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σ-δ^*-OPEN SETS AND σ-δ-OPEN SETS ON σ-STRUCTURES

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Abstract: The purpose of this paper is to introduce the notions of σ-δ^*-open sets and σ-δ-open sets, and investigate basic properties on σ-structures. We also study the relations among σ-δ^*-open sets, σ-δ-open sets, σ-regular open sets and σ-open sets.

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1. Introduction

The notions of generalized topology and generalized open sets are introduced by Császár [1] as the following: Let X be a nonempty set and μ be a collection of subsets of X. Then μ is called a *generalized topology* (briefly GT) on X iff $\emptyset \in \mu$ and $G_i \in \mu$ for $i \in I \neq \emptyset$ implies $G = \cup_{i \in I} G_i \in \mu$. The elements of μ are

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called *g-open* sets and the complements are called *g-closed* sets. Kim and Min [2] introduced the notion of σ-structures which is an extended notion of generalized topology : $s \subseteq 2^X$ is called a *σ-structure* on X if for $i \in I \neq \emptyset$, $U_i \in s$ implies $\cup_{i \in I} U_i \in s$. The elements of s are called *σ-open* sets and the complements are called *σ-closed* sets. We also introduced the notions of σ-semiopen sets [3], σ-preopen sets [4] and σ-β-open sets [5]. In particular, we showed that the family $\sigma PO(X)$ (resp., $\sigma\beta O(X)$) of all σ-preopen sets (resp., σ-β-open sets) in (X, σ) is a generalized topology in sense of Császár. In [4], we showed that the family $\sigma SO(X)$ of all σ-semiopen subsets is strong but $\sigma PO(X)$ may not be strong in (X, σ). In [5], we studied σ-β-open sets which are the generalized σ-open sets of σ-semiopen sets and σ-preopen sets, and showed that the family $\sigma\beta O(X)$ of all σ-β-open sets in (X, σ) is a strong σ-structure. In [6], we also introduced the notion of σ-regular open set and studied basic properties. The purpose of this paper is to study σ-δ^*-open sets and σ-δ-open sets which are the generalized σ-open sets of σ-regular open sets. First, we introduce the notions of σ-δ^*-open sets and σ-δ-open sets, and investigate basic properties. Finally, we study the relations among σ-δ^*-open sets, σ-δ-open sets and the other generalized σ-open sets.

2. Preliminaries

Theorem 2.1 ([2]). *Let s be a σ-structure on a nonempty set X and $A, B \subseteq X$. Then*

(1) $i_s\emptyset = \emptyset$ and $c_sX = X$.

(2) $i_sA \subseteq A$ and $A \subseteq c_sA$.

(3) If $A \subseteq B$, then $i_sA \subseteq i_sB$ and $c_sA \subseteq c_sB$.

(4) $i_si_sA = i_sA$ and $c_sc_sA = c_sA$.

(5) $c_s(A) = X - i_s(X - A)$ and $i_s(A) = X - c_s(X - A)$.

(6) A is σ-open iff $A = i_sA$ for $A \neq \emptyset$; A is σ-closed iff $A = c_sA$ for $A \neq X$.

Theorem 2.2 ([2]). *Let s be a σ-structure on a nonempty set X and $A \subseteq X$. Then*

(1) The collection $\mu = \{A \subseteq X : i_sA = A\}$ is a generalized topology on X.

(2) $x \in i_sA$ iff there exists a σ-open set S containing x such that $S \subseteq A$.

(3) $x \in c_sA$ iff $S \cap A \neq \emptyset$ for every σ-open set S containing x.

3. σ-δ^*-Open Sets

First, we introduce the notion of δ-σ^*-open sets using by σ-closed sets and σ-interior operator i_s.

Definition 3.1. Let s be a σ-structure on a nonempty set X and $A \subseteq X$. Then a subset A in a σ-structure σ is said to be *δ-σ^*-open* if for each $x \in A$, there exists a σ-closed set F such that $x \in i_s(F) \subseteq A$. The collection of all σ-δ^*-open sets is denoted by δ_{σ^*}

Remark 3.2. Every nonempty σ-δ^*-open set is clearly σ-open but the converse is not true in general as shown in the next example.

Example 3.3. Let $X = \{a, b, c\}$ and a σ-structure $s = \{\{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$. Consider a σ-open set $A = \{b\}$: For $b \in A$, there exists the only σ-closed set $F = \{b, c\}$ containing b. But since $i_s(F) = \{b, c\}$, the σ-open set A can not be σ-δ^*-open.

Let s be a σ-structure on a nonempty set X and $A \subseteq X$. Then A is said to be *σ-regular open* [6] if $A = i_s(c_s(A))$.

Theorem 3.4. *Let s be a σ-structure on a nonempty set X. If $\emptyset \in s$, then every σ-regular open set is σ-δ^*-open.*

Proof. Let A be a nonempty σ-regular open set. Since $\emptyset \in s$, $c_s(A)$ is σ-closed. Since $A = i_s(c_s(A))$, A is σ-δ^*-open. □

Remark 3.5. In the above theorem, the condition $\emptyset \in s$ is essential, and a σ-δ^*-open set may not be σ-regular open as shown in the next examples.

Example 3.6. (1) Let $X = \{a, b, c\}$ and $s = \{\{a, c\}, \{b, c\}, X\}$ a σ-structure: Let $A = X$. Then A is σ-regular open. For $a \in A$, $\{a\}$ is the only σ-closed set containing a and $i_s(\{a\}) = \emptyset$. So A is not σ-δ^*-open.

(2) Let $X = \{a, b, c\}$ and $s = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ a σ-structure: Let $A = \{a, b\}$. Note that $\{a\} = i_s(c_s(\{a\}))$ and $\{b\} = i_s(c_s(\{b\}))$. Then A is σ-δ^*-open but not σ-regular open since $i_s(c_s(\{a, b\})) = X \neq \{a, b\}$.

Remark 3.7. Let s be a σ-structure on a nonempty set X. Let $T = \cup_{G \in s} G$: Then $T = X$ or $T \neq X$. If $\emptyset \notin s$, then T is always σ-regular open. But since there is no any σ-closed set containing T, finally T is not σ-δ^*-open.

Remark 3.8. Let s be a σ-structure on a nonempty set X. The σ-structure s is said to be *strong* [4] if $X \in \sigma$. Then from Remark 3.2, if ${\delta^*}_\sigma$ is strong, s is also strong. But, although s is strong, from (1) of Example 3.6, we know that ${\delta^*}_\sigma$ may not be strong.

Theorem 3.9. *Let s be a σ-structure on a nonempty set X. Then*
(1) the empty set is σ-δ^*-open;
(2) the any union of σ-δ^*-open sets is σ-δ^*-open.

Proof. (1) Obvious.
(2) Let $A = \cup_{i\in I} A_i$ where A_i is a σ-δ^*-open-open set. For each $x \in A$, there is some $i \in I$ such that $x \in A_i \in \delta^*_\sigma$. Then there exists a σ-closed set F such that $x \in i_s(F) \subseteq A_i \subseteq A$. This implies that A is σ-δ^*-open. □

Let s be a σ-structure on a nonempty set X and $A \subseteq X$. We define the operators $i_{\delta^*}, c_{\delta^*}$ as the following:

$$i_{\delta^*} A = \cup\{S \subseteq X : S \subseteq A, \quad S \text{ is } \sigma\text{-}\delta^*\text{-open }\};$$

$$c_{\delta^*} A = \cap\{F \subseteq X : A \subseteq F, \quad F \text{ is } \sigma\text{-}\delta^*\text{-closed }\}.$$

From Theorem 3.9, we know that particularly, δ^*_σ is a generalized topology in sense of Császár. So the following theorem is obviously obtained:

Theorem 3.10. *Let s be a σ-structure on a nonempty set X and $A \subseteq X$. Then*
(1) $i_{\delta^*}\emptyset = \emptyset$ and $c_{\delta^*} X = X$.
(2) $i_{\delta^*} A \subseteq A$ and $A \subseteq c_{\delta^*} A$.
(3) If $A \subseteq B$, then $i_{\delta^*} A \subseteq i_{\delta^*} B$ and $c_{\delta^*} A \subseteq c_{\delta^*} B$.
(4) $i_{\delta^*} i_{\delta^*} A = i_{\delta^*} A$ and $c_{\delta^*} c_{\delta^*} A = c_{\delta^*} A$.
(5) $c_{\delta^*}(A) = X - i_{\delta^*}(X - A)$ and $i_{\delta^*}(A) = X - c_{\delta^*}(X - A)$.
(7) A is σ-δ^*-open iff $A = i_{\delta^*}(A)$.
(8) A is σ-δ^*-closed iff $A = c_{\delta^*}(A)$.

4. σ-δ-Open Sets

Lemma 4.1 ([6]). *Let s be a σ-structure on a nonempty set X and $A \subseteq X$. Then if A is σ-closed, then $i_s(A)$ is σ-regular open.*

Definition 4.2. Let s be a σ-structure on a nonempty set X and $A \subseteq X$. Then a subset A in a σ-structure σ is said to be *δ-σ-open* if for each $x \in A$, there exists a σ-regular open set G such that $x \in G \subseteq A$. The collection of all σ-δ-open sets is denoted by δ_σ

Remark 4.3. Every non-empty σ-δ-open set is clearly σ-open but the converse is not true in general as shown in the next example.

Example 4.4. Let $X = \{a, b, c\}$ and a σ-structure $s = \{\{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$. Consider a σ-open set $A = \{b\}$: For $b \in A$, there exists the only σ-closed set $F = \{b, c\}$. By Lemma 4.1, $i_s(F) = \{b, c\}$ is σ-regular open set. We know that $i_s(F) = \{b, c\}$ is the only σ-regular open set containing b, and it is impossible that the σ-regular open set is contained in A. Consequently, the σ-open set A is not σ-δ-open.

Theorem 4.5. *Let s be a σ-structure on a nonempty set X. Then every σ-regular open set is σ-δ-open.*

Proof. Obvious. □

Remark 4.6. In the above theorem, the converse may not be true as shown in (2) of Example 3.6.

Theorem 4.7. *Let s be a σ-structure on a nonempty set X. Then every σ-δ^*-open set is σ-δ-open.*

Proof. It follows from Lemma 4.1. □

In the above theorem, the converse is not true in general as shown in the next example.

Example 4.8. Let $X = \{a, b, c, d\}$ and a σ-structure $s = \{\{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$. Consider a σ-open set $A = X$: Then A is σ-regular open set and so A is σ-δ-open. For $d \in A$, there is no a σ-closed set F such that $d \in i_s(F) \subseteq A$. So A is not σ-δ^*-open.

Theorem 4.9. *Let s be a σ-structure on a nonempty set X. Then*
(1) the empty set is σ-δ-open.
(2) the any union of σ-δ-open sets is σ-δ-open.

Proof. (1) Obvious.
(2) Let $A = \cup_{i \in I} A_i$ where A_i is a σ-δ-open set. For each $x \in A$, there is some $i \in I$ such that $x \in A_i \in \delta_\sigma$. Then there exists a σ-regular open set G_i such that $x \in G_i \subseteq A_i \subseteq A$. This implies that A is σ-δ-open. □

Theorem 4.10. *Let s be a σ-structure on a nonempty set X. If s is strong, then δ_σ is also strong.*

Proof. Since $X \in \sigma$, X is a σ-regular open set and so X is σ-δ-open. So δ_σ is strong. □

Definition 4.11. Let s be a σ-structure on a nonempty set X and $A \subseteq X$. Then

$$i_\delta A = \cup\{S \subseteq X : S \subseteq A, \quad S \text{ is } \sigma\text{-}\delta\text{-open }\};$$

$$c_\delta A = \cap\{F \subseteq X : A \subseteq F, \quad F \text{ is } \sigma\text{-}\delta\text{-closed }\}.$$

Theorem 4.12. *Let s be a σ-structure on a nonempty set X and $A \subseteq X$. Then*

(1) $i_\delta \emptyset = \emptyset$ and $c_\delta X = X$.
(2) $i_\delta A \subseteq A$ and $A \subseteq c_\delta A$.
(3) If $A \subseteq B$, then $i_\delta A \subseteq i_\delta B$ and $c_\delta A \subseteq c_\delta B$.
(4) $i_\delta i_\delta A = i_\delta A$ and $c_\delta c_\delta A = c_\delta A$.
(5) $c_\delta(A) = X - i_\delta(X - A)$ and $i_\delta(A) = X - c_\delta(X - A)$.
(7) A is σ-δ-open iff $A = i_\delta(A)$.
(8) A is σ-δ-closed iff $A = c_\delta(A)$.

Proof. Obvious. □

Theorem 4.13. *Let s be a σ-structure on a nonempty set X. Then the non-empty elements of δ_σ coincide with the unions of σ-regular open sets.*

Proof. By Theorem 4.5 and Theorem 4.9, it is obtained that any union of σ-regular open sets is δ-σ-open.

□

Corollary 4.14. *Let s be a σ-structure on a nonempty set X. If $\emptyset \in s$, then the elements of δ^*_σ coincide with the unions of σ-regular open sets.*

Proof. From Theorem 3.4, obviously $\delta^*_\sigma = \delta_\sigma$, and so it is obtained from Theorem 4.13. □

Theorem 4.15. *Let s be a σ-structure on a nonempty set X and $A \subseteq X$. Then*

(1) $x \in i_\delta A$ iff there exists a σ-regular open set S containing x such that $S \subseteq A$.
(2) $x \in c_\delta A$ iff $S \cap A \neq \emptyset$ for every σ-regular open set S containing x.

Proof. From Theorem 2.2, these are easily obtained. □

Corollary 4.16. *Let s be a σ-structure on a nonempty set X and $A \subseteq X$. If $\emptyset \in s$, then*

(1) $x \in i_{\delta^*} A$ iff there exists a σ-regular open set S containing x such that $S \subseteq A$.
(2) $x \in c_{\delta^*} A$ iff $S \cap A \neq \emptyset$ for every σ-regular open set S containing x.

Proof. From Corollary 4.14 and Theorem 4.15, obviously it is obtained. □

Remark 4.17. Let s be a σ-structure on a nonempty set X and $A \subseteq X$. Then A is said to be *σ-semiopen* [3] (resp., *σ-preopen* [4], *σ-β-open* [5], *σ-regular open* [6]) if $A \subseteq c_s(i_s(A))$ (resp., $A \subseteq i_s(c_s(A))$, $A \subseteq c_s(i_s(c_s(A)))$, $A = i_s(c_s(A))$). The family of all σ-preopen sets (resp., σ-semiopen sets, σ-β-open sets, σ-regular open sets) in X is denoted by $\sigma PO(X)$ (resp., $\sigma SO(X)$, $\sigma\beta O(X)$, $\sigma RO(X)$). Then, in case $\emptyset \in s$, we obtained the following diagram in [6]:

$$\begin{array}{ccccccccc} & & & & & & \sigma\text{-semiopen} & & \\ & & & & & \nearrow & & \searrow & \\ \sigma\text{-regular open} & \rightarrow & \sigma\text{-}\delta\text{-open} & \rightarrow & \sigma\text{-open} & & & & \sigma\text{-}\beta\text{-open} \\ & & & & & \searrow & & \nearrow & \\ & & & & & & \sigma\text{-preopen} & & \end{array}$$

In summary, we have the following results about this investigation:

(1) In case $X \notin s$ and $\emptyset \notin s$:

$$\sigma RO(X) \subseteq \sigma_{\delta^*} \subseteq \sigma_\delta;$$
$$\sigma RO(X) - \{\emptyset\} \subseteq \sigma_{\delta^*} - \{\emptyset\} \subseteq \sigma_\delta - \{\emptyset\} \subseteq \sigma;$$
$$\sigma \subseteq \sigma SO(X) \subseteq \sigma\beta O(X);$$
$$\sigma \subseteq \sigma PO(X) \subseteq \sigma\beta O(X).$$

(2) In case $X \in s$ and $\emptyset \notin s$:

$$\sigma_{\delta^*} \subseteq \sigma_\delta;$$
$$\sigma RO(X) - \{X\} \subseteq \sigma_{\delta^*} \subseteq \sigma_\delta;$$
$$\sigma RO(X) - \{\emptyset\} \subseteq \sigma_\delta - \{\emptyset\} \subseteq \sigma;$$
$$\sigma_{\delta^*} - \{\emptyset\} \subseteq \sigma_\delta - \{\emptyset\} \subseteq \sigma;$$
$$\sigma RO(X) - \{X, \emptyset\} \subseteq \sigma_{\delta^*} - \{\emptyset\} \subseteq \sigma_\delta - \{\emptyset\} \subseteq \sigma;$$
$$\sigma \subseteq \sigma SO(X) \subseteq \sigma\beta O(X);$$
$$\sigma \subseteq \sigma PO(X) \subseteq \sigma\beta O(X).$$

(3) In case $\emptyset \in s$:

$$\sigma RO(X) \subseteq \sigma_{\delta^*} = \sigma_\delta \subseteq \sigma \subseteq \sigma SO(X) \subseteq \sigma\beta O(X);$$
$$\sigma \subseteq \sigma PO(X) \subseteq \sigma\beta O(X).$$

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