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SOME GRAPHS IN C_f2 BASED ON f-COLORING

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Abstract: Let $G = (V, E)$ be a graph and $f : V \to Z^+$a positive integer be a function. An *f-coloring* of G is a coloring of the edges such that every vertex $v \in V$ is incident to at most $f(v)$ edges of the same color. The minimum number of colors of an f-coloring of G is the *f-chromatic index* $\chi'_f(G)$ of G. Based on the f-chromatic index, a graph G can be either in class C_f1, if $\chi'_f(G) = \Delta_f(G)$, or in class C_f2, if $\chi'_f(G) = \Delta_f(G) + 1$, where $\Delta_f(G) = \max_{x\in V} \lceil d(v)/f(v)\rceil$. In this paper, we give some sufficient conditions for a graph to be in C_f2. One of the results is a generalization of a theorem by Zhang *et al.* (2008). Moreover, we show that, when f is constant and a divisor of $(n-1)$, a maximal subgraph of the complete graph K_n which is in class C_f1 has precisely $\binom{n}{2} - \Delta_f(K_n)/2$ edges.

AMS Subject Classification: 05C15
Key Words: edge coloring, f-coloring, f-chromatic index

1. Introduction

Let $G = (V, E)$ be a finite and simple graph and let f be a function from V to a positive integer set. An f-*coloring* of G is a coloring of the edges such that

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each vertex v has at most $f(v)$ edges colored with a same color. The minimum number of colors needed to define an f-coloring of G is called the f-*chromatic index* of G, denoted by $\chi'_f(G)$. In case $f(v) = 1$ for every $v \in V$, an f-coloring is just a proper edge-coloring of the graph.

The f–colorings arise in many applications, including network design, scheduling problems and the file transfer problem in computer networks. For instance, the file transfer problem in a computer network is modeled as follows. Each computer is represented by a vertex and the file transfer process between two computers is represented by an edge. Each computer v has a limited number $f(v)$ of communication ports. If we assume that the transfer time is constant for every file, we can use an f-coloring to manage transferring all files with minimum time. Being a generalization of the proper edge–coloring, which was shown to be an NP–complete problem by Holyer [5], the f-coloring problem is also NP-complete.

Let $d(v)$ denote the degree of $v \in V$. By extending the well–known theorem of Vizing (1965) to f-colorings, Hakimi and Kariv [4] showed that

$$\Delta_f(G) \leq \chi'_f(G) \leq \Delta_f(G) + 1, \tag{1}$$

where

$$\Delta_f(G) = \Sigma_{i=1}^{n} \left\lceil \frac{d(v)}{f(v)} \right\rceil. \tag{2}$$

According to (1), G is said to be in *class-1*, denoted by C_f1, if $\chi'_f(G) = \Delta_f(G)$; otherwise G is in *class-2*, denoted by C_f2. As for the corresponding classification for proper edge–colorings, graphs in class C_f2 are much less common than the ones in class C_f1.

Hakimi and Kariv [4] showed that any bipartite graph is in C_f1. Moreover, for any graph, they showed that, if $f(v)$ is even for each $v \in G$, then G is in C_f1. Zhang and Liu [6] gave the following classification of complete graphs based on f-colorings.

Theorem 1 (Zhang and Liu [6]). *If k and n are odd integers with $n \geq 3$, $f(v) = k$ for all $v \in V(K_n)$, and k divides $n-1$, then the complete graph K_n is in C_f2. Otherwise, K_n is in C_f1.*

Zhang , Wang and Liu [7] gave the following sufficient condition for a regular graph to be in C_f2.

Theorem 2 (Zhang, Wang and Liu [7]). *Let $n \geq 1$. Let G be a Δ-regular* *graph of order $2n+1$ and let k be an odd positive integer such that k divides Δ. If $f(v) = k$ for each $v \in V$, then $G \in C_f2$.*

In [1] we gave a classification of some graphs containing wheels, namely the corona product of either the complement of a complete graph, or a path, or a star with a cycle, based on f-coloring. As well known, the corona product is not commutative. Hence, in [2] we provide a characterization of the corona product of a cycle with some graphs based f-chromatic index.

In this paper we give a sufficient condition for a graph to be in C_f2 (Theorem 3). We also generalize Theorem 2 of Zhang, Wang and Liu [7], by dropping the condition of regularity of a graph (see Theorem 4). Finally, in Theorem 5, we provide an edge-reduction of a complete graph which is in C_f2 in order to get a maximal subgraph of K_n which is in C_f1.

2. Main Results

Theorem 3 below provides a sufficient condition for a graph containing odd number of edges to be in C_f2. It can be seen as an extension to f-colorings of the fact that odd cycles belong to class 2 under ordinary proper edge-colorings.

Theorem 3. *Let $G = (V, E)$. Let $f : V \to$ a positive integer be a function defined as $f(v) = d(v)/k$ for each $v \in V$, where k is even. If $|E|$ is odd, then $G \in C_f2$.*

Proof. Suppose on the contrary that $G \in C_f1$. Let C be an f-coloring which uses $\Delta_f(G) = k$ colors.

We have

$$\sum_{v \in V} f(v) = \frac{1}{k} \sum_{v \in V} d(v) = \frac{2}{k}|E|.$$

Since $|E|$ is odd and k is even, we have $\sum_{v \in V} f(v) \equiv 1 \pmod 2$. Since the f–coloring C uses k–colors, each vertex v is incident with $d(v)/k$ colors. But this means that each color class contains $(1/2)\sum_{v \in V} d(v)/k = (1/2)\sum_{v \in V} f(v)$ edges, which is impossible since $\sum_{v \in V} f(v)$ is odd. □

Figure 1 shows an example of a graph $G \in C_f2$ where $f(v) = d(v)/2$. The fact that $G \in C_f2$ follows from Theorem 3. However, this fact is not implied by Theorem 2 of Zhang, Wang and Liu [7], which applies only to regular graphs. However, this fact is not implied by Theorem 2 of Zhang, Wang and

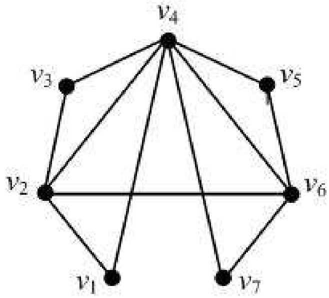

Figure 1: A non-regular graph that is in C_f2

Liu [7], which applies only to regular graphs. In Theorem 4, we give a sufficient condition for a graph to be in C_f2 which generalizes Theorem 2 since it does not require the graph to be regular and $f(v)$ is not necessarily odd for each vertex.

Theorem 4. *Let $G = (V, E)$ be a graph and let $g = \gcd(d(v) : v \in V)$. Let $f : V \to$ a positive integer be defined as $f(v) = d(v)/k$ for each $v \in V$, where k divides g. If $V_1 = \{v \in V : f(v) \equiv 1 \pmod 2\}$ and $|V_1| \equiv 1 \pmod 2$, then $G \in C_f2$.*

Proof. Suppose on the contrary that $G \in C_f1$. Let C be an f-coloring that uses $\Delta_f(G) = k$ colors. Let, $V_2 = V \setminus V_1$. Since the f–coloring C uses k–colors, then, by the definition of f, each vertex v is incident with $d(v)/k$ colors. Therefore each color class E_i must contain

$$|E_i| = \frac{1}{2}\sum_{v\in V} d(v)/k = \frac{1}{2}\sum_{v\in V} f(v) = \frac{1}{2}\left(\sum_{v\in V_1} f(v) + \sum_{v\in V_2} f(v)\right)$$

edges. By the definition of V_2 we have $\sum_{v\in V_2} f(v) \equiv 0 \pmod 2$. Since $|V_1|$ is odd then, again by the definition of the set V_1, we have $\sum_{v\in V_1} f(v) \equiv 1 \pmod 2$ and therefore $\sum_{v\in V_1} f(v) + \sum_{v\in V_2} f(v)$ is an odd number. But then $|E_i|$ is not an integer, a contradiction. □

According to the classification of Zhang *et al.* [6] as we rewrite in Theorem 1, the only functions f for which complete graphs with order odd n are in class C_f2 are the ones defined by $f(v) = k$ for some odd divisor k of $n-1$. A natural question is how many edges have to be removed from K_n for such an f so that the remaining graph moves from C_f2 to C_f1. The following theorem answers the question.

Theorem 5. *Let k and n be odd integers with $n \geq 3$ and k divides $(n-1)$, and let $f(v) = k$ for all $v \in V(K_n)$. If H is a maximal subgraph of K_n in C_f1 then*

$$|E(H)| = |E(K_n)| - (n-1)/2k.$$

Proof. From Theorem 1 we know that $K_n \in C_f2$. We divide the proof into two cases as follows.

Case 1. $f(v) = 1$ for all $v \in V(K_n)$.

In this case an f–coloring is a proper edge–coloring. Since H is in C_f1 it can be f–colored with $(n-1)$ colors. This means that H can be decomposed into $(n-1)$ edge–disjoint matchings. Since each matching has at most $(n-1)/2$ edges, we have $|E(H)| \leq (n-1)^2/2 = |E(K_n)| - (n-1)/2$. On the other hand, it is well–known that the complete graph K_n where n is odd, admits a decomposition into n edge–disjoint matchings with $(n-1)/2$ edges each. By removing one of the matchings we obtain a subgraph H which can be properly edge colored with $(n-1)$ colors and has $|E(K_n)| - (n-1)/2$ edges.

Case 2. We have $f(v) = 2r+1$ for all $v \in V$ and some positive integer r.

Let $\Delta = \Delta_f = (n-1)/(2r+1)$ and let H be a subgraph of K_n obtained by removing a matching M of size $\Delta/2$. We will show that $H \in C_f1$. This will show that there is a subgraph of K_n with $\binom{n}{2} - (n-1)/(4r+2)$ edges belonging to the class C_f1.

Let $W = \{w_1, \ldots, w_\Delta\}$ be the set vertices incident to the edges in M and let $U = \{u_1, \ldots, u_{n-\Delta}\}$ be the set remaining vertices of K_n. Consider the complete subgraph $K_{n-\Delta}$ of K_n induced by U. It is well–known that the edge set of a complete graph of odd order can be decomposed into edge–disjoint hamiltonian cycles (Lucas Theorem). Consider such a decomposition of $K_{n-\Delta}$ into $(n-\Delta-1)/2 = r(n-1)/(2r+1)$ edge–disjoint hamiltonian cycles. Partition this set of hamiltonian cycles into $(n-1)/(2r+1)$ classes where the cardinality of every class is r. Give a different color to each class. This results an f-coloring of the edges of $K_{n-\Delta}$ with Δ colors in such a way that each vertex is incident with $2r$ edges of each color.

We next partition the vertices of $U \setminus \{u_{n-\Delta}\}$ into Δ subsets $U_1, \ldots, U_\Delta$ with $(n-\Delta-1)/\Delta = 2r$ elements each. Color the edges joining w_i with U_j with color $(i+j) \pmod{\Delta}$ (in case $(i+j) \equiv 0 \pmod{\Delta}$, we color the edges by color Δ). At this point every vertex in $U \setminus \{u_{n-\Delta}\}$ is incident with precisely $2r+1$ edges of each color, and every vertex in W is incident with $2r$ edges of each color. We next color the edge $u_{n-\Delta}w_i$ with color i, so that $u_{n-\Delta}$ is incident to $2r+1$ edges of each color and vertex w_i is incident with $2r+1$ edges of color i and $2r$ edges of color j for each $j \neq i$. It remains to show that we can color the edges of the subgraph $H[W]$ of H induced by W in such a way that vertex w_i is not incident with color i for each i.

Consider the complete graph $K_{\Delta+1}$. It admits a decomposition into $\Delta+1$ matchings of size $\Delta/2$ each. Consider the proper edge coloring of $K_{\Delta+1}$ given by this decomposition. By removing one color class, we obtain a graph with a proper edge–coloring where one vertex incident to Δ colors and the Δ remaining vertices each adjacent to $\Delta-1$ colors. The subgraph induced by these Δ vertices is isomorphic to $H[W]$, and, out of the Δ colors, each vertex misses one. Moreover, no two vertices have the same missing color. By appropriately renaming the colors, one may require that this missing color at vertex w_i is color i. By using this coloring in the graph $H[W]$ above we complete a coloring of H with Δ colors such that every vertex is incident with precisely $2r+1$ edges of each color. Therefore $H \in C_f1$.

To complete the proof we need to show that every subgraph $H \subset K_n$ that belongs to C_f1 has at most $\binom{n}{2} - (n-1)/2k$ edges. But this follows from the fact that in an f–coloring with Δ colors of any graph H with $\binom{n}{2} - (n-1)/2k$ edges, every vertex must be incident with precisely k edges of each color, so that the addition of any edges increases the frequency of one color above k. This completes the proof. □

References

[1] Adiwijaya, A.N.M. Salman, D. Suprijanto, E.T. Baskoro, A classification of some graphs containing wheels based on f-coloring, *East West Journal of Mathematics* Special volume (2010) 200-207.

[2] Adiwijaya, A.N.M. Salman, D. Suprijanto, E.T. Baskoro, A characterization of the corona product of a cycle with some graphs based f-chromatic index, *AIP Conference Proceedings 1450 (2012) 155-158*

[3] Adiwijaya, A.N.M. Salman, D. Suprijanto, E.T. Baskoro, On the f-colorings of the corona product of a cycle with some graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing* **71** (2009) 235-241.

[4] S.L. Hakimi, O. Kariv, A generalization of edge-coloring in graphs, *Journal of Graph Theory* **10** (1986) 139-154.

[5] I. Holyer, The NP-completness of edge-coloring, *SIAM Journal of Computation* **10:4** (1981) 718 - 720.

[6] X. Zhang, G. Liu, The classification of K_n on f-colorings, *Journal of Applied Mathematics and Computing* **19:1-2** (2006) 127-133.

[7] X. Zhang, J. Wang, G. Liu, The classification of regular graphs on f-colorings, *Ars Combinatoria* **86** (2008) 273-280.