the pair (A, S) is compatible of type (B) and $Sv = z = Av$, it follows from Proposition 2.10 that $Sz = SAv = ASv = Az$. Therefore, $Az = Bz = Sz = Tz = z$ and hence z is common fixed point of S, T, A and B.

Now suppose that A is continuous. Then AAx_{2n} and ASx_{2n} converge to Az as $n \to \infty$. Since (A,S) is compatible of type (B), it follows from Proposition 2.11 that SSx_{2n} converges to Az as $n \to \infty$. Consider

$$\begin{aligned} d(SSx_{2n}, Tx_{2n+1}) \leq [\max\{&d(ASx_{2n}, Bx_{2n+1}), d(ASx_{2n}, SSx_{2n}),\\ &d(Bx_{2n+1}, Tx_{2n+1}), d(SSx_{2n}, Bx_{2n+1}),\\ &d(ASx_{2n}, Tx_{2n+1})\}]^{\lambda}. \end{aligned}$$

Letting $n \to \infty$, we get

$$d(Az, z) \leq d^{\lambda}(Az, z).$$

This implies $Az = z$. Consider

$$\begin{aligned} d(Sz, Tx_{2n+1}) \leq [\max\{&d(Az, Bx_{2n+1}), d(Az, Sz),\\ &d(Bx_{2n+1}, Tx_{2n+1}), d(Sz, Bx_{2n+1}),\\ &d(Az, Tx_{2n+1})\}]^{\lambda}. \end{aligned}$$

Letting $n \to \infty$, we get

$$d(Sz, z) \leq d^{\lambda}(Sz, z).$$

This implies $Sz = z$. Since $SX \subset BX$, there exists a point $w \in X$ such that $z = Sz = Bw$. Consider

$$\begin{aligned} d(z, Tw) &= d(Sz, Tu)\\ &\leq [\max\{d(Az, Bw), d(Az, Sz),\\ &\qquad d(Bw, Tw), d(Sz, Bw), d(Az, Tw)\}]^{\lambda}\\ &= [\max\{d(z, z), d(z, z), d(z, Tw), d(z, z), d(z, Tw)\}]^{\lambda}. \end{aligned}$$

This implies that $z = Tw$. Since (B,T) is compatible of type (B) and $Bw = z = Tw$, from Proposition 2.10, $TBw = BTw$ and so $Bz = BTw = TBw = Tz$. Consider

$$\begin{aligned} d(Sz, Tz) &\leq [\max\{d(z, Tz), d(z, z), d(Tz, Tz), d(z, Tz), d(z, Tz)\}]^{\lambda}\\ &= d^{\lambda}(z, Tz). \end{aligned}$$

This implies that $z = Tz$. Therefore, z is common fixed point of S, T, A and B.

Similarly, we can complete the proof when B or T is continuous.

Finally, if z and w $(z \neq w)$ are two common fixed points, then we have

$$\begin{aligned} d(z, w) &= d(Sz, Tw)\\ &\leq [\max\{d(Az, Bw), d(Az, Sz),\\ &\qquad d(Bw, Tw), d(Sz, Bw), d(Az, Tw)\}]^{\lambda}\\ &= d^{\lambda}(z, w). \end{aligned}$$

This implies that $z = w$. Therefore, z is a unique common fixed point of S, T, A and B. This completes the proofs. □

Now we give the following theorem for compatible mappings of type (C).

Theorem 3.5. *Let S, T, A and B be mappings of a multiplicative metric space (X, d) into itself satisfying $(C1)$-$(C3)$.*

Assume that the pair (A, S) and (B, T) are compatible of type (C). Then S, T, A and B have a unique common fixed point.

Proof. From the proof of Theorem 3.2, $\{y_n\}$ is multiplicative Cauchy sequence in X. Consequently, the subsequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ of $\{y_n\}$ also converge to z.

Suppose that S is continuous. Then SSx_{2n}, SAx_{2n} converge to Sz as $n \to \infty$. Since the pair (A, S) is compatible of type (C), it follows from Remark 2.13 that AAx_{2n} converges to Sz as $n \to \infty$.

We claim that $Sz = z$. Consider

$$\begin{aligned} d(SAx_{2n}, Tx_{2n+1}) \leq [\max\{&d(AAx_{2n}, Bx_{2n+1}), d(AAx_{2n}, SAx_{2n}), \\ &d(Bx_{2n+1}, Tx_{2n+1}), d(SAx_{2n}, Bx_{2n+1}), \\ &d(AAx_{2n}, Tx_{2n+1})\}]^{\lambda}. \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} d(Sz, z) &\leq [\max\{d(Az, z), d(Sz, Sz), d(z, z), d(Sz, z), d(Sz, z)\}]^{\lambda} \\ &= d^{\lambda}(Sz, z). \end{aligned}$$

This implies that $Sz = z$. Since $SX \subset BX$, there exists a point $u \in X$ such that $z = Sz = Bu$. Consider

$$\begin{aligned} d(SAx_{2n}, Tw) \leq [\max\{&d(AAx_{2n}, Bw), d(AAx_{2n}, SAx_{2n}), \\ &d(Bu, Tu), d(SAx_{2n}, Bu), d(AAx_{2n}, Tu)\}]^{\lambda}. \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} d(Sz, Tu) &\leq [\max\{d(Sz, Sz), d(Sz, Sz), \\ &\qquad d(Sz, Tu), d(Bu, Bu), d(Sz, Tu)\}]^{\lambda} \\ &= d^{\lambda}(Sz, Tu). \end{aligned}$$

This implies that $Sz = Tu$ ($z = Tu$). Since the pair (B, T) is compatible of type (C) and $Bu = z = Tu$, by Remark 2.12, we get $TBu = BTu$ and so $Bz = BTu = TBu = Tz$. Consider

$$\begin{aligned} d(Sx_{2n}, Tz) \leq [\max\{&d(Ax_{2n}, Bz), d(Ax_{2n}, Sx_{2n}), \\ &d(Bz, Tz), d(Sx_{2n}, Bz), d(Ax_{2n}, Tz)\}]^{\lambda}. \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(z, Tz) &\leq [\max\{d(z, Tz), d(z, z), 1, d(z, Tz), d(z, Tz)\}]^{\lambda} \\ &= d^{\lambda}(z, Tz). \end{aligned}$$

This implies that $Tz = z$. Since $TX \subset AX$, there exists a point $v \in X$ such that $z = Tz = Av$. Consider

$$\begin{aligned} d(Sv, z) &= d(Sv, Tz) \\ &\leq [\max\{d(Av, Bz), d(Av, Sv), \\ &\qquad d(Bz, Tz), d(Sv, Bz), d(Av, Tz)\}]^{\lambda} \\ &\leq [\max\{d(z, z), d(Sv, z), d(z, z), d(Sv, z), d(z, z)\}]^{\lambda} \\ &= d^{\lambda}(Sv, z). \end{aligned}$$

This implies that $z = Sv$. Since the pair (A, S) is compatible of type (C) and $Sv = z = Av$, by Remark 2.12, $ASv = SAv$ we have $Sz = SAv = ASv = Az$. Therefore $Bz = Az = Tz = Sz = z$ and hence z is common fixed point of S, T, A and B.

Suppose that A is continuous. Then AAx_{2n} and ASx_{2n} converge to Az as $n \to \infty$. Since the pair (A, S) is compatible of type (C), it follows from Remark 2.13 that SSx_{2n} converges to Az as $n \to \infty$.

Also we have

$$\begin{aligned} d(SSx_{2n}, Tx_{2n+1}) \leq [\max\{&d(ASx_{2n}, Bx_{2n+1}), d(ASx_{2n}, SSx_{2n}), \\ &d(Bx_{2n+1}, Tx_{2n+1}), d(SSx_{2n}, Bx_{2n+1}), \\ &d(ASx_{2n}, Tx_{2n+1})\}]^{\lambda}. \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} d(Az, z) &\leq [\max\{d(Az, z), d(Az, Av), d(z, z), d(Az, z), d(Az, z)\}]^{\lambda} \\ &= d^{\lambda}(Az, z). \end{aligned}$$

This implies that $Az = z$. Consider

$$\begin{aligned} d(Sz, Tx_{2n+1}) \leq [\max\{&d(Az, Bx_{2n+1}), d(Az, Sz), \\ &d(Bx_{2n+1}, Tx_{2n+1}), d(Sz, Bx_{2n+1}), \\ &d(Az, Tx_{2n+1})\}]^{\lambda}. \end{aligned}$$

Letting $n \to \infty$, we get

$$d(Sz, z) \leq d^{\lambda}(Sz, z).$$

This implies $Sz = z$. Since $SX \subset BX$, there exists a point $w \in X$ such that $z = Sz = Bw$. Again, we have

$$\begin{aligned} d(z, Tw) &= d(Sz, Tw) \\ &\leq [\max\{d(Az, Bw), d(Az, Sz), \\ &\qquad d(Bw, Tw), d(Sz, Bw), d(Az, Tw)\}]^{\lambda} \\ &= d^{\lambda}(z, Tw). \end{aligned}$$

This implies that $Tw = z$. Since (B, T) is compatible of type (C) and $Bw = z = Tw$, by Remark 2.12 we have $TBw = BTw$ and so $Bz = BTw = TBw = Tz$. Also consider

$$d(z, Tz) = d(Sz, Tz) \leq d^{\lambda}(z, Tz).$$

Thus implies that $Tz = z$. Hence $Tz = Bz = Sz = Az = z$. Therefore, z is the common fixed point of S, T, A and B.

Similarly, we can complete the proof when B or T is continuous.

Uniqueness follows easily. This completes the proofs. □

Finally we give the following theorem for compatible mappings of type (P).

Theorem 3.6. *Let S, T, A and B be mappings of a complete multiplicative metric space (X, d) into itself satisfying $(C1)$-$(C3)$.*

Assume that the pairs (A, S) and (B, T) are compatible of type (P). Then S, T, A and B have a unique common fixed point.

Proof. From the proof of Theorem 3.2, $\{y_n\}$ is multiplicative Cauchy sequence in X. Consequently, the subsequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ of $\{y_n\}$ converge to z as $n \to \infty$.

Suppose that S is continuous. Then SSx_{2n}, SAx_{2n} converge to Sz as $n \to \infty$. Since (A, S) is compatible of type (P), it follows from Remark 2.13 that AAx_{2n} converges to Sz as $n \to \infty$.

We claim $Sz = z$. Consider

$$d(SAx_{2n}, Tx_{2n+1}) \leq [\max\{d(AAx_{2n}, Bx_{2n+1}), d(AAx_{2n}, SAx_{2n}x), d(Bx_{2n+1}, Tx_{2n+1}), d(SAx_{2n}x, Bx_{2n+1}), d(AAx_{2n}, Tx_{2n+1})\}]^{\lambda}.$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(Sz, z) &\leq [\max\{d(Sz, z), d(Sz, Sz), d(z, z), d(Sz, z), d(Sz, z)\}]^{\lambda} \\ &= d^{\lambda}(Sz, z). \end{aligned}$$

This implies that $Sz = z$. Since $SX \subset BX$, so there exists a point $u \in X$ such that $z = Sz = Bu$.

Now we claim that $Tu = z$. Consider

$$d(Sx_{2n}, Tu) \leq [\max\{d(Ax_{2n}, Bu), d(Ax_{2n}, Sx_{2n}), d(Bu, Tu), d(Sx_{2n}, Bu), d(Ax_{2n}, Tu)\}]^{\lambda}.$$

Letting $n \to \infty$, we get

$$\begin{aligned} d(z, Tu) &\leq [\max\{d(z, z), d(z, z), d(z, Tu), d(z, z), d(z, Tu)\}]^{\lambda} \\ &= d^{\lambda}(z, Tu). \end{aligned}$$

This implies that $z = Tu$. Therefore, $Bu = Tu = z$. Since (B, T) is compatible of type (P), by Remark 2.12, we have $TTu = BBu$, which implies that $d(Bz, Tz) = 1$. Hence $Tz = Bz$.

Now we claim that $Tz = z$. Consider

$$d(Sx_{2n}, Tz) \leq [\max\{d(Ax_{2n}, Bz), d(Ax_{2n}, Sx_{2n}), d(Bz, Tz), d(Sx_{2n}, Bz), d(Ax_{2n}, Tz)\}]^{\lambda}.$$

Letting $n \to \infty$, we have

$$d(z, Tz) \leq d^{\lambda}(z, Tz),$$

which implies that $z = Tz$. Therefore, $Bz = Tz = z$. Since $TX \subset AX$, so there exists a point $v \in X$ such that $z = Tz = Av$.

Now to claim that $Sv = z$. Consider

$$\begin{aligned} d(Sv, z) &= d(Sv, Tz) \\ &\leq [\max\{d(Av, Bz), d(Av, Sv), d(Bz, Tz), d(Sv, Bz), d(Av, Tz)\}]^{\lambda} \\ &= d^{\lambda}(Sv, z). \end{aligned}$$

This implies that $z = Sv$. Therefore $z = Sv = Av$. Since (A, S) is compatible of type (P), by Remark 2.12, we have $SSv = AAv$, which implies that $d(Sz, Az) = 1$. Hence $Sz = Az$. Since $Az = Bz = Sz = Tz = z$, z is common fixed point of S, T, A and B.

Similarly, we can complete the proof when A or B or T is continuous.

The uniqueness follows easily. This completes the proofs. □

Now we give an example in support of our main theorems.

Example 3.7. Let $X = [1, \infty)$ with usual multiplicative metric $d(x, y) = \left|\frac{x}{y}\right|$. Consider the following self-mappings $Sx = x$, $Tx = x^2$, $Bx = 2x^4 - 1$ and $Ax = 2x^2 - 1$ for all $x \geq 1$.

$SX = TX = BX = AX = X$, $SX \subset BX, TX \subset AX$;

S, T, A and B are all continuous mappings;

(iii) the pairs (A, S) and (B, T) are compatible, and they are compatible mappings of type (A), of type (B), of type (C) and of type (P).

Consider $x_n = 1 + \frac{1}{n}$ for $n \geq 1$. Then $x_n \to 1$ as $n \to \infty$. Now we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = 1 = t \in X$$

as $n \to \infty$. Also we have

$$\begin{aligned}
&\lim_{n\to\infty} d(ASx_n, SAx_n) = 1, \quad \lim_{n\to\infty} d(BTx_n, TBx_n) = 1,\\
&\lim_{n\to\infty} d(ASx_n, SSx_n) = 1, \quad \lim_{n\to\infty} d(SAx_n, AAx_n) = 1,\\
&\lim_{n\to\infty} d(BTx_n, TTx_n) = 1, \quad \lim_{n\to\infty} d(TBx_n, BBx_n) = 1.
\end{aligned}$$

(iv) For $\lambda = \frac{1}{3}$,

$$\begin{aligned}
d(Sx, Ty) \leq [\max\{&d(Ax, By), d(Ax, Sx),\\
&d(By, Ty), d(Sx, By), d(Ax, Ty)\}]^{\lambda}
\end{aligned}$$

is satisfied for all $x, y \in X$. Therefore, all conditions of main theorems are satisfied, and 1 is a unique common fixed point of S, T, A and B.

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