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COMMON FIXED POINTS FOR WEAKLY COMPATIBLE MAPPINGS SATISFYING IMPLICIT FUNCTIONS IN MULTIPLICATIVE METRIC SPACES

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Abstract: In this paper we prove common fixed point theorems for weakly compatible mapping in multiplicative metric spaces. Next, we prove common fixed point theorems for weakly compatible mappings along with E.A and common limit range properties.

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1. Introduction and Preliminaries

It is well know that the set of positive real numbers $\mathbb{R}_+$ is not complete according

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to the usual metric. To overcome this problem, in 2008, Bashirov et al. [4] introduced the concept of multiplicative metric spaces as follows:

Definition 1.1. Let X be a nonempty set. A multiplicative metric is a mapping $d : X \times X \to \mathbb{R}_+$ satisfying the following conditions:

(i) $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) \cdot d(z, y)$ for all $x, y, z \in X$ (multiplicative triangle inequality).

Example 1.2. ([11]) Let $\mathbb{R}^n_+$ be the collection of all n-tuples of positive real numbers. Let $d : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ be defind as follows:

$$d(x, y) = \left(\left| \frac{x_1}{y_1} \right| \cdot \left| \frac{x_2}{y_2} \right| \cdots \left| \frac{x_n}{y_n} \right| \right),$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$ and $| \cdot | : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$|a| = \begin{cases} a & \text{if } a \geq 1; \\ \frac{1}{a} & \text{if } a < 1. \end{cases}$$

Then it is obvious that all conditions of a multiplicative metric are satisfied.

Example 1.3. ([15]) Let $d : \mathbb{R} \times \mathbb{R} \to [1, \infty)$ be defined as $d(x, y) = a^{|x-y|}$, where $x, y \in \mathbb{R}$ and $a > 1$. Then d is a multiplicative metric.

Remark 1.4. We note that the Example 1.2 is valid for positive real numbers and Example 1.3 is valid for all real numbers.

One can refer to [6, 11] for detailed multiplicative metric topology.

Definition 1.5. Let (X, d) be a multiplicative metric space. Then a sequence $\{x_n\}$ in X said to be

(1) a *multiplicative convergent* to x if for every multiplicative open ball $B_\epsilon(x) = \{y \mid d(x, y) < \epsilon\}$, $\epsilon > 1$, there exists a natural number N such that $n \geq N$, then $x_n \in B_\epsilon(x)$, that is, $d(x_n, x) \to 1$ as $n \to \infty$.

(2) a *multiplicative Cauchy sequence* if for all $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $m, n > N$, that is, $d(x_n, x_m) \to 1$ as $n \to \infty$.

(3) We call a multiplicative metric space *complete* if every multiplicative Cauchy sequence in it is multiplicative convergent to $x \in X$.

Remark 1.6. The set of positive real numbers $\mathbb{R}_+$ is not complete according to the usual metric. Let $X = \mathbb{R}_+$ and the sequence $\{x_n\} = \{\frac{1}{n}\}$. It is obvious $\{x_n\}$ is a Cauchy sequence in X with respect to usual metric and X is not a complete metric space, since $0 \notin \mathbb{R}_+$. In case of a multiplicative metric space, we take a sequence $\{x_n\} = \{a^{\frac{1}{n}}\}$, where $a > 1$. Then $\{x_n\}$ is a Cauchy sequence since for $n \geq m$,

$$\begin{aligned} d(x_n, x_m) &= \left|\frac{x_n}{x_m}\right| = \left|\frac{a^{\frac{1}{n}}}{a^{\frac{1}{m}}}\right| = \left|a^{\frac{1}{n}-\frac{1}{m}}\right| \\ &\leq a^{\frac{1}{m}-\frac{1}{n}} < a^{\frac{1}{m}} < \epsilon \quad \text{if } m > \frac{\log a}{\log \epsilon}, \end{aligned}$$

where $|a| = \begin{cases} a & \text{if } a \geq 1, \\ \frac{1}{a} & \text{if } a < 1. \end{cases}$ Also, $\{x_n\} \to 1$ as $n \to \infty$ and $1 \in \mathbb{R}_+$. Hence (X, d) is a complete multiplicative metric space.

In 2012, Özavsar and Çevikel [11] gave the concept of multiplicative contraction mappings and proved some fixed point theorem of such mappings in a multiplicative metric space.

Definition 1.7. Let f be a mapping of a multiplicative metric space (X, d) into itself. Then f is said to be a *multiplicative contraction* if there exists a real constant $\lambda \in [0, 1)$ such that

$$d(fx, fy) \leq d^{\lambda}(x, y) \quad \text{for all } x, y \in X.$$

Gu et. al. [5] introduced the notion of commutative and weak commutative mappings in a multiplicative metric space and proved some fixed point theorems for these mappings.

Definition 1.8. Let f and g be two mappings of a multiplicative metric space (X, d) into itself. Then f and g are said to be

(1) *commutative mappings* if $fgx = gfx$ for all $x \in X$.

(2) *weak commutative mappings* if $d(fgx, gfx) \leq d(fx, gx)$ for all $x \in X$.

Notice that commuting mappings are obviously weakly commuting. However, the converse need not be true.

Example 1.9. Let $X = [0, 1]$ be a multiplicative metric d on X defined by $d(x, y) = a^{|x-y|}$, where for all $x, y \in X$ and $a > 1$. Define mappings f and $g : X \to X$ by $fx = \frac{x}{3-x}$ and $gx = \frac{x}{3}$ for all $x \in X$. For any $x \in X$,

$$d(fgx, gfx) = a^{\left|\frac{2x^2}{(9-x)(9-3x)}\right|} \leq a^{\left|\frac{x^2}{9-3x}\right|} = d(fx, gx).$$

Then f and g are weakly commuting but f and g are not commuting since

$$fgx = \frac{x}{9-x} < \frac{x}{9-3x} = gfx$$

for any non-zero $x \in X$.

In metric spaces, they introduced the notions of weak compatibility [9, 10], E.A. property [1] and common limit range property [8, 17].

Now, we introduce the notions in multiplicative metric spaces

Definition 1.10. Let f and g be two mappings of a multiplicative metric space (X, d) into itself. Then f and g are said to be *weakly compatible* if they commute at coincidence points, that is, if $ft = gt$ for some $t \in X$ implies that $fgt = gft$.

Notice that weakly commuting mappings are obviously weakly compatible. However, the converse need not be true.

Example 1.11. Let $X = [0, \infty)$ be a multiplicative metric d on X defined by $d(x, y) = a^{|x-y|}$, where for all $x, y \in X$ and $a > 1$. Define mappings f and $g : X \to X$ by $fx = x^2$ and $gx = 2x^2$ for all $x \in X$. So we have $fgx = 4x^4$ and $gfx = 2x^4$ for all $x \in X$. For any $x \in X$,

$$d(fgx, gfx) = a^{2x^4} \not\leq a^{x^2} = d(fx, gx).$$

Then f and g are not weakly commuting but f and g are weakly compatible since $f0 = g0$ for some $0 \in X$ implies $fg0 = gf0$.

Definition 1.12. Let f and g be two mappings of a multiplicative metric space (X, d) into itself. Then f and g are said to satisfy *E.A property* if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Example 1.13. Let $X = [1, \infty)$. Define $d : X^2 \to R_+$ by $d(x, y) = a^{|x-y|}$, $a > 1$. Then (X, d) be a multiplicative metric space. Define $f, g : X \to X$ as $fx = \frac{1}{x}$ and $gx = \frac{1}{x^2}$. Consider the sequence $\{x_n\} = \{a^{1/n}\}$, $a > 1$ in X. Now $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 1 \in X$. Hence f and g satisfies E.A. property.

Definition 1.14. Let f and g be two mappings of a multiplicative metric space (X, d) into itself. Then f and g are said to satisfy *CLR_g property* (common limit range of g property) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gt$ for some $t \in X$.

Example 1.15. Let $X = [1, \infty)$. Define $d : X^2 \to R_+$ by $d(x, y) = a^|x-y|$, $a > 1$. Then (X, d) be a multiplicative metric space. Define $f, g : X \to X$ as $fx = x^2$ and $gx = x^3$. Consider the sequence $\{x_n\} = \{a^{1/n}\}$, $a > 1$ in X. Now $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 1 = g1$ and $1 \in X$. Hence f and g satisfies CLR_g property.

Definition 1.16. Let f, g and h, k be mappings of a multiplicative metric space (X, d) into itself. Then the pairs f, g and h, k are said to share *common limit in the range of g property* if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} hy_n = \lim_{n\to\infty} ky_n = gt$ for some $t \in X$.

2. Main Results

Recently, Popa [12] used the implicit function rather than contraction conditions to prove fixed point theorems in metric spaces. The strength of implicit relation unifies several contraction conditions at the same time. This fact is seen from examples furnished in Popa [12]. Implicit relations on metric spaces have been used by many authors (for details see [2, 3, 7, 13, 14, 16, 18] and their references therein).

In this section, we define a suitable class of the implicit function involving five real non-negative arguments as follows:

Let Φ denote the family of functions such that $\phi : \mathbb{R}_+^5 \to \mathbb{R}_+$ is continuous and increasing in each coordinate variable and $\phi(t, 1, 1, t, t) \leq t$, $\phi(1, t, 1, t, 1) \leq t$, $\phi(1, 1, t, 1, t) \leq t$, $\phi(t_1, t_1, t, 1, t_1t) \leq t_1t$, $\phi(t_1, t, t_1, t_1t, 1) \leq t_1t$ for every $t, t_1 \in \mathbb{R}_+$ $(t, t_1 \geq 1)$.

Obviously $\phi(1, 1, 1, 1, 1) = 1$.

There exists many functions ϕ which belongs to Φ:

Example 2.1. Let $\phi : \mathbb{R}_+^5 \to \mathbb{R}_+$ be defined by

$$\phi(t_1, t_2, t_3, t_4, t_5) = t_1 + t_2 + t_3 - t_4 - t_5,$$

then $\phi \in \Phi$.

Example 2.2. Let $\phi : \mathbb{R}_+^5 \to \mathbb{R}^+$ be defined by

$$\phi(t_1, t_2, t_3, t_4, t_5) = t_1 + 1 - \max\{t_2, t_3, t_4, t_5\},$$

then $\phi \in \Phi$.

Example 2.3. Let $\phi : \mathbb{R}_+^5 \to \mathbb{R}^+$ be defined by

$$\phi(t_1, t_2, t_3, t_4, t_5) = \max\{t_1, t_2, t_3, t_4, t_5\},$$

then $\phi \in \Phi$.

Example 2.4. Let $\phi : \mathbb{R}_+^5 \to \mathbb{R}^+$ be defined by

$$\phi(t_1, t_2, t_3, t_4, t_5) = [\max\{t_1, t_2, t_3, t_4, t_5\}]^{1/2},$$

then $\phi \in \Phi$.

Now we prove the following theorems for weakly compatible mappings satisfying the implicit function in a multiplicative metric space as follow:

Theorem 2.5. *Let A, B, S and T be mappings of a multiplicative metric space (X, d) into itself satisfying*

$$(C1) \qquad SX \subset BX, \quad TX \subset AX;$$

$$(C2) \qquad d(Sx, Ty) \leq [\phi\big(d(Ax, By), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ax, Ty)\big)]^{\lambda}$$

for all $x, y \in X$, where $\lambda \in (0, \frac{1}{2})$ and $\phi \in \Phi$;

$(C3)$ Assume that the pairs A, S and B, T are weakly compatible;

$(C4)$ One of the subspace AX or BX or SX or TX is complete.

Then A, B, S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. Since $SX \subset BX$, there exists $x_1 \in X$ such that $Sx_0 = Bx_1 = y_0$. Now for this x_1 there exists $x_2 \in X$ such that $Tx_1 = Ax_2 = y_1$. Similarly, we can inductively define a sequence $\{y_n\}$ such that

$$Sx_{2n} = Bx_{2n+1} = y_{2n}, \quad Tx_{2n+1} = Ax_{2n+2} = y_{2n+1}.$$

From $(C2)$, we have

$$\begin{aligned}
&d(y_{2n}, y_{2n+1}) \\
&= d(Sx_{2n}, Tx_{2n+1}) \\
&\leq [\phi\big(d(Ax_{2n}, Bx_{2n+1}), d(Ax_{2n}, Sx_{2n}), \\
&\quad d(Bx_{2n+1}, Tx_{2n+1}), d(Sx_{2n}, Bx_{2n+1}), d(Ax_{2n}, Tx_{2n+1})\}]^{\lambda} \\
&= [\phi\big(d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), \\
&\quad d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n}), d(y_{2n-1}, y_{2n+1})\}]^{\lambda} \\
&\leq [\phi\big(d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), \\
&\quad d(y_{2n}, y_{2n+1}), 1, d(y_{2n-1}, y_{2n}) \cdot d(y_{2n}, y_{2n+1})\}]^{\lambda} \\
&\leq d^{\lambda}(y_{2n-1}, y_{2n}) \cdot d^{\lambda}(y_{2n}, y_{2n+1}).
\end{aligned}$$

This implies that $d(y_{2n}, y_{2n+1}) \leq d^{\frac{\lambda}{1-\lambda}}(y_{2n-1}, y_{2n})$. On putting $\frac{\lambda}{1-\lambda} = h$.

$$d(y_{2n}, y_{2n+1}) \leq d^{h}(y_{2n-1}, y_{2n}).$$

Similarly we obtain

$$d(y_{2n+1}, y_{2n+2}) \leq d^{h}(y_{2n}, y_{2n+1}).$$

Hence

$$d(y_n, y_{n+1}) \leq d^{h}(y_{n-1}, y_n) \leq d^{h^2}(y_{n-2}, y_{n-1}) \leq \cdots \leq d^{h^n}(y_0, y_1)$$

for all $n \geq 2$. Let $m, n \in \mathbb{N}$ such that $m \geq n$. Then we get

$$\begin{aligned}
d(y_m, y_n) &\leq d(y_m, y_{m-1}) \cdot d(y_{m-1}, y_{m-2}) \cdots d(y_{n+1}, y_n) \\
&\leq d^{h^{m-1}}(y_1, y_0) \cdot d^{h^{m-2}}(y_1, y_0) \cdots d^{h^n}(y_1, y_0) \\
&\leq d^{\frac{h^n}{1-h}}(y_1, y_0).
\end{aligned}$$

Letting limit as $m, n \to \infty$, we have $d(y_m, y_n) \to 1$. Therefore $\{y_n\}$ is a multiplicative Cauchy sequence.

Now, suppose that AX is complete there exists $u \in AX$ such that

$$y_{2n+1} = Tx_{2n+1} = Ax_{2n+2} \to u \quad \text{as } n \to \infty.$$

Consequently, we can find $v \in X$ such that $Av = u$. Further a multiplicative Cauchy sequence $\{y_n\}$ has a convergent subsequence $\{y_{2n+1}\}$, therefore, the sequence $\{y_n\}$ converges and hence a subsequence $\{y_{2n}\}$ also converges. Thus we have

$$y_{2n} = Sx_{2n} = Bx_{2n+1} \to u \quad \text{as } n \to \infty.$$

We claim $Sv = u$. Putting $x = v$ and $y = x_{2n+1}$ in $(C2)$, we get

$$\begin{aligned} d(Sv, y_{2n+1}) &= d(Sv, Tx_{2n+1}) \\ &\le [\phi\big(d(Av, Bx_{2n+1}), d(Av, Sv), \\ &\qquad d(Bx_{2n+1}, Tx_{2n+1}), d(Sv, Bx_{2n+1}), d(Av, Tx_{2n+1})\big)]^{\lambda}. \end{aligned}$$

Letting $n \to \infty$, we have

$$d(Sv, u) \le [\phi\big(1, d(u, Sv), 1, d(Sv, u), 1\big)]^{\lambda} \le d^{\lambda}(Sv, u),$$

this implies that $d(Sv, u) = 1$ and hence $u = Sv$. Since $u = Sv \in SX \subset BX$ there exists $w \in X$ such that $u = Bw$.

We claim $Tw = u$. Putting $x = v$ and $y = w$ in $(C2)$, we have

$$\begin{aligned} d(u, Tw) &= d(Sv, Tw) \\ &\le [\phi\big(d(Av, Bw), d(Av, Sv), \\ &\qquad d(Bw, Tw), d(Sv, Bw), d(Av, Tw)\big)]^{\lambda} \\ &= [\phi\big(1, 1, d(u, Tw), 1, d(u, Tw)\big)]^{\lambda} \\ &\le d^{\lambda}(u, Tw), \end{aligned}$$

this implies that $d(u, Tw) = 1$ and hence $u = Tw$. Hence we get $u = Av = Sv$, that is, v is a coincidence point of A and S and $u = Bw = Tw$, that is w is a coincidence point of B and T. Therefore $Av = Sv = Bw = Tw = u$.

Since the pairs A, S and B, T are weakly compatible, we have

$$Su = S(Av) = A(Sv) = Au = w_1 \text{ (say)}$$

and

$$Tu = T(Bw) = B(Tw) = Bu = w_2 \text{ (say)}.$$

From $(C2)$, we have

$$\begin{aligned} d(w_1, w_2) &= d(Su, Tu) \\ &\leq [\phi\big(d(Au, Bu), d(Au, Su), \\ &\qquad d(Bu, Tu), d(Su, Bu), d(Au, Tu)\big)]^{\lambda} \\ &= [\phi\big(d(w_1, w_2), 1, 1, d(w_1, w_2), d(w_1, w_2)\big)]^{\lambda} \\ &\leq d^{\lambda}(w_1, w_2), \end{aligned}$$

this implies that $w_1 = w_2$. and hence we have $Su = Au = Tu = Bu$.

Again using $(C2)$, we have

$$\begin{aligned} d(Sv, Tu) &\leq [\phi\big(d(Av, Bu), d(Av, Sv), \\ &\qquad d(Bu, Tu), d(Sv, Bu), d(Av, Tu)\big)]^{\lambda} \\ &= [\phi\big(d(Sv, Tu), 1, 1, d(Sv, Tu), d(Sv, Tu)\big)]^{\lambda} \\ &\leq d^{\lambda}(Sv, Tu), \end{aligned}$$

this implies that $Sv = Tu$ $(u = Tu)$ and hence we have $u = Su = Au = Tu = Bu$. Therefore u is a common fixed point of A, B, S and T.

Similarly, we can complete the proof for cases in which BX or TX or SX is complete.

The uniqueness can be easily follows from $(C2)$. This completes the proof. □

In Theorem 2.3. if we put $S = T$, then we obtains the following corollary.

Corollary 2.6. *Let A, B and S be mappings of a multiplicative metric space (X, d) into itself satisfying*

$(c1)$
$$SX \subset BX, \quad SX \subset AX;$$

$(c2)$
$$\begin{aligned} d(Sx, Sy) &\leq [\phi\big(d(Ax, By), d(Ax, Sx), \\ &\qquad d(By, Sy), d(Sx, By), d(Ax, Sy)\}]^{\lambda} \end{aligned}$$

for all $x, y \in X$, where $\lambda \in (0, \frac{1}{2})$ and $\phi \in \Phi$;

$(c3)$ *the pairs A, S and B, S are weakly compatible;*

$(c4)$ *one of the subspace AX or BX or SX is complete.*

Then A, B and S have a unique common fixed point.

In Theorem 2.3, if we put $A = B = I$, then we obtains the corollary.

Corollary 2.7. *Let S and T be mappings of a multiplicative metric space (X, d) into itself satisfying*

$$(c5)\qquad \begin{aligned} d(Sx, Ty) \leq [\phi\big(&d(x, y), d(x, Sx), \\ &d(y, Ty), d(Sx, y), d(x, Ty)\big)]^{\lambda} \end{aligned}$$

for all $x, y \in X$, where $\lambda \in (0, \frac{1}{2})$ and $\phi \in \Phi$;

$(c6)$ one of the subspace SX or TX is complete.

Then S and T have a unique common fixed point.

Next we prove the following theorems for weakly compatible mappings with E.A. property satisfying the implicit function in a multiplicative metric space as follow:

Theorem 2.8. *Let A, B, S and T be mappings of a multiplicative metric space (X, d) into itself satisfying the conditions $(C1)$, $(C2)$, $(C3)$ and the following conditions:*

$(C5)$ One of the subspace AX or BX or SX or TX is closed subset of X;

$(C6)$ the pairs A, S and B, T satisfies the E.A. property.

Then A, B, S and T have a unique common fixed point.

Proof. Suppose that the pair A, S satisfies the E.A property. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$. Since $SX \subset BX$, there exists a sequence $\{y_n\}$ in X such that $Sx_n = By_n$. Hence $\lim_{n\to\infty} By_n = z$.

Now, suppose that BX is a closed subset of X, there exists a point $u \in X$ such that $Bu = z$.

We will show that $\lim_{n\to\infty} Ty_n = z$. From inequality $(C2)$, we have

$$\begin{aligned} d(Sx_n, Ty_n) \leq [\phi\big(&d(Ax_n, By_n), d(Ax_n, Sx_n), \\ &d(By_n, Ty_n), d(Sx_n, By_n), d(Ax_n, Ty_n)\big)]^{\lambda}. \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(z, Ty_n) &\leq [\phi\big(1, 1, d(z, Ty_n), 1, d(z, Ty_n)\big)]^{\lambda} \\ &\leq d^{\lambda}(z, Ty_n), \end{aligned}$$

which implies that $\lim_{n\to\infty} d(z, Ty_n) = 1$. Thus we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = z = Bu$$

for some $u \in X$.

Putting $x = x_n$ and $y = u$ in $(C2)$, we have

$$\begin{aligned} d(Sx_n, Tu) \le [\phi\big(&d(Ax_n, Bu), d(Ax_n, Sx_n),\\ &d(Bu, Tu), d(Sx_n, Bu), d(Ax_n, Tu)\big)]^\lambda. \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(Bu, Tu) &\le [\phi\big(1, 1, d(Bu, Tu), 1, d(Bu, Tu)\big)]^\lambda \\ &\le d^\lambda(Bu, Tu), \end{aligned}$$

which implies that $Bu = Tu$. Since the pair B, T is weakly compatible, we have $BTu = TBu$ and then $BBu = BTu = TBu = TTu$.

On the other way, since $TX \subset AX$, there exists $v \in X$ such that $Tu = Av$.

Next we claim that $Av = Sv$. Putting $x = v$ and $y = u$, we have

$$\begin{aligned} d(Sv, Tu) \le [\phi\big(&d(Av, Bu), d(Av, Sv),\\ &d(Bu, Tu), d(Sv, Bu), d(Av, Tu)\big)]^\lambda. \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(Sv, Av) &\le [\phi\big(1, d(Av, Sv), 1, d(Av, Sv), 1\big)]^\lambda \\ &\le d^\lambda(Sv, Av), \end{aligned}$$

which implies that $Sv = Av$ and hence $Bu = Tu = Av = Sv$. Since the pair A, S is weakly compatible, we have $ASv = SAv$ and then $SSv = SAv = ASv = AAv$.

Next we claim that $SAv = Av$. Putting $x = Av$ and $y = u$, we have

$$\begin{aligned} d(SA, Tu) &= d(SAv, Tu) \\ &\le [\phi\big(d(AAv, Bu), d(AAv, SAv),\\ &\qquad d(Bu, Tu), d(SAv, Bu), d(AAvv, Tu)\big)]^\lambda \\ &\le d^\lambda(SAv, Av), \end{aligned}$$

which implies that $SAv = Av$ and hence $SAv = Av = AAv$. Hence Av is common fixed point of A and S.

Also, one can easily prove that $BBu = Bu = TBu$, that is, Bu is common fixed point of B and T. As $Av = Bu$, Av is a common fixed point of A, B, S and T.

Similarly we can complete the proof for cases in which AX, or TX, or SX is a closed subset of X.

The uniqueness follows easily from inequality $(C2)$. This completes the proof. □

Finally, we prove the following theorems for weakly compatible mappings with common limit range property satisfying the implicit function in a multiplicative metric space.

Lemma 2.9. *Let A, B, S and T be mappings of a multiplicative metric space (X, d) satisfying the conditions $(C1)$, $(C2)$ and the following condition:*

$(C7)$ the pair A, S satisfies CLR_A property or the pair B, T satisfies CLR_B property.

Then the pairs A, S and B, T share the common limit in the range of A property or B property.

Proof. Suppose that the pair A, S satisfies common limit range of A property. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = Az$ for some $z \in X$. Since $SX \subset BX$, so for each x_n there exists y_n in X such that $Sx_n = By_n$. Then $\lim_{n\to\infty} By_n = Az$. Hence, we have $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = Az$.

Now we claim that $\lim_{n\to\infty} Ty_n = Az$. Putting $x = x_n$ and $y = y_n$ in $(C2)$, we have

$$\begin{aligned} d(Sx_n, Ty_n) \leq [\phi\big(&d(Ax_n, By_n), d(Ax_n, Sx_n), \\ &d(By_n, Ty_n), d(Sx_n, By_n), d(Ax_n, Ty_n)\big)]^\lambda. \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(Az, Ty_n) &\leq [\phi\big(1, 1, d(Az, Ty_n), 1, d(Az, Ty_n)\big)]^\lambda \\ &= d^\lambda(Az, Ty_n), \end{aligned}$$

which implies that $\lim_{n\to\infty} Ty_n = Az$.

Then the pairs A, S and B, T share the common limit in the range of A property.

Similarly we can complete the proof for cases in which the pair B, T satisfies common limit in the range of B property. This completes the proof. □

Theorem 2.10. *Let A, B, S and T be mappings of a multiplicative metric space (X, d) satisfying the conditions $(C1)$, $(C2)$ and $(C7)$.*

Then the pairs A, S and B, T have a coincidence point.

Moreover, assume that the pairs A, S and B, T are weakly compatible. Then A, B, S and T have a unique common fixed point.

Proof. From Lemma 2.9, the pairs A, S and B, T share the common limit in the range of A property, that is, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = Av$$

for some $v \in X$.

Firstly, we claim that $Av = Sv$. Putting $x = v$ and $y = y_n$ in $(C2)$, we have

$$\begin{aligned} d(Sv, Ty_n) \le [\phi\big(&d(Av, By_n), d(Av, Sv),\\ &d(By_n, Ty_n), d(Sv, By_n), d(Av, Ty_n)\big)]^\lambda. \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(Av, Av) &\le [\phi\big(1, d(Av, Sv), 1, d(Sv, Av), 1\big)]^\lambda\\ &\le d^\lambda(Av, Sv), \end{aligned}$$

which implies that $Sv = Av$. Since $SX \subset BX$, there exists $w \in X$ such that $Bw = Sv$.

Now we claim that $Bw = Tw$. Putting $x = v$ and $y = w$, we have

$$\begin{aligned} d(Bw, Tw) &= d(Sv, Tw)\\ &\le [\phi\big(d(Av, Bw), d(Av, Sv),\\ &\qquad d(Bw, Tw), d(Sv, Bw), d(Av, Tw)\big)]^\lambda\\ &= [\phi\big(1, 1, d(Bw, Tw), 1, d(Bw, Tw)\big)]^\lambda\\ &= d^\lambda(Bw, Tw), \end{aligned}$$

this implies that $Bw = Tw$ and hence $Tw = Bw = Av = Sv$. Since the pairs A, S and B, T are weakly compatible and $Av = Sv$ and $Tw = Bw$. Hence

$$ASv = SAv = AAv = SSv, \quad TBw = BTw = BBw = TTw.$$

Finally, we claim that $SAv = Av$. Putting $x = Av$ and $y = w$, we have

$$\begin{aligned} d(SAv, Av) &= d(SAv, Tw) \\ &\leq [\phi\big(d(AAv, Bw), d(AAv, SAv), \\ &\qquad d(Bw, Tw), d(SAv, Bw), d(AAv, Tw)\big)]^{\lambda} \\ &\leq d^{\lambda}(SAv, Av), \end{aligned}$$

this implies that $SAv = Av$ and hence $SAv = Av = AAv$, which implies that Av is a common fixed point of A and S.

Also, one can easily prove that $BBw = Bw = TBw$, that is, Bw is a common fixed point of B and T. As $Av = Bw$, Av is common fixed point of A, B, S and T.

The uniqueness follows easily from $(C2)$. This completes the proof. □

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