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DYNAMIC PROPERTIES FOR AN EPIDEMIC MODEL WITH PARTIAL IMMUNITY

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Abstract: In this paper, an epidemic model with waning preventive vaccine is formulated. The analysis of the model is presented in terms of the basic reproduction number. If the basic reproduction number is less than unity, the disease-free equilibrium is globally asymptotically stable. If the basic reproduction number is greater than unity, the system is permanent and there is a unique endemic equilibrium. In this case, sufficient conditions are obtained for the global attractiveness of the endemic equilibrium. Numerical simulations are carried out to illustrate the main results.

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1. Introduction

Over the past few decades, many epidemic models have been proposed and analyzed to investigate the transmission dynamics of infectious diseases (see, for

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instance, [1], [2], [3] and the the references therein). In these studies, it was assumed that the infection is acquired following effective contact with infected population. However, some patients with infectious diseases can discharge infectious pathogens at the end of the latent period, such as tuberculosis, measles and chicken pox. Hence, the infection can also be acquired following effective contact with the latened population(see, for instance, [4] and the references therein).

Vaccination has been useful in controlling diseases spread. The mathematical models with vaccination have been studied by many authors(see, for instance, [5], [6], [7] and the the references therein). These articles all assumed that the vaccinees obtained the permanent immunity. However, some clinical studies have shown that the permanent immunity induced by the preventive vaccines may wane over time(see, for instance, [8], [9] and the the references therein).

Motivated by the work of Lianbing Li et al.[4] and Jianwen Jia et al.[9], we study the following differential equations:

$$\dot{S}(t) = (1-p)\Pi - \beta SI - \eta\beta ES - \mu S + \omega V, \tag{1.1a}$$

$$\dot{V}(t) = p\Pi - (1-\tau)\beta VI - \mu V - \omega V, \tag{1.1b}$$

$$\dot{E}(t) = \beta SI + \eta\beta ES + (1-\tau)\beta VI - \alpha E - \mu E, \tag{1.1c}$$

$$\dot{I}(t) = \alpha E - \delta I - dI - \mu I, \tag{1.1d}$$

$$\dot{R}(t) = \delta I - \mu R, \tag{1.1e}$$

where S, V, E, I and R denote the susceptible, vaccinated, exposed, infectious and recovered individuals, respectively. Π is the constant recruitment rate of individuals, p is the fraction of recruited individuals who are vaccinated, β is the effective contact rate, α is the rate at which exposed individuals become infectious, δ is the recovery rate, μ is the natural mortality rate, d is the disease-induced mortality rate, ω is the rate at which vaccine wanes, (that is $1/\omega$ is the duration of the loss of immunity acquired by preventive vaccine or by infection), $0 < \eta \leq 1$ is the constant describing the decrease in the relative infectiousness of population in the exposed individuals E in comparison to those in the infectious individuals I, and $0 \leq \tau \leq 1$ is the vaccine efficacy ($\tau = 1$ represents a vaccine that offers 100% protection against infection, $\tau = 0$ represents a vaccine that offers no protection at all).

The initial conditions for the model $(1.1a) - (1.1e)$ take the form

$$S(0) > 0, \;\; V(0) > 0, \;\; E(0) > 0, \;\; I(0) > 0, \;\; R(0) > 0. \tag{1.2}$$

Notice that the recovered population $R(t)$ does not feature in the first four equations of the model, we will only discuss Equations $(1.1a) - (1.1d)$ in the following. The dynamic behaviors of $R(t)$ can be obtained from Equation $(1.1e)$.

The paper is organized as follows. In the next section, some basic properties are presented. In Section 3, the local stability and the global asymptotic stability of a disease-free equilibrium of the model $(1.1a) - (1.1d)$ are discussed. The permanence of the model $(1.1a) - (1.1d)$ is given by means of the persistence theory on infinite dimensional systems in Section 4. In Section 5, sufficient conditions are received for the global attractiveness of the endemic equilibrium by using the theory of compound matrices. Numerical simulations are carried out in Section 6 to illustrate the main theoretical results. A brief conclusion is given in Section 7.

2. Basic Properties

In this section, we study the basic properties of the model $(1.1a) - (1.1d)$.

Theorem 2.1. *The arbitrary positive solution of the model* $(1.1a)-(1.1d)$ *with initial conditions in* R_4^+ *is ultimately bounded.*

Proof. Suppose $(S(t), V(t), E(t), I(t))$ be a positive solution of the model $(1.1a) - (1.1d)$ with initial conditions in R_4^+. Define

$$L(t) = S(t) + V(t) + E(t) + I(t).$$

Calculating the derivative of $L(t)$ along the solutions of the model $(1.1a) - (1.1d)$, it follows that

$$\dot{L}(t) = \Pi - \mu L - dI - \delta I \leq \Pi - \mu L,$$

a standard comparison argument shows that $\limsup_{t\to+\infty} L(t) \leq \Pi/\mu$. That is,

$$\limsup_{t\to+\infty} E(t) \leq \frac{\Pi}{\mu}, \quad \limsup_{t\to+\infty} I(t) \leq \frac{\Pi}{\mu}.$$

Hence, for $\varepsilon > 0$ sufficiently small, there exists a $T_1 > 0$ such that if $t > T_1$,

$$E(t) \leq \frac{\Pi}{\mu} + \varepsilon, \quad I(t) \leq \frac{\Pi}{\mu} + \varepsilon.$$

By Equation $(1.1b)$,

$$\dot{V}(t) \leq p\Pi - (\mu + \omega)V,$$

which yields

$$\limsup_{t\to+\infty} V(t) \le \frac{p\Pi}{\mu+\omega}. \tag{2.1}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_2 > T_1$ such that if $t > T_2$,

$$V(t) < p\Pi/(\mu+\omega) + \varepsilon.$$

We therefore derive from Equation (1.1a) that, for $t > T_2$,

$$\dot{S}(t) \le (1-p)\Pi + \omega\left(\frac{p\Pi}{\mu+\omega} + \varepsilon\right) - \mu S,$$

which yields

$$\limsup_{t\to+\infty} S(t) \le \frac{(1-p)\Pi}{\mu} + \frac{\omega}{\mu}\left(\frac{p\Pi}{\mu+\omega} + \varepsilon\right).$$

Since this inequality holds for arbitrary $\varepsilon > 0$ sufficiently small, it follows that

$$\limsup_{t\to+\infty} S(t) \le \frac{\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)}. \tag{2.2}$$

Hence, for $\varepsilon > 0$ sufficiently small, there exists a $T_3 > T_2$ such that if $t > T_3$,

$$S(t) \le \frac{\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)} + \varepsilon.$$

That is, the arbitrary positive solution $(S(t), V(t), E(t), I(t))$ of the model (1.1a) − (1.1d) is ultimately bounded. This completes the proof. □

Denote

$$D = \Big\{(S, V, E, I) \in R^4_{+0} : S + V + E + I \le \frac{\Pi}{\mu}, \\ S \le \frac{\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)}, V \le \frac{p\Pi}{\mu+\omega}\Big\}.$$

Theorem 2.1 implies that the set D is a positively invariant and the attracting region for the disease transmission model given by the model (1.1a) − (1.1d) with initial conditions in R^4_+.

3. Existence and Stability of the Disease-Free Equilibrium

In this section, we study the local stability of the disease-free equilibrium of the model $(1.1a)-(1.1d)$ by analyzing the characteristic equation and discuss the global stability of the disease-free equilibrium by using comparison arguments.

The model $(1.1a)-(1.1d)$ always has a disease-free equilibrium $P_0=(S_0,V_0,E_0,I_0)=(\Pi((1-p)\mu+\omega)/(\mu(\mu+\omega)), p\Pi/(\mu+\omega),0,0)$. Following the method of next generation matrix by van den Driessche and Watmough [10], one obtains the basic reproduction number for the model $(1.1a)-(1.1d)$ as

$$R_0=\frac{\eta\beta\Pi(\delta+d+\mu)((1-p)\mu+\omega)+\alpha\beta\Pi((1-\tau p)\mu+\omega)}{\mu(\mu+\omega)(\alpha+\mu)(\delta+d+\mu)}.$$

Theorem 3.1. *The disease-free equilibrium P_0 is locally asymptotically stable if $R_0<1$ and unstable if $R_0>1$.*

Proof. The characteristic equation of the model $(1.1a)-(1.1d)$ at the disease-free equilibrium P_0 is of the form

$$\begin{aligned}(\lambda+\mu)(\lambda+\mu+\omega)\Big[\lambda^2+\Big((\alpha+\mu)+(\delta+d+\mu)\\ -\frac{\eta\beta\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)}\Big)\lambda+(\alpha+\mu)(\delta+d+\mu)(1-R_0)\Big]=0.\end{aligned}\tag{3.1}$$

Clearly, Equation (3.1) always has two negative real roots $\lambda_1=-\mu$, $\lambda_2=-\mu-\omega$. The other roots λ_3,λ_4 of Equation (3.1) are determined by the following equation

$$\begin{aligned}\lambda^2+\left((\alpha+\mu)+(\delta+d+\mu)-\frac{\eta\beta\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)}\right)\lambda\\ +(\alpha+\mu)(\delta+d+\mu)(1-R_0)=0.\end{aligned}\tag{3.2}$$

If $R_0>1$, Equation (3.2) has one positive real part root. Hence, P_0 is unstable.

If $R_0<1$,

$$\lambda_3\lambda_4=(\alpha+\mu)(\delta+d+\mu)(1-R_0)>0,$$

$$\begin{aligned}\lambda_3+\lambda_4&=\frac{\eta\beta\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)}-(\alpha+\mu)-(\delta+d+\mu)\\ &\le\left(1-\frac{\alpha\beta\Pi((1-\tau p)\mu+\omega)}{\mu(\mu+\omega)(\alpha+\mu)(\delta+d+\mu)}\right)(\alpha+\mu)-(\alpha+\mu)-(\delta+d+\mu)\\ &=-\frac{\alpha\beta\Pi((1-\tau p)\mu+\omega)}{\mu(\mu+\omega)(\delta+d+\mu)}-(\delta+d+\mu)<0.\end{aligned}$$

Hence, the characteristic roots of Equation (3.2) have negative real parts. Therefore, P_0 is locally asymptotically stable. □

Theorem 3.1 implies that the disease can be eliminated when $R_0 < 1$ if the initial sizes of the sub-populations of the model are in the basin of attraction of P_0. To ensure the disease eradication is independent of the initial sizes of the sub-populations of the model, we study the global stability of the disease-free equilibrium P_0 in the following.

Theorem 3.2. *The disease-free equilibrium P_0 is globally asymptotically stable if $R_0 < 1$.*

Proof. Let $(S(t), V(t), E(t), I(t))$ be any positive solution of the model $(1.1a) - (1.1d)$ with initial conditions in R_4^+.

Since $R_0 < 1$, we can choose $\varepsilon > 0$ small enough such that

$$R_0 + \frac{\eta\beta(\delta+\mu+d) + (2-\tau)\alpha\beta}{(\alpha+\mu)(\delta+\mu+d)}\varepsilon < 1. \tag{3.3}$$

By (2.1) and (2.2), for $\varepsilon > 0$ satisfying (3.3), there exists a $T_4 > 0$ such that if $t > T_4$,

$$S(t) \le \frac{\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)} + \varepsilon, \quad V(t) \le \frac{p\Pi}{\mu+\omega} + \varepsilon.$$

From Equation(1.1c), it is easy to know that if $t > T_4$,

$$\begin{aligned}\dot{E}(t) \le & \left(\eta\beta\left(\frac{\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)} + \varepsilon\right) - (\alpha+\mu)\right)E \\ & + \left(\beta\left(\frac{\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)} + \varepsilon\right) + (1-\tau)\beta\left(\frac{p\Pi}{\mu+\omega} + \varepsilon\right)\right)I.\end{aligned}$$

Consider the following auxiliary system

$$\begin{aligned}\dot{u_1}(t) = & \left(\eta\beta\left(\frac{\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)} + \varepsilon\right) - (\alpha+\mu)\right)u_1 \\ & + \left(\beta\left(\frac{\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)} + \varepsilon\right) + (1-\tau)\beta\left(\frac{p\Pi}{\mu+\omega} + \varepsilon\right)\right)u_2, \\ \dot{u_2}(t) = & \alpha u_1 - (\delta+d+\mu)u_2.\end{aligned} \tag{3.4}$$

It is easy to prove that the equilibrium $(0,0)$ of system (3.4) is globally asymptotically stable for (3.3). By comparison, it follows that

$$\lim_{t\to\infty} E(t) = 0, \quad \lim_{t\to\infty} I(t) = 0. \tag{3.5}$$

Hence, for arbitrary $\varepsilon > 0$, there exists a $T_5 > 0$ such that if $t > T_5$, $E(t) < \varepsilon, I(t) < \varepsilon$. From Equations $(1.1a) - (1.1b)$, it is easy to know that if $t > T_5$,

$$\begin{aligned}\dot{S}(t) \geq& (1-p)\Pi - (\beta\varepsilon + \eta\beta\varepsilon + \mu)S + \omega V,\\ \dot{V}(t) \geq& p\Pi - ((1-\tau)\beta\varepsilon + \mu + \omega)V.\end{aligned}$$

Consider the following auxiliary system

$$\begin{aligned}\dot{v_1}(t) =& (1-p)\Pi - (\beta\varepsilon + \eta\beta\varepsilon + \mu)v_1 + \omega v_2,\\ \dot{v_2}(t) =& p\Pi - ((1-\tau)\beta\varepsilon + \mu + \omega)v_2.\end{aligned} \tag{3.6}$$

It is easy to prove that the equilibrium $(((1-p)\Pi((1-\tau)\beta\varepsilon+\mu)+\omega\Pi)/((\beta\varepsilon+\eta\beta\varepsilon+\mu)((1-\tau)\beta\varepsilon+\mu+\omega)), p\Pi/((1-\tau)\beta\varepsilon+\mu+\omega))$ of system (3.6) is globally asymptotically stable. By comparison, it follows that

$$\begin{aligned}\liminf_{t\to\infty} S(t) \geq& \frac{(1-p)\Pi((1-\tau)\beta\varepsilon+\mu)+\omega\Pi}{(\beta\varepsilon+\eta\beta\varepsilon+\mu)((1-\tau)\beta\varepsilon+\mu+\omega)},\\ \liminf_{t\to\infty} V(t) \geq& \frac{p\Pi}{(1-\tau)\beta\varepsilon+\mu+\omega}.\end{aligned}$$

Since this inequality holds for arbitrary $\varepsilon > 0$ sufficiently small, it follows that

$$\liminf_{t\to\infty} S(t) \geq \frac{\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)}, \quad \liminf_{t\to\infty} V(t) \geq \frac{p\Pi}{\mu+\omega}. \tag{3.7}$$

By (2.1), (2.2) and (3.7), it follows that

$$\lim_{t\to\infty} S(t) = \frac{\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)}, \quad \lim_{t\to\infty} V(t) = \frac{p\Pi}{\mu+\omega}. \tag{3.8}$$

By (3.5) and (3.8), P_0 is globally asymptotically stable when $R_0 < 1$. The proof is complete. □

4. Permanence

In this section, we study the permanence of the model $(1.1a) - (1.1d)$ by the persistence theory on infinite dimensional systems developed by Hale and Waltman[11].

Let X be a complete metric space. Suppose that $X^0 \subset X, X_0 \subset X, X^0 \cap X_0 = \phi$. Assume that $T(t)$ is a C_0 semigroup on X satisfying

$$T(t) : X^0 \to X^0, \quad T(t) : X_0 \to X_0. \tag{4.1}$$

Let $T_b(t) = T(t)|_{X_0}$ and A_b be the global attractor for $T_b(t)$. Define $W^s(A)$ the strong stable set of a compact invariant set A as $W^s(A) = \{x : x \in X, \omega(x) \neq \varnothing, \omega(x) \subset A\}$.The following is a small variant of Theorem 4.1 in [11].

Lemma 4.1. *(see [11]) Suppose that $T(t)$ satisfies (4.1) and we have the following:*
(i) *There is a $t_0 \geq 0$ such that $T(t)$ is compact for $t > t_0$;*
(ii)*$T(t)$ is point dissipative in X;*
(iii) *$\widetilde{A}_b = \bigcup_{x \in A_b} \omega(x)$ is isolated and has an acyclic covering $\widetilde{M}$, where*

$$\widetilde{M} = \{M_1, M_2, \cdots, M_n\};$$

(iv) *$W^s(M_i) \cap X^0 = \varnothing$ for $i = 1, 2, ..., n$.*
Then X_0 is a uniform repeller with respect to X^0, that is, there is an $\varepsilon > 0$ such that for any $x \in X^0, \liminf_{t\to+\infty} \mathrm{d}(T(t)x, X_0) \geq \varepsilon$, where d is the distance of $T(t)x$ from X_0.

Theorem 4.1. *If $R_0 > 1$, then the model $(1.1a) - (1.1d)$ is permanent.*

Proof. Let $(S(t), V(t), E(t), I(t))$ be any solution of the model $(1.1a) - (1.1d)$ with initial conditions in R^4_+. By Equation $(1.1b)$, we obtain that

$$\dot{V}(t) \geq p\Pi - \left((1-\tau)\beta\left(\frac{\Pi}{\mu} + \varepsilon\right) + \mu + \omega\right)V,$$

for $t > T_1$, which yields

$$\liminf_{t\to+\infty} V(t) \geq \frac{p\Pi}{(1-\tau)\beta(\Pi/\mu + \varepsilon) + \mu + \omega}.$$

Since this inequality holds for arbitrary $\varepsilon > 0$ sufficiently small, it follows that

$$\liminf_{t\to+\infty} V(t) \geq \frac{\mu p\Pi}{(1-\tau)\beta\Pi + \mu(\mu + \omega)} := m_1. \tag{4.2}$$

By (4.2), for $\varepsilon > 0$ sufficiently small, there is a $T_6 > T_5$ such that if $t > T_6$, $V(t) > m_1 - \varepsilon$. We therefore derive from Equation $(1.1a)$ that, for $t > T_6$,

$$\dot{S}(t) \geq (1-p)\Pi + \omega(m_1 - \varepsilon) - \left(\beta\left(\frac{\Pi}{\mu} + \varepsilon\right) + \eta\beta\left(\frac{\Pi}{\mu} + \varepsilon\right) + \mu\right)S,$$

which yields

$$\liminf_{t\to+\infty} S(t) \geq \frac{(1-p)\Pi + \omega(m_1 - \varepsilon)}{\beta(\Pi/\mu + \varepsilon) + \eta\beta(\Pi/\mu + \varepsilon) + \mu}.$$

Since this inequality holds for arbitrary $\varepsilon > 0$ sufficiently small, it follows that

$$\liminf_{t\to+\infty} S(t) \geq \frac{((1-p)\Pi + \omega m_1)\mu}{(1+\eta)\beta\Pi + \mu^2} := m_2. \tag{4.3}$$

If there exists $m_3 > 0$ such that $\liminf\limits_{t\to+\infty} E(t) \geq m_3$, then for $\varepsilon > 0$ sufficiently small, there is a $T_7 > T_6$ such that if $t > T_7$, $E(t) > m_3 - \varepsilon$. We therefore derive from Equation (1.1d) that, for $t > T_7$,

$$\dot{I}(t) \geq \alpha(m_3 - \varepsilon) - (\delta + d + \mu)I,$$

which yields

$$\liminf_{t\to+\infty} I(t) \geq \frac{\alpha(m_3 - \varepsilon)}{\delta + d + \mu}.$$

Since this inequality holds for arbitrary $\varepsilon > 0$ sufficiently small, it follows that

$$\liminf_{t\to+\infty} I(t) \geq \frac{\alpha m_3}{\delta + d + \mu} := m_4. \tag{4.4}$$

Hence, it suffices to prove that $\liminf\limits_{t\to+\infty} E(t) \geq m_3$ for the permanence of the model (1.1a) – (1.1d).

Let

$$X = \{(S(t), V(t), E(t), I(t)) | S(t) \geq 0, V(t) \geq 0, E(t) \geq 0, I(t) \geq 0\},$$

$$X_0 = \{(S(t), V(t), E(t), I(t)) | S(t) \geq 0, V(t) \geq 0, E(t) > 0, I(t) \geq 0\},$$

$$\partial X_0 = X \backslash X_0 = \{(S(t), V(t), E(t), I(t)) | S(t) \geq 0, V(t) \geq 0, E(t) = 0, I(t) \geq 0\}.$$

It is easy to show that X_0 and ∂X_0 are positively invariant and the condition (ii) in Lemma 4.1 is clearly satisfied. The solution of system (1.1) is ultimately bounded by Theorem 2.1. Thus, the condition (i) in Lemma 4.1 is satisfied by the smoothing property of solutions of delay differential equations introduced in Kuang[12] (Theorem 2.2.8).

Next, we verify that the condition (iii) in Lemma 4.1 is satisfied.

Denote

$$M_\partial = \{(S(0), V(0), E(0), I(0)) | (S(t), V(t), E(t), I(t)) \in \partial X_0, t \geq 0\},$$

and

$$\Omega = \cup \{\omega(S(0), V(0), E(0), I(0)) | (S(0), V(0), E(0), I(0)) \in M_\partial\},$$

here, $\omega(S(0), V(0), E(0), I(0))$ denotes the $\omega-$limit set of the solution of the model $(1.1a)-(1.1d)$ starting in $(S(0), V(0), E(0), I(0)) \in X$.

Restrict the model $(1.1a)-(1.1d)$ to M_∂, then $I = 0$ by Equation $(1.1d)$ and

$$\begin{aligned} \dot{S}(t) &= (1-p)\Pi - \mu S + \omega V, \\ \dot{V}(t) &= p\Pi - \mu V - \omega V. \end{aligned} \tag{4.5}$$

System (4.5) has a unique equilibrium $P_1(((1-p)\mu+\omega)\Pi/\mu(\mu+\omega), p\Pi/(\mu+\omega))$. It is easy to prove that the equilibrium P_1 is globally asymptotically stable. Hence, $\Omega = \{P_0\}$. That is, $\{P_0\}$ is a covering of Ω. Therefore, Ω is isolated and has an acyclic covering satisfying the conditions (iii) in Lemma 4.1.

We now show that $W^s(P_0) \cap X_0 = \varnothing$. Assume $W^s(P_0) \cap X_0 \neq \varnothing$. Then there exists a solution $(S(t), V(t), E(t), I(t)) \in X_0$ satisfying

$$\lim_{t\to\infty} S(t) = \frac{((1-p)\mu+\omega)\Pi}{\mu(\mu+\omega)}, \lim_{t\to\infty} V(t) = \frac{p\Pi}{\mu+\omega}, \lim_{t\to\infty} E(t) = 0, \lim_{t\to\infty} I(t) = 0.$$

Since $R_0 > 1$, we can choose $\varepsilon > 0$ small enough such that

$$R_0 - \frac{\eta\beta(\delta+\mu+d) + (2-\tau)\alpha\beta}{(\alpha+\mu)(\delta+\mu+d)}\varepsilon > 1, \tag{4.6}$$

and for $\varepsilon > 0$ satisfying (4.6), there exists a $T_8 > T_7$ such that if $t > T_8$,

$$S(t) \geq \frac{((1-p)\mu+\omega)\Pi}{\mu(\mu+\omega)} - \varepsilon, \quad V(t) \geq \frac{p\Pi}{\mu+\omega} - \varepsilon.$$

From Equation $(1.1c)$, it is easy to know that if $t > T_8$,

$$\begin{aligned} \dot{E}(t) \geq& \left(\eta\beta\left(\frac{((1-p)\mu+\omega)\Pi}{\mu(\mu+\omega)} - \varepsilon\right) - (\alpha+\mu)\right)E \\ &+ \left(\beta\left(\frac{((1-p)\mu+\omega)\Pi}{\mu(\mu+\omega)} - \varepsilon\right) + (1-\tau)\beta\left(\frac{p\Pi}{\mu+\omega} - \varepsilon\right)\right)I. \end{aligned}$$

Consider the following auxiliary system

$$\begin{aligned} \dot{z}_1(t) =& \left(\eta\beta\left(\frac{((1-p)\mu+\omega)\Pi}{\mu(\mu+\omega)} - \varepsilon\right) - (\alpha+\mu)\right)z_1 \\ &+ \left(\beta\left(\frac{((1-p)\mu+\omega)\Pi}{\mu(\mu+\omega)} - \varepsilon\right) + (1-\tau)\beta\left(\frac{p\Pi}{\mu+\omega} - \varepsilon\right)\right)z_2, \\ \dot{z}_2(t) =& \alpha z_1 - (\delta+d+\mu)z_2. \end{aligned} \tag{4.7}$$

The Jacobian matrix of system (4.7) is

$$J_\varepsilon=\begin{pmatrix}\eta\beta\left(\frac{((1-p)\mu+\omega)\Pi}{\mu(\mu+\omega)}-\varepsilon\right)-(\alpha+\mu) & \beta\left(\frac{((1-p)\mu+\omega)\Pi}{\mu(\mu+\omega)}-\varepsilon\right)+(1-\tau)\beta\left(\frac{p\Pi}{\mu+\omega}-\varepsilon\right)\\ \alpha & -(\delta+d+\mu)\end{pmatrix}.$$

Since J_ε admits positive off-diagonal elements, the Perron-Frobenius theorem implies that there is a positive eigenvector $p=(p_1,p_2)$ for the maximum root γ of J_ε.

The characteristic equation of system (4.7) is

$$\lambda^2+b_1\lambda+b_2=0, \tag{4.8}$$

where

$$\begin{aligned} b_1&=\eta\beta\left(\frac{((1-p)\mu+\omega)\Pi}{\mu(\mu+\omega)}-\varepsilon\right)-\alpha-\delta-d-2\mu,\\ b_2&=(\alpha+\mu)(\delta+d+\mu)\left(1-\left(R_0-\frac{\eta\beta(\delta+\mu+d)+(2-\tau)\alpha\beta)}{(\alpha+\mu)(\delta+\mu+d)}\varepsilon\right)\right).\end{aligned} \tag{4.9}$$

Since (4.6) holds, it is shown that the maximum root γ of J_ε is positive by a simple computation.

Let $z(t)=(z_1(t),z_2(t))$ be a solution of system (4.7) through (lp_1,lp_2) at $t=t_0$, where $l>0$ satisfies $lp_1<E(t_0),lp_2<I(t_0)$.

We know that

$$\begin{pmatrix}z_1\\ z_2\end{pmatrix}=\begin{pmatrix}le^{\gamma(t-t_0)}p_1\\ le^{\gamma(t-t_0)}p_2\end{pmatrix}.$$

Clearly, $z_i(t)$ is strictly increasing and $z_i(t)\to+\infty$ as $t\to+\infty$, $i=1,2$. Consequently, $E(t)\to+\infty$, $I(t)\to+\infty$ as $t\to+\infty$, contradicting $E(t)\to 0$, $I(t)\to 0$ as $t\to+\infty$. Hence, $W^s(P_0)\cap X_0=\varnothing$.

By Lemma 4.1, we conclude that ∂X_0 repels positive solutions of the model $(1.1a)-(1.1d)$ uniformly. That is, there exists a $m_3>0$ such that

$$\liminf_{t\to+\infty}E(t)\geq m_3. \tag{4.10}$$

By Theorem 2.1, (4.2), (4.3), (4.4) and (4.10), the model $(1.1a)-(1.1d)$ is permanent for $R_0>1$. This proof is complete. □

5. Existence and Stability of the Endemic Equilibrium

In this section, we are concerned with the existence of the endemic equilibrium and prove that the endemic equilibrium is globally attractive by the theory of the compound matrices.

To find conditions for the existence of the endemic equilibrium $P^*(S^*, V^*, E^*, I^*)$, we solve

$$(1-p)\Pi - \beta S^* I^* - \eta\beta E^* S^* - \mu S^* + \omega V^* = 0, \tag{5.1a}$$
$$p\Pi - (1-\tau)\beta V^* I^* - \mu V^* - \omega V^* = 0, \tag{5.1b}$$
$$\beta S^* I^* + \eta\beta E^* S^* + (1-\tau)\beta V^* I^* - \alpha E^* - \mu E^* = 0, \tag{5.1c}$$
$$\alpha E^* - \delta I^* - dI^* - \mu I^* = 0. \tag{5.1d}$$

From Equations (5.1a), (5.1b) and (5.1d), we get

$$S^* = \frac{(1-p)\Pi + \omega V^*}{\beta I^* + \eta\beta E^* + \mu}, V^* = \frac{p\Pi}{(1-\tau)\beta I^* + \mu + \omega}, E^* = \frac{\mu+\delta+d}{\alpha} I^*. \tag{5.2}$$

Equation (5.1c) yields

$$E^* = \frac{\beta S^* I^* + (1-\tau)\beta V^* I^*}{\alpha + \mu - \eta\beta S^*}. \tag{5.3}$$

Substituting the expressions of S^*, E^*, V^* in Equation (5.2) into Equation (5.3), which gives

$$Q(I^*) = AI^{*2} + BI^* + C = 0, \tag{5.4}$$

where

$$\begin{aligned} A =& (\alpha+\mu)(\mu+\delta+d)(\alpha+\eta(\mu+\delta+d))(1-\tau)\beta^2, \\ B =& \beta(\alpha+\mu)(\mu+\omega)(\mu+\delta+d)(\alpha+\eta(\mu+\delta+d))\Big[1+ \\ & \frac{(1-\tau)\alpha\mu}{(\mu+\omega)(\alpha+\eta(\mu+\delta+d))}\Big(1-R_0-\frac{(\mu+\delta+d)\eta\Pi\beta p + \alpha\tau\beta p\Pi}{(\alpha+\mu)(\mu+\omega)(\mu+\delta+d)}\Big)\Big], \\ C =& \alpha\mu(\alpha+\mu)(\mu+\omega)(\mu+\delta+d)(1-R_0). \end{aligned} \tag{5.5}$$

The endemic equilibrium of the model (1.1a) − (1.1d) are given by Equation (5.2) with the positive root of Equation (5.4).

Suppose $\tau = 1$, then $A = 0, B > 0$. Hence, $Q(I) = BI + C$, with the root $I = -C/B$. If $R_0 > 1$, $C < 0$, then Equation (5.4) has a unique positive root.

Suppose $0 \le \tau < 1$, then $A > 0$. If $R_0 > 1$, $C < 0$, then Equation (5.4) has a unique positive root.

In conclusion, we have the following results.

Theorem 5.1. *If $R_0 > 1$, the model* $(1.1a)-(1.1d)$ *has a unique endemic equilibrium* $P^*(S^*, V^*, E^*, I^*)$.

In the following, the method developed in [13], [14] is used to discuss the global attraction of the endemic equilibrium $P^*(S^*,V^*,E^*,I^*)$. First, we introduce this method briefly.

Let $G \subset R^n$ be an open set. Consider the differential equation:

$$\dot{x} = f(x), \tag{5.6}$$

where, the function $f : x \to f(x) \in R^n, x \in G$ is continuous on G.

Let $Q(x)$ be a $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function that is continuous on G and consider

$$A = Q_f Q^{-1} + Q J^{[2]} Q^{-1},$$

where the matrix Q_f is the derivative of Q in the direction of the vector field f in system (5.6), and $J^{[2]}$ is the second additive compound matrix of the Jacobian matrix of system (5.6).

Lemma 5.1. [13] *If G_1 is a compact absorbing subset in the interior of G, and there exist a $\gamma > 0$ and a Lozinskiĭ measure $\bar{\mu}(A) \leq -\gamma$ for all $x \in G_1$, then every omega limit point of system (5.6) in the interior of G is an equilibrium in G_1.*

According to [15], the *Lozinskiĭ* measure in Lemma 5.1 can be evaluated as:

$$\bar{\mu}(A) = \inf\{\bar{k} : D_+\|z\| \leq \bar{k}\|z\|, for\ \ all\ \ solutions\ \ of\ \ z' = Az\}, \tag{5.7}$$

where D_+ is the right-hand derivative.

Theorem 5.2. *If $R_0 > 1$, then the endemic equilibrium P^* is globally attractive provided that:*

$$u > \max\left\{\alpha - \omega,\ \frac{2\eta\beta\Pi((1-p)\mu + \omega)}{\mu(\mu + \omega)} + \omega\right\}. \tag{5.8}$$

Proof. Let $(S(t), V(t), E(t), I(t))$ be any positive solution of the model $(1.1a) - (1.1d)$ with initial conditions in R_4^+. The second additive compound matrix $J^{[2]}$ associated with the solution $(S(t), V(t),$
$E(t), I(t))$ is

$$J^{[2]} = -diag\begin{pmatrix} 2\mu+\omega+\eta\beta E+(2-\tau)\beta I \\ 2\mu+\alpha+\beta I+\eta\beta E-\eta\beta S \\ 2\mu+\delta+d+\beta I+\eta\beta E \\ 2\mu+\omega+\alpha+(1-\tau)\beta I-\eta\beta S \\ 2\mu+\delta+d+\omega+(1-\tau)\beta I \\ 2\mu+\delta+d+\alpha-\eta\beta S \end{pmatrix}$$

$$+\begin{pmatrix} 0 & 0 & -(1-\tau)\beta V & \eta\beta S & \beta S & 0 \\ (1-\tau)\beta I & 0 & \beta S+(1-\tau)\beta V & \omega & 0 & \beta S \\ 0 & \alpha & 0 & 0 & \omega & -\eta\beta S \\ -\beta I-\eta\beta E & 0 & 0 & 0 & \beta S+(1-\tau)\beta V & (1-\tau)\beta V \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta I+\eta\beta E & 0 & (1-\tau)\beta I & 0 \end{pmatrix}.$$

Define

$$Q = \begin{pmatrix} 1/E & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/E & 0 & 0 \\ 0 & 0 & 1/I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/I & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/I \end{pmatrix},$$

then $Q_f Q^{-1} = -diag\{\dot{E}/E, \dot{E}/E, \dot{E}/E, \dot{I}/I, \dot{I}/I, \dot{I}/I\}$, and
$A = Q_f Q^{-1} + QJ^{[2]}Q^{-1}$

$$= -diag\begin{pmatrix} (2-\tau)\beta I+\eta\beta E+\eta\beta S+\frac{\beta SI+(1-\tau)\beta VI}{E}+\omega+\mu-\alpha \\ \beta I+\eta\beta E+\frac{\beta SI+(1-\tau)\beta VI}{E}+\mu \\ (1-\tau)\beta I+\frac{\beta SI+(1-\tau)\beta VI}{E}+\mu+\omega \\ \beta I+\eta\beta E+\frac{\alpha E}{I}+\mu \\ (1-\tau)\beta I+\frac{\alpha E}{I}+\mu+\omega \\ \frac{\alpha E}{I}+\mu+d-\eta\beta S \end{pmatrix}$$

$$+\begin{pmatrix} 0 & 0 & \eta\beta S & -\frac{(1-\tau)\beta VI}{E} & \frac{\beta SI}{E} & 0 \\ (1-\tau)\beta I & 0 & \omega & \frac{\beta SI+(1-\tau)\beta VI}{E} & 0 & \frac{\beta SI}{E} \\ -\beta I-\eta\beta E & 0 & 0 & 0 & \frac{\beta SI+(1-\tau)\beta VI}{E} & \frac{(1-\tau)\beta VI}{E} \\ 0 & \frac{\alpha E}{I} & 0 & 0 & \omega & -\eta\beta S \\ 0 & 0 & \frac{\alpha E}{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta I+\eta\beta E & (1-\tau)\beta I & 0 \end{pmatrix}.$$

Define the norm on R^6[6]:

$$\|z\| = \max\{U_1, U_2\},$$

where $z \in R^6$ with component $z_i, i = 1, 2, \cdots, 6$, and

$$U_1(z_1,z_2,z_3)=\begin{cases}\max\{|z_1|, |z_2|+|z_3|\} & \text{if } \operatorname{sgn}(z_1)=\operatorname{sgn}(z_2)=\operatorname{sgn}(z_3)\\ \max\{|z_2|, |z_1|+|z_3|\} & \text{if } \operatorname{sgn}(z_1)=\operatorname{sgn}(z_2)=-\operatorname{sgn}(z_3)\\ \max\{|z_1|, |z_2|, |z_3|\} & \text{if } \operatorname{sgn}(z_1)=-\operatorname{sgn}(z_2)=\operatorname{sgn}(z_3)\\ \max\{|z_1|+|z_3|, |z_2|+|z_3|\} & \text{if } -\operatorname{sgn}(z_1)=\operatorname{sgn}(z_2)=\operatorname{sgn}(z_3)\end{cases},$$

$$U_2(z_4,z_5,z_6)=\begin{cases}|z_4|+|z_5|+|z_6| & \text{if } \operatorname{sgn}(z_4)=\operatorname{sgn}(z_5)=\operatorname{sgn}(z_6)\\ \max\{|z_4|+|z_5|, |z_4|+|z_6|\} & \text{if } \operatorname{sgn}(z_4)=\operatorname{sgn}(z_5)=-\operatorname{sgn}(z_6)\\ \max\{|z_5|, |z_4|+|z_6|\} & \text{if } \operatorname{sgn}(z_4)=-\operatorname{sgn}(z_5)=\operatorname{sgn}(z_6)\\ \max\{|z_4|+|z_6|, |z_5|+|z_6|\} & \text{if } -\operatorname{sgn}(z_4)=\operatorname{sgn}(z_5)=\operatorname{sgn}(z_6)\end{cases}.$$

Clearly,

$$|z_2| \le U_1(z), |z_3| \le U_1(z), |z_2 + z_3| \le U_1(z),$$

and

$$|z_i| \le U_2(z), |z_i + z_j| \le U_2(z), |z_4 + z_5 + z_6| \le U_2(z), \quad i = 4, 5, 6, i \ne j.$$

for all $z = (z_1, z_2, z_3, z_4, z_5, z_6) \in R^6$.

Case 1: $U_1(z) > U_2(z)$, $\operatorname{sgn}(z_1) = \operatorname{sgn}(z_2) = \operatorname{sgn}(z_3)$ and $|z_1| > |z_2| + |z_3|$, then $\|z\| = |z_1|$. Taking the right-hand derivative of $\|z\|$, we get

$$\begin{aligned} D_+\|z\| &= D_+(|z_1|)\\ &\le \left(-(2-\tau)\beta I - \eta\beta E - \eta\beta S - \frac{\beta SI + (1-\tau)\beta VI}{E} - \omega - \mu + \alpha\right)|z_1|\\ &\quad + \eta\beta S|z_3| + \frac{(1-\tau)\beta VI}{E}|z_4| + \frac{\beta SI}{E}|z_5|. \end{aligned}$$

Since $|z_3| < |z_1|$ and $|z_4|, |z_5| \le U_2(z) < U_1(z) \le |z_1|$, we have

$$D_+\|z\| \le (-(2-\tau)\beta I - \eta\beta E - \omega - \mu + \alpha)\,|z_1|.$$

Thus,

$$D_+\|z\| \le (\alpha - \omega - \mu)\|z\|. \tag{5.9}$$

Case 2: $U_1(z) > U_2(z)$, $\operatorname{sgn}(z_1) = \operatorname{sgn}(z_2) = \operatorname{sgn}(z_3)$ and $|z_2| + |z_3| > |z_1|$, then $\|z\| = |z_2| + |z_3|$. Taking the right-hand derivative of $\|z\|$, we get

$$\begin{aligned} D_+\|z\| &= D_+(|z_2| + |z_3|) \\ &\le(-\tau\beta I - \eta\beta E)|z_1| + \left(-\beta I - \eta\beta E - \frac{\beta SI + (1-\tau)\beta VI}{E} - \mu\right)|z_2| \\ &\quad + \left(-(1-\tau)\beta I - \frac{\beta SI + (1-\tau)\beta VI}{E} - \mu\right)|z_3| \\ &\quad + \frac{\beta SI + (1-\tau)\beta VI}{E}|z_4 + z_5 + z_6|. \end{aligned}$$

Since $|z_4 + z_5 + z_6| \le U_2(z) < U_1(z) \le |z_2| + |z_3|$ and $(-\tau\beta I - \eta\beta E)|z_1| < 0$, we have

$$D_+\|z\| \le (-\beta I - \eta\beta E - \mu)|z_2| + (-(1-\tau)\beta I - \mu)|z_3| \le -\mu(|z_2| + |z_3|).$$

Thus,

$$D_+\|z\| \le -\mu\|z\|. \tag{5.10}$$

<u>Case 3</u>: $U_1(z) > U_2(z)$, $\operatorname{sgn}(z_1) = \operatorname{sgn}(z_2) = -\operatorname{sgn}(z_3)$ and $|z_2| > |z_1| + |z_3|$, then $\|z\| = |z_2|$. Taking the right-hand derivative of $\|z\|$, we get

$$\begin{aligned} D_+\|z\| &= D_+(|z_2|) \\ &\le(1-\tau)\beta I|z_1| + \left(-\beta I - \eta\beta E - \frac{\beta SI + (1-\tau)\beta VI}{E} - \mu\right)|z_2| \\ &\quad + \frac{\beta SI}{E}|z_4 + z_6| + \frac{(1-\tau)\beta VI}{E}|z_4|. \end{aligned}$$

Since $|z_4|, |z_4 + z_6| \le U_2(z) < U_1(z) \le |z_2|$, and $|z_1| < |z_2|$, we have

$$D_+\|z\| \le (-\tau\beta I - \eta\beta E - \mu)|z_2| \le -\mu|z_2|.$$

Thus,

$$D_+\|z\| \le -\mu\|z\|. \tag{5.11}$$

<u>Case 4</u>: $U_1(z) > U_2(z)$, $\operatorname{sgn}(z_1) = \operatorname{sgn}(z_2) = -\operatorname{sgn}(z_3)$ and $|z_1| + |z_3| > |z_2|$, then $\|z\| = |z_1| + |z_3|$. Taking the right-hand derivative of $\|z\|$, we get

$$\begin{aligned} D_+\|z\| &= D_+(|z_1| + |z_3|) \\ &\le \left(-\eta\beta S - (1-\tau)\beta I - \frac{\beta SI + (1-\tau)\beta VI}{E} - \omega - \mu + \alpha\right)|z_1| \\ &\quad + \left(-\eta\beta S - (1-\tau)\beta I - \frac{\beta SI + (1-\tau)\beta VI}{E} - \mu - \omega\right)|z_3| \\ &\quad + \frac{(1-\tau)\beta VI}{E}|z_4 + z_5 + z_6|. \end{aligned}$$

Since $|z_4 + z_5 + z_6| \leq U_2(z) < U_1(z) \leq |z_1| + |z_3|$, we have

$$\begin{aligned} D_+\|z\| \leq & \left(-\eta\beta S - (1-\tau)\beta I - \frac{\beta SI}{E} - \omega - \mu + \alpha\right)|z_1| \\ & + \left(-\eta\beta S - (1-\tau)\beta I - \frac{\beta SI}{E} - \mu - \omega\right)|z_3| \\ \leq & (\alpha - \mu - \omega)(|z_1| + |z_3|). \end{aligned}$$

Thus,

$$D_+\|z\| \leq (\alpha - \mu - \omega)\|z\|. \tag{5.12}$$

Case 5: $U_1(z) > U_2(z)$, $\operatorname{sgn}(z_1) = -\operatorname{sgn}(z_2) = \operatorname{sgn}(z_3)$ and $|z_1| > |z_2|, |z_1| > |z_3|$, then $\|z\| = |z_1|$. Taking the right-hand derivative of $\|z\|$, we get

$$\begin{aligned} & D_+\|z\| = D_+(|z_1|) \\ \leq & \left(-\eta\beta E - \eta\beta S - (2-\tau)\beta I - \frac{\beta SI + (1-\tau)\beta VI}{E} - \omega - \mu + \alpha\right)|z_1| \\ & + \eta\beta S|z_3| + \frac{(1-\tau)\beta VI}{E}|z_4| + \frac{\beta SI}{E}|z_5|. \end{aligned}$$

Since $|z_3| < |z_1|$ and $|z_4|, |z_5| \leq U_2(z) < U_1(z) \leq |z_1|$, we have

$$D_+\|z\| \leq (-\eta\beta E - (2-\tau)\beta I - \omega - \mu + \alpha)|z_1| \leq (\alpha - \omega - \mu)|z_1|.$$

Thus,

$$D_+\|z\| \leq (\alpha - \omega - \mu)\|z\|. \tag{5.13}$$

Case 6: $U_1(z) > U_2(z)$, $\operatorname{sgn}(z_1) = -\operatorname{sgn}(z_2) = \operatorname{sgn}(z_3)$ and $|z_2| > |z_1|, |z_2| > |z_3|$, then $\|z\| = |z_2|$. Taking the right-hand derivative of $\|z\|$, we get

$$\begin{aligned} & D_+\|z\| = D_+(|z_2|) \\ \leq & -(1-\tau)\beta I|z_1| + \left(-\beta I - \eta\beta E - \frac{\beta SI + (1-\tau)\beta VI}{E} - \mu\right)|z_2| \\ & - \omega|z_3| + \frac{\beta SI}{E}|z_4 + z_6| + \frac{(1-\tau)\beta VI}{E}|z_4|. \end{aligned}$$

Since $|z_4|, |z_4 + z_6| \leq U_2(z) < U_1(z) \leq |z_2|$, $-(1-\tau)\beta I|z_1| < 0$ and $-\omega|z_3| < 0$, we have

$$D_+\|z\| \leq (-\beta I - \eta\beta E - \mu)|z_2| \leq -\mu|z_2|.$$

Thus,

$$D_+\|z\| \leq -\mu\|z\|. \tag{5.14}$$

Case 7: $U_1(z) > U_2(z)$, $\operatorname{sgn}(z_1) = -\operatorname{sgn}(z_2) = \operatorname{sgn}(z_3)$ and $|z_3| > |z_1|, |z_3| > |z_2|$, then $\|z\| = |z_3|$. Taking the right-hand derivative of $\|z\|$, we get

$$\begin{aligned}
&D_+\|z\| = D_+(|z_3|) \\
&\leq(-\beta I - \eta\beta E)|z_1| + \left(-(1-\tau)\beta I - \frac{\beta SI + (1-\tau)\beta VI}{E} - \mu - \omega\right)|z_3| \\
&\quad + \frac{\beta SI}{E}|z_5| + \frac{(1-\tau)\beta VI}{E}|z_5 + z_6|
\end{aligned}$$

Since $|z_5|, |z_5 + z_6| \leq U_2(z) < U_1(z) \leq |z_3|$ and $(-\beta I - \eta\beta E)|z_1| < 0$, we have

$$D_+\|z\| \leq (-(1-\tau)\beta I - \mu - \omega)|z_3| \leq -(\mu + \omega)|z_3|.$$

Thus,

$$D_+\|z\| \leq -(\mu + \omega)\|z\|. \tag{5.15}$$

Case 8: $U_1(z) > U_2(z)$, $-\operatorname{sgn}(z_1) = \operatorname{sgn}(z_2) = \operatorname{sgn}(z_3)$ and $|z_1| + |z_3| > |z_2| + |z_3|$, then $\|z\| = |z_1| + |z_3|$. Taking the right-hand derivative of $\|z\|$, we get

$$\begin{aligned}
&D_+\|z\| = D_+(|z_1| + |z_3|) \\
&\leq \left(-\eta\beta S - (1-\tau)\beta I - \frac{\beta SI + (1-\tau)\beta VI}{E} - \mu - \omega + \alpha\right)|z_1| \\
&\quad + \left(-\eta\beta S - (1-\tau)\beta I - \frac{\beta SI + (1-\tau)\beta VI}{E} - \mu - \omega\right)|z_3| \\
&\quad + \frac{(1-\tau)\beta VI}{E}|z_4 + z_5 + z_6|.
\end{aligned}$$

Since $|z_4 + z_5 + z_6| \leq U_2(z) < U_1(z) \leq |z_1| + |z_3|$ and $|z_2| < |z_1|$, we have

$$\begin{aligned}
D_+\|z\| \leq &\left(-\eta\beta S - (1-\tau)\beta I - \frac{\beta SI}{E} - \mu - \omega + \alpha\right)|z_1| \\
&+ \left(-\eta\beta S - (1-\tau)\beta I - \frac{\beta SI}{E} - \mu - \omega\right)|z_3| \\
\leq &(\alpha - \mu - \omega)(|z_1| + |z_3|).
\end{aligned}$$

Thus,

$$D_+\|z\| \leq (\alpha - \mu - \omega)\|z\|. \tag{5.16}$$

Case 9: $U_1(z) > U_2(z)$, $-\operatorname{sgn}(z_1) = \operatorname{sgn}(z_2) = \operatorname{sgn}(z_3)$ and $|z_2| + |z_3| > |z_1| + |z_3|$, then $\|z\| = |z_2| + |z_3|$. Taking the right-hand derivative of $\|z\|$, we get

$$\begin{aligned}
D_+\|z\| &= D_+(|z_2| + |z_3|) \\
&\leq (\beta I + \eta\beta E - (1-\tau)\beta I)|z_1| + \frac{\beta SI + (1-\tau)\beta VI}{E}|z_4 + z_5 + z_6| \\
&\quad + \left(-\beta I - \eta\beta E - \frac{\beta SI + (1-\tau)\beta VI}{E} - \mu\right)|z_2| \\
&\quad + \left(-(1-\tau)\beta I - \frac{\beta SI + (1-\tau)\beta VI}{E} - \mu\right)|z_3|.
\end{aligned}$$

Since $|z_4+z_5+z_6| \leq U_2(z) < U_1(z) \leq |z_2|+|z_3|$, $|z_1| < |z_2|$ and $-(1-\tau)\beta I|z_1| < 0$, we have

$$D_+\|z\| \leq -\mu|z_2| + (-(1-\tau)\beta I - \mu)\,|z_3| \leq -\mu(|z_2| + |z_3|).$$

Thus,

$$D_+\|z\| \leq -\mu\|z\|. \tag{5.17}$$

<u>Case 10</u>: $U_2(z) > U_1(z)$, $\operatorname{sgn}(z_4) = \operatorname{sgn}(z_5) = \operatorname{sgn}(z_6)$, then $\|z\| = |z_4| + |z_5| + |z_6|$. Taking the right-hand derivative of $\|z\|$, we get

$$\begin{aligned}
D_+\|z\| &= D_+(|z_4| + |z_5| + |z_6|) \\
&\leq \left(-\mu - \frac{\alpha E}{I}\right)|z_4| + \left(-\mu - \frac{\alpha E}{I}\right)|z_5| + \left(-\alpha - \mu - \frac{\alpha E}{I}\right)|z_6| \\
&\quad + \frac{\alpha E}{I}|z_2 + z_3|.
\end{aligned}$$

Since $|z_2 + z_3| \leq U_1(z) < U_2(z) \leq |z_4| + |z_5| + |z_6|$, we have

$$D_+\|z\| \leq -\mu|z_4| - \mu|z_5| + (-\alpha - \mu)|z_6| \leq -\mu(|z_4| + |z_5| + |z_6|).$$

Thus,

$$D_+\|z\| \leq -\mu\|z\|. \tag{5.18}$$

<u>Case 11</u>: $U_2(z) > U_1(z)$, $\operatorname{sgn}(z_4) = \operatorname{sgn}(z_5) = -\operatorname{sgn}(z_6)$ and $|z_4| + |z_5| > |z_4| + |z_6|$, then $\|z\| = |z_4| + |z_5|$. Taking the right-hand derivative of $\|z\|$, we get

$$\begin{aligned}
D_+\|z\| &= D_+(|z_4| + |z_5|) \\
&\leq \left(-\beta I - \eta\beta E - \mu - \frac{\alpha E}{I}\right)|z_4| + \left(-(1-\tau)\beta I - \mu - \frac{\alpha E}{I}\right)|z_5| \\
&\quad + \frac{\alpha E}{I}|z_2 + z_3| + \eta\beta S|z_6|.
\end{aligned}$$

Since $|z_2 + z_3| \le U_1(z) < U_2(z) \le |z_4| + |z_5|$, $|z_6| < |z_5|$ and $S \le \Pi((1-p)\mu + \omega)/(\mu(\mu+\omega))$ in D, we have

$$\begin{aligned} D_+\|z\| &\le (-\beta I - \eta\beta E - \mu)|z_4| + (\eta\beta S - (1-\tau)\beta I - \mu)|z_5| \\ &\le \left(\frac{\eta\beta\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)} - \mu\right)(|z_4| + |z_5|). \end{aligned}$$

Thus,

$$D_+\|z\| \le \left(\frac{\eta\beta\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)} - \mu\right)\|z\|. \tag{5.19}$$

<u>Case 12</u>: $U_2(z) > U_1(z)$, $\operatorname{sgn}(z_4) = \operatorname{sgn}(z_5) = -\operatorname{sgn}(z_6)$ and $|z_4| + |z_6| > |z_4| + |z_5|$, then $\|z\| = |z_4| + |z_6|$. Taking the right-hand derivative of $\|z\|$, we get

$$\begin{aligned} D_+\|z\| &= D_+(|z_4| + |z_6|) \\ &\le \frac{\alpha E}{I}|z_2| + \left(-2\beta I - 2\beta E - \mu - \frac{\alpha E}{I}\right)|z_4| + (\omega - (1-\tau)\beta I)|z_5| \\ &\quad + \left(2\eta\beta S - \alpha - \mu - \frac{\alpha E}{I}\right)|z_6|. \end{aligned}$$

Since $|z_2| \le U_1(z) < U_2(z) \le |z_4| + |z_6|$, $-(1-\tau)\beta I|z_5| < 0$, $|z_5| < |z_6|$ and $S \le \Pi((1-p)\mu + \omega)/(\mu(\mu+\omega))$ in D, we have

$$\begin{aligned} D_+\|z\| &\le (-2\beta I - 2\eta\beta E - \mu)|z_4| + (2\eta\beta S + \omega - \alpha - \mu)|z_6| \\ &\le \left(\frac{2\eta\beta\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)} + \omega - \mu\right)(|z_4| + |z_5|). \end{aligned}$$

Thus,

$$D_+\|z\| \le \left(\frac{2\eta\beta\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)} + \omega - \mu\right)\|z\|. \tag{5.20}$$

<u>Case 13</u>: $U_2(z) > U_1(z)$, $\operatorname{sgn}(z_4) = -\operatorname{sgn}(z_5) = \operatorname{sgn}(z_6)$ and $|z_5| > |z_4| + |z_6|$, then $\|z\| = |z_5|$. Taking the right-hand derivative of $\|z\|$, we get

$$D_+\|z\| = D_+(|z_5|) \le \left(-(1-\tau)\beta I - \mu - \omega - \frac{\alpha E}{I}\right)|z_5| + \frac{\alpha E}{I}|z_3|.$$

Since $|z_3| \le U_1(z) < U_2(z) \le |z_5|$, we have

$$D_+\|z\| \le (-(1-\tau)\beta I - \mu - \omega)|z_5| \le -(\mu+\omega)|z_5|.$$

Thus,

$$D_+\|z\| \le -(\mu+\omega)\|z\|. \tag{5.21}$$

Case 14: $U_2(z) > U_1(z)$, $\operatorname{sgn}(z_4) = -\operatorname{sgn}(z_5) = \operatorname{sgn}(z_6)$ and $|z_4|+|z_6| > |z_5|$, then $\|z\| = |z_4| + |z_6|$. Taking the right-hand derivative of $\|z\|$, we get

$$\begin{aligned} D_+\|z\| &= D_+(|z_4| + |z_6|) \\ &\leq \frac{\alpha E}{I}|z_2| + \left(-\mu - \frac{\alpha E}{I}\right)|z_4| + (-(1-\tau)\beta I - \omega)|z_5| \\ &+ \left(-\alpha - \mu - \frac{\alpha E}{I}\right)|z_6|. \end{aligned}$$

Since $|z_2| \leq U_1(z) < U_2(z) \leq |z_4| + |z_6|$ and $(-(1-\tau)\beta I - \omega)|z_5| < 0$, we have

$$D_+\|z\| \leq -\mu|z_4| + (-\alpha - \mu)|z_6| \leq -\mu(|z_4| + |z_6|).$$

Thus,

$$D_+\|z\| \leq -\mu\|z\|. \tag{5.22}$$

Case 15: $U_2(z) > U_1(z)$, $-\operatorname{sgn}(z_4) = \operatorname{sgn}(z_5) = \operatorname{sgn}(z_6)$ and $|z_4| + |z_6| > |z_5| + |z_6|$, then $\|z\| = |z_4| + |z_6|$. Taking the right-hand derivative of $\|z\|$, we get

$$\begin{aligned} D_+\|z\| &= D_+(|z_4| + |z_6|) \\ &\leq \frac{\alpha E}{I}|z_2| + \left(-2\beta I - 2\eta\beta E - \mu - \frac{\alpha E}{I}\right)|z_4| + ((1-\tau)\beta I - \omega)|z_5| \\ &+ \left(2\eta\beta S - \alpha - \mu - \frac{\alpha E}{I}\right)|z_6|. \end{aligned}$$

Since $|z_2| \leq U_1(z) < U_2(z) \leq |z_4| + |z_6|$, $|z_5| < |z_4|$ and $S \leq \Pi((1-p)\mu + \omega)/(\mu(\mu+\omega))$ in D, we have

$$\begin{aligned} D_+\|z\| &\leq (-(1+\tau)\beta I - 2\eta\beta E - \mu - \omega)|z_4| + (2\eta\beta S - \alpha - \mu)|z_6| \\ &\leq \left(\frac{2\eta\beta\Pi((1-p)\mu + \omega)}{\mu(\mu+\omega)} - \mu\right)(|z_4| + |z_6|) \end{aligned}$$

Thus,

$$D_+\|z\| \leq \left(\frac{2\eta\beta\Pi((1-p)\mu + \omega)}{\mu(\mu+\omega)} - \mu\right)\|z\|. \tag{5.23}$$

Case 16: $U_2(z) > U_1(z)$, $-\operatorname{sgn}(z_4) = \operatorname{sgn}(z_5) = \operatorname{sgn}(z_6)$ and $|z_5| + |z_6| > |z_4| + |z_6|$, then $\|z\| = |z_5| + |z_6|$. Taking the right-hand derivative of $\|z\|$, we get

$$\begin{aligned} D_+\|z\| &= D_+(|z_5| + |z_6|) \\ &\leq \frac{\alpha E}{I}|z_3| - (\beta I + \eta\beta E)|z_4| + \left(-\mu - \omega - \frac{\alpha E}{I}\right)|z_5| \\ &+ \left(\eta\beta S - \alpha - \mu - \frac{\alpha E}{I}\right)|z_6|. \end{aligned}$$

Since $|z_3| \leq U_1(z) < U_2(z) \leq |z_5| + |z_6|$, $-(\beta I + \eta\beta E)|z_4| < 0$ and $S \leq \Pi((1-p)\mu+\omega)/(\mu(\mu+\omega))$in D, we have

$$\begin{aligned} D_+\|z\| \leq & -(\mu+\omega)|z_5| + (\eta\beta S - \alpha - \mu)|z_6| \\ \leq & \left(\frac{\eta\beta\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)} - \mu\right)(|z_5|+|z_6|). \end{aligned}$$

Thus,

$$D_+\|z\| \leq \left(\frac{\eta\beta\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)} - \mu\right)\|z\|. \tag{5.24}$$

Combing the results of (5.9) − (5.24), we obtain

$$D_+\|z\| \leq \max\{\alpha-\mu-\omega,\ \frac{2\eta\beta\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)} + \omega - \mu\}\|z\|.$$

By (5.7),

$$\bar{\mu}(A) \leq \max\{\alpha-\mu-\omega,\ \frac{2\eta\beta\Pi((1-p)\mu+\omega)}{\mu(\mu+\omega)} + \omega - \mu\}.$$

Thus, $\bar{\mu}(A) < 0$ by (5.8).

Since $R_0 > 1$, the model $(1.1a)-(1.1d)$ is permanent by Theorem 4.1. Hence, there exists a compact absorbing set in D. By Lemma 5.1, the endemic equilibrium is globally attractive in the interior of D . The proof is complete. □

6. Numerical Simulations

In this section, we show the feasibility of the conditions of Theorem 5.2.

Example In $(1.1a)-(1.1e)$, let $\mu = 1.0977, \beta = 2.6918, p = 0.2622, \eta = 0.0049, d = 0.2455, \delta = 0.1485, \alpha = 1.0778, \tau = 0.0439, \omega = 0.7504, \Pi = 1.9758$. The model $(1.1a)-(1.1e)$ with above coefficients has an endemic equilibrium $P^*(0.9113,$
$0.2082, 0.3433, 0.2485, 0.0330)$. A direct calculation shows that $R_0 = 1.6 > 1, 2\eta\beta\Pi((1-p)\mu+\omega)/(\mu(\mu+\omega)) + \omega - \mu = -0.3 < 0, \alpha - \mu - \omega = -0.8 < 0$. By Theorem 5.1, we see that the endemic equilibrium P^* is globally attractive. Numerical simulation illustrates our result (see Fig.1).

7. Conclusion

In this paper, the dynamics of a SEIR epidemic model with vaccination is investigated. It is shown that if the basic reproductive number $R_0 < 1$, the disease-free equilibrium is globally asymptotically stable while the endemic equilibrium is not feasible. In this case, the disease dies out. If $R_0 > 1$, the system is permanent, and the endemic equilibrium is globally attractive provided $u > \max\{\alpha + \omega,\ 2\eta\beta\Pi((1-p)\mu + \omega)/(\mu(\mu + \omega)) + \omega\}$. To control the disease, a strategy should reduce the basic reproduction number to below unity. Clearly, if τ, p or $1/\omega$ increases, the basic reproduction number decreases. Hence, it is useful to control the disease by increasing the rate τ, p or $1/\omega$.

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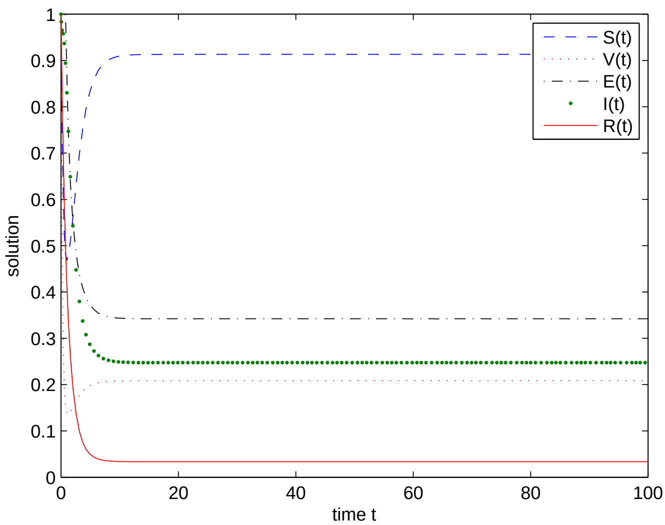

Fig.1 The temporal solution found by numerical integration of the model $(1.1a)-(1.1e)$ with $\mu = 1.0977, \beta = 2.6918, p = 0.2622, \eta = 0.0049, d = 0.2455, \delta = 0.1485, \alpha = 1.0778, \tau = 0.0439, \omega = 0.7504, \Pi = 1.9758$ and initial conditions $S(0) = 1, V(0) = 1, E(0) = 1, I(0) = 1, R(0) = 1$.