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ON *b*-CONVERGENCE OF *p*-STACKS

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Abstract: We introduce the notion of b-convergence of p-stacks and by using that notion we characterize the b-interior, b-closure, separation axioms and b-irresoluteness on a topological space. Also we introduce a new notion of p-b-compactness and investigate its properties in terms of b-convergence of p-stacks.

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1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. In 1996, Andrijevic [1] introduced a new class of generalized open sets called b-open sets into the field of topology. Andrijevic studied several fundamental and interesting properties of b-open sets. In this paper, we introduce the notion of b-

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convergence of p-stacks and by using that notion we characterize the b-interior, b-closure, separation axioms and b-irresoluteness on a topological space. Also we introduce a new notion of p-b-compactness and investigate its properties in terms of b-convergence of p-stacks.

2. Preliminaries

Throughout this paper, spaces always means topological spaces on which no separation axioms are assumed unless otherwise mentioned and $f : (X, \tau) \to (Y, \sigma)$ (or simply $f : X \to Y$) denotes a function f of a space (X, τ) into a space (Y, σ). Let A be a subset of a space X. The closure and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of X is said to be b-open [1] (= γ-open [4]) $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$. The complement of a b-open set is called a b-closed set [1] (= γ-closed set [4])). The union of all b-open sets contained in $A \subset X$ is called the b-interior of A, and is denoted by $b\,\mathrm{Int}(A)$. The intersection of all b-closed sets containing A is called the b-closure [1] of A and is denoted by $b\,\mathrm{Cl}(A)$. A subset $M(x)$ of a topological space X is called a b-neighbourhood of a point $x \in X$ if there exists a b-open set S such that $x \in S \subset M(x)$. Given a set X, a collection $\mathbf{C}$ of subsets of X is called a stack if $A \in \mathbf{C}$ whenever $B \in \mathbf{C}$ and $B \subset A$. A stack $\mathbf{H}$ on a set X is called a p-stack if it satisfies the following condition: (P) $A, B \in \mathbf{H} \Rightarrow A \cap B \neq \emptyset$. Condition (P) is called the pairwise intersection property $(P.I.P)$. A collection $\mathbf{B}$ of subsets of X with the $P.I.P$ is called a p-stack base. For any collection B, we denote by $< \mathbf{B} >= \{A \subset X$: there exists $B \in \mathbf{B}$ such that $B \subset A\}$ the stack generated by $\mathbf{B}$, and if $\{B\}$ is a p-stack base, then $< \{B\} >$ is a p-stack. We will denote simply $< \{B\} >=< B >$. In case $x \in X$ and $B = \{x\}$, $< x >$ is usually denoted by x. Let $pS(X)$ denote the collection of all p-stacks on X, partially ordered by inclusion. The maximal elements in $pS(X)$ are called ultra p-stacks is contained in an ultra p-stack. For a function $f : X \to Y$ and $\mathbf{H} \in pS(X)$, the image stack $f(\mathbf{H})$ in $pS(Y)$ has p-stack base $\{f(H) : H \in \mathbf{H}\}$. Likewise, if $\mathbf{G} \in pS(Y)$, $f^{-1}(\mathbf{G})$ denotes the p-stack on X generated by $\{f^{-1}(G) : G \in \mathbf{G}\}$.

Definition 2.1. Let (X, τ) be a topological space. A class G_i of b-open subsets of X is said to be b-open cover of X if each point in X belongs to at least one G_i that is $\bigcup\limits_i G_i = X$.

Definition 2.2. A subset K of a nonempty set X is said to be b-compact relative to (X, τ) if every cover of K by b-open sets of X has a finite subcover.

We say that (X, τ) is b-compact if X is b-compact.

Definition 2.3. A topological space (X, τ) is said to be:

(i) b-T_1 [2] if for each pair of distinct points x and y of X, there exist b-open sets U and V containing x and y, respectively such that $y \notin U$ and $x \notin V$;

(ii) b-T_2 [2] if for each pair of distinct points x and y of X, there exist b-open sets U and V such that $x \in U, y \in V$ and $U \cap V = \varnothing$;

(iii) b-regular [6] if for any closed set $F \subset X$ and any point $x \in X \backslash F$, there exists disjoint b-open sets U and V such that $x \in U$ and $F \subset V$.

Lemma 2.4. *[5] For $\mathbf{H} \in pS(X)$, the following are equivalent:*

(i) $\mathbf{H}$ is an ultra p-stack;

(ii) If $A \cap H = \emptyset$ for all $H \in \mathbf{H}$, then $A \in \mathbf{H}$;

(iii) $B \in H$ implies $X \backslash B \in H$.

Theorem 2.5. *[5] Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $\mathbf{H} \in pS(X)$.*

(i) If $\mathbf{H}$ is a filter, so is $f(\mathbf{H})$;

(ii) If $\mathbf{H}$ is a ultra filter, so is $f(\mathbf{H})$;

(iii) If $\mathbf{H}$ is a ultra p-stack, so is $f(\mathbf{H})$.

3. b-Convergence of p-Stacks

Definition 3.1. Let X be a topological space, $x \in X$ and let $\mathbf{B}(x) = \{V \subset X : V$ is a b-neighbourhood of $x\}$. Then we call the family $\mathbf{B}(x)$ the b-neighbourhood stack at x.

Definition 3.2. Let X be a topological space, $x \in X$ and let $\mathbf{B}(x) = \{V \subset X : V$ is a b-neighbourhood of $x\}$. Then we call the family $\mathbf{B}(x)$ the b-neighbourhood stack at x.

Theorem 3.3. *Let $f : (X, \tau)$, be a topological space. Then we have the following*

(i) $\dot{x}$ b-converges to x for all $x \in X$;

(ii) If $\mathbf{F}$ *b-converges to* x *and* $\mathbf{F} \subset \mathbf{G}$ *for* $\mathbf{F}; \mathbf{G} \in pS(X)$*, then* $\mathbf{G}$ *b-converges to* x*;*

(iii) If both $\mathbf{F}$ *and* $\mathbf{G}$ *are p-stacks b-converging to* x*, then* $\mathbf{F} \cap \mathbf{G}$ *b-converges to* x*;*

(iv) If p-stacks $\mathbf{F}_i$*b-converge to* x *for all* $i \in J$*, then* $\cap \mathbf{F}_i$ *b-converges to* x*.*

Proof. Follows from the definitions. □

Theorem 3.4. *Let* (X, τ) *be a topological space and* $A \subset X$*. Then the following are equivalent:*

(i) $x \in b\,\mathrm{Cl}(A)$

(ii) There is $\mathbf{F} \in pS(X)$ *such that* $A \in \mathbf{F}$ *and* $\mathbf{F}$ *b-converges to x;*

(iii) For all $V \in \mathbf{B}(x), A \cap V = \emptyset$*.*

Proof. (i) $\Rightarrow$ (ii): Let x be an element in $b\,\mathrm{Cl}(A)$, then $U(x) \cap A = \emptyset$ for each b-open $U(x)$ of x. Let $\mathbf{F} = \mathbf{B}(x) \cup < A >$. Then the p-stack $\mathbf{F}$ b-converges to x and $A \in \mathbf{F}$. (ii)$\Rightarrow$ (iii): Let F be a p-stack and $A \in \mathbf{F}$ and p-stack $\mathbf{F}$ b-converge to x. Then $\mathbf{B}(x) \subset F$. Thus since $\mathbf{B}(x)$ is a p-stack, we get $U \cap A \neq \emptyset$ for all $U \in B(x)$. (iii) $\Rightarrow$ (i): It is obvious. □

Theorem 3.5. *Let* (X, τ) *be a topological space and* $A \subset X$*. Then the following are equivalent:*

(i) $x \in b\,\mathrm{Int}(A)$

(ii) For every p-stack $\mathbf{F}$ *b-converging to* x*,* $A \in \mathbf{F}$*;*

(iii) $A \in \mathbf{B}(x)$*.*

Proof. (i) $\Rightarrow$ (ii): Let x be an element in $b\,\mathrm{Int}(A)$,and let $\mathbf{F}$ b-stack b-converges to x. Since $x \in b\,\mathrm{Int}(A)$, there is a b-open subset U such that $x \in U \subset A$, so $A \in \mathbf{B}(x)$. Thus by the definition of b-convergence of p-stack, we can say $A \in \mathcal{F}$. (ii) $\Rightarrow$ (iii): The b-neighborhood stack $\mathbf{B}(x)$ is always b-converges to x. Thus by (ii), $A \in \mathbf{B}(x)$. (iii) $\Rightarrow$ (i): It is obvious. □

Now by using b-convergence of p-stacks, we characterize the properties of b-T_1, b-T_2 and b-regular induced by b-open subsets on a topological space.

Theorem 3.6. *Let* (X, τ) *be a topological space. Then the following statements are equivalent:*

(i) (X, τ) *is* b-T_1

(ii) $\cap \mathbf{B}(x) = \{x\}$ *for* $x \in X$

(iii) if $\dot{x}$ b*-converges to* y*, then* $x = y$*.*

Proof. (i)⇒)(ii): Let y be an element in $\cap \mathbf{B}(x)$, then $y \in U$ for each b-open neighborhood U of x. Since X is b-T_1, we get $y = x$. (ii)⇒(iii): Let $\dot{x}$ b-converge to y. Since $\mathbf{B}(y) \subset x$, x is an element in $\cap \mathbf{B}(y)$. Thus $x = y$. (iii)⇒(i): Suppose that X is not b-T_1, then there are distinct x and y such that every b-open neighborhood of x contains y. Thus $\mathbf{B}(x) \subset \dot{y}$ and $\dot{y}$ b-converges to x. This contradicts the hypothesis. □

Theorem 3.7. *Let* (X, τ) *be a topological space. Then the following statements are equivalent:*

(i) (X, τ) *is* b-T_2*;*

(ii) Every b*-convergent* p*-stack* $\mathbf{F}$ *on* X b*-converges to exactly one point;*

(iii) Every b*-convergent ultra* p*-stack* $\mathbf{F}$ *on* X b*-converges to exactly one point.*

Proof. (i) ⇒ (ii): Suppose that X is b-T_2 and a p-stack $\mathbf{F}$ b-converges to x. For any $y \neq x$, there are disjoint b-open sets $U(x)$ and $U(y)$ containing x and y, respectively. Since $\mathbf{B}(x) \subset F$ and $\mathbf{F}$ is a p-stack, both $U(x)$ and $X \backslash U(y)$ are elements of $\mathbf{F}$. Thus $\mathbf{F}$ is not finer than (y), so $\mathbf{F}$ doesn't b-converge to y. (ii) ⇒ (iii): It is obvious. (iii) ⇒ (i): Suppose that X is not b-T_2. Then there must exist x, y such that $U(x) \cap U(y) \neq \emptyset$ for every b-open sets $U(x)$ and $U(y)$ of x and y, respectively. Let $\mathbf{F}$ be a ultra p-stack finer than a p-stack $\mathbf{B}(x) \subset B(y)$. Then $\mathbf{F}$ is finer than $B(x)$ and (y), so the ultra p-stack F b-converges to both x and y. This contradicts (ii). □

If (X, τ) is a topological space and $F \in pS(X)$, then $\mathbf{B} = \{b\,\mathrm{Cl}(F)\} : F \in \mathbf{F}$ is a p-stack base on X, and the b-closure p-stack generated by B is denoted by $b\,\mathrm{Cl}(\mathbf{F})$.

Theorem 3.8. *Let* (X, τ) *be a topological space. Then the following statements are equivalent:*

(i) (X, τ) *is* b*-regular;*

(ii) For every $x \in X, \mathbf{B}(x) = b\,\mathrm{Cl}(\mathbf{B}(x))$*;*

(iii) If a p*-stack* F b*-converges to* x*, then the* b*-closure* p*-stack* $b\,\mathrm{Cl}(\mathbf{F})$ b*-converges to* x*.*

Proof. (i)⇒(*ii*): Let F be an element in $\mathbf{B}(x)$. There exists a b-open neighborhood $U(x)$ such that $U(x) \subset F$. Since X is b-regular, there is a b-open neighborhood $W(x)$ of x such that $W(x) \subset b\,\mathrm{Cl}(W(x)) \subset U(x) \subset F$. Since $b\,\mathrm{Cl}(W(x)) \in b\,\mathrm{Cl}(\mathbf{B}(x))$ and $b\,\mathrm{Cl}(\mathbf{B}(x))$ is a p-stack, $F \in b\,\mathrm{Cl}(\mathbf{B}(x))$. $(ii) \Rightarrow (iii)$: Let a p-stack $\mathbf{F}$ b-converge to x. Then $\mathbf{B}(x) \subset F$, and so $b\,\mathrm{Cl}(\mathbf{B}(x)) \subset b\,\mathrm{Cl}(\mathbf{F})$. By (ii), we get that $b\,\mathrm{Cl}(\mathbf{F})$ b-converges to x. $(iii) \Rightarrow (i)$: Let U be a b-open set containing $x \in X$. Since $\mathbf{B}(x)$ b-converges to x, by (iii) $b\,\mathrm{Cl}(\mathbf{B}(x))$ b-converges to x, and so $U \in b\,\mathrm{Cl}(\mathbf{B}(x))$. Then by the definition of the b-closure of p-stacks, we can get a b-open neighborhood V of x such that $V \subset b\,\mathrm{Cl}(V) \subset U$. □

Definition 3.9. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be b-irresolute [4] if $f^{-1}(V)$ is b-closed (resp. b-open) in X for every b-closed (resp. b-open) subset V of Y

Theorem 3.10. *Let X and Y be topological spaces. Then a function $f : (X, \tau) \to (Y, \sigma)$ is b-irresolute if and only if for each x in X and each b-neighborhood U of $f(x)$, there is a b-neighborhood V of x such that $f(V) \subset U$.*

Now we get another characterization of the b-irresolute function on a topological space using the notion of p-stacks.

Theorem 3.11. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(i) f is b-irresolute;

(ii) $\mathbf{B}(f(x)) \subset f(\mathbf{B}(x))$ for all $x \in X$;

(iii) If a p-stack $\mathbf{F}$ b-converges to x, then the image p-stack $f(\mathbf{F})$ b-converges to $f(x)$.

Proof. $(i) \Rightarrow (ii)$: Let V be any member of $\mathbf{B}(f(x))$ in Y . Then there is a b-open set W such that $W \subset V$. Since f is b-irresolute, there exists a b-open neighborhood $U \in \mathbf{B}(x)$ such that $f(U) \subset W \subset V$, thus $V \in f(\mathbf{B}(x))$. $(ii) \Rightarrow (iii)$: It is obvious. $(iii) \Rightarrow (i)$: If f is not b-irresolute, then for some $x \in X$, there is a b-open neighborhood $V \in \mathbf{B}(f(x))$ such that for all b-open neighborhood $U \in \mathbf{B}(x)$, $f(U)$ is not included in V . For all $U \in \mathbf{B}(x)$, since $f(U) \cap (Y \backslash V) \neq \emptyset$, we get a p-stack $\mathbf{F} = f(\mathbf{B}(x)) \cup < Y \backslash V >$. And since $U \cap f^{-1}(Y \backslash V) \neq \emptyset$, also we get a p-stack $\mathbf{G} = \mathbf{B}(x) \cup f^{-1} < Y \backslash V >$ which b-converges to x.But since $f(\mathbf{G})$ is a finer p-stack than F and $Y \backslash V \in \mathbf{F}$, $f(\mathbf{G})$

can't b-converge to $f(x)$, contradicting to (iii).
Now we introduce a new notion of p-b-compactness by p-stacks and investigate the related properties. $\square$

Definition 3.12. Let (X, τ) be a topological space and A be a subset of X. A subset A of a topological space (X, τ) is p-b-compact if every ultra p-stack containing A b-converges to a point in A. A topological space (X, τ) is p-b-compact if X is p-b-compact.

Let $X = \{a, b, c\}$. In case τ is the discrete topology, let $\mathbf{H}$ be an ultra p-stack containing a p-stack $\mathbf{F}$ generated by {{a,b},{b,c},{a,c}}. Then it does not b-converge to any point in X. Thus the topological space (X, τ) is not p-b-compact. But in case $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ the topological space (X, τ) is p-b-compact.

Theorem 3.13. *If a topological space (X, τ) is p-b-compact and $A \subset X$ is b-closed, then A is p-b-compact.*

Proof. Let $\mathbf{F}$ be an ultra p-stack containing A. From Definition 3.12, there is $x \in X$ such that $\mathbf{F}$ b-converges to x. Thus $\mathbf{B}(x) \subset F$, and since $A \in \mathbf{F}$ and $\mathbf{F}$ is a p-stack, $A \cap V \neq \emptyset$ for all $V \in \mathbf{B}(x)$. So by Theorem 9, we can say $x \in b\,\mathrm{Cl}(A) = A$. $\square$

Theorem 3.14. *The b-irresolute image of a p-b-compact set is p-b-compact.*

Proof. Let a function $f : (X, \tau) \to (Y, \sigma)$ be b-irresolute, let $A \subset X$ be p-b-compact, and let $\mathbf{H}$ be an ultra p-stack containing $f(A)$. If $\mathbf{G}$ is an ultra p-stack containing the p-stack base $\{f^{-1}(H) : H \in \mathbf{H}\} \cup < A >$, then for some $x \in A$, $\mathbf{G}$ b-converges to x, and $\mathbf{H} = f(\mathbf{G})$ b-converges to $f(x)$. Thus, $f(A)$ is p-b-compact. $\square$

Theorem 3.15. *A topological space (X, τ) is p-b-compact if and only if each b-open cover of X has a two-element subcover.*

Proof. Suppose $\mathbf{H}$ is an ultra p-stack in X such that it does not b-converge to any point in X. Then for each $x \in X$, there is a b-open subset $U_x \in \mathbf{B}(x)$ such that $U_x \notin H$. By Lemma 2.4(iii), $X \backslash U_x \in H$,for all $x \in X$. Thus $\mathbf{U} = \{U_x : x \in X\}$ is a b-open cover of X. But $\mathbf{U}$ has no two-element subcover of X, for if $U, V \in \mathbf{U}$ and $X \subset U \cup V$,then $(X\backslash U) \cap (X\backslash V) = X\backslash(U \cup V) = \emptyset$, contradicting the assumption that $\mathbf{H}$ is a p-stack. Conversely, let $\mathbf{U}$ be a b-open cover of X with no two-element subcover of X. Then $\mathbf{B} = \{X\backslash U : U \in \mathbf{U}\}$ is p-stack base, and any ultra p-stack containing $\mathbf{B}$ cannot b-converge to any point in X. $\square$

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