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HYERS-ULAM STABILITY OF HEAT-CONDUCTION EQUATION

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Abstract: We prove the Hyers-Ulam stability of a partial differential equation. That is, if u is an approximate solution of the heat-conduction equation $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t)$ and $\frac{\partial u}{\partial t} = a^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + f(x,y,t)$, then there exists an exact solution of the differential equation near to u.

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1. Introduction and Preliminaries

Since S. M. Ulam [24] gave the stability of group homomorphisms in 1940, the stability of many function have been investigated by several mathematicians (cf. [5], [6], [19], and [20]).

During last two decades, the Hyers-Ulam stability of differential equations also has been extensively investigated by several mathematicians (see [1, 17, 18]). Furthermore, the results of Hyers-Ulam stability for some ordinary differential equations have been generalized (see [3, 7, 8, 11, 13, 15, 16, 22, 23, 25]).

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The investigation of Hyers-Ulam stability of partial differential equation started recently and we should mention the results by Jung [9], Lungu et. al.[12], Cimpean et. al. [2] and Hegyi et. al.[4].

The aim of this paper is to study the Hyers-Ulam stability of the following partial differential equations :

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2} + f(x,t), \\ u(x,0) = \varphi(x), \end{cases} \tag{1}$$

where $a \in \mathbb{R}$ is a fixed number.

2. Fourier Transform and Inverse Transform

Firsly, we should have the following prerequisite knowledge.

Fourier transform:

$$F(f) = \int_{-\infty}^{+\infty} f(\xi) e^{-i\lambda\xi} d\xi;$$

inverse transform:

$$F^{-1}(g) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(\lambda) e^{i\lambda x} d\lambda;$$

we make the notation that:

$$f_1 * f_2(x) = \int_{-\infty}^{+\infty} f_1(x-t) f_2(t) dt.$$

Hence, we have

$$F(f_1 * f_2) = F(f_1) \cdot F(f_2) \tag{2.1}$$

and if both $f'(x)$ and $f(x)$ is suitable to fourier transform, we have:

$$F(f'(x)) = i\lambda F[f(x)] \tag{2.2}$$

3. Main Results

To prove the stability of heat-conduction equation, we need to prove the following Lemmas firstly. As we all know, fourier transform can be use to solve some partial differential equation:

Lemma 3.1. *Suppose that* $u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$, *and* $\varphi(x) : \mathbb{R} \to \mathbb{R}$ *is continuous and bounded. Then*

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} \varphi(\xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi$$

is the solution of the equations

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2}, \\ u(x,0) = \varphi(x). \end{cases}$$

Proof. We make the notation that:

$$\begin{cases} F[u(x,t)] = \tilde{u}(\lambda,t) \\ F[\varphi(x)] = \tilde{\varphi}(\lambda). \end{cases}$$

And then apply Fourier transform to

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2}, \\ u(x,0) = \varphi(x). \end{cases}$$

and by using Eq.(2.2), we have

$$\begin{cases} \dfrac{d\tilde{u}}{dt} = -a^2 \lambda^2 \tilde{u}, \\ \tilde{u}(\lambda,0) = \tilde{\varphi}(\lambda). \end{cases}$$

It is an ordinary differential equation, the solution is:

$$\tilde{u}(\lambda,t) = \tilde{\varphi}(\lambda) e^{-a^2\lambda^2 t}$$

By using the inverse Fourier transform:

$$F^{-1}[e^{-a^2\lambda^2 t}] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-(a^2\lambda^2 t - i\lambda x)} d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-a^2t(\lambda-\frac{ix}{2a^2t})^2}d\lambda\cdot e^{-\frac{x^2}{4a^2t}}$$

$$= \frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}$$

By the properties of the Fourier transform Eq.(2.1), we get

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi.$$

We can verify that it is exactly the solution.

□

Lemma 3.2. *Suppose that* $u(x,t):\mathbb{R}\times[0,\infty)\to\mathbb{R}$, *and* $f(x):\mathbb{R}\to\mathbb{R}$ *is continuous and bounded. Then*

$$u(x,t) = \frac{1}{2a\sqrt{\pi}}\int_0^t\int_{-\infty}^{+\infty}\frac{f(\xi,\tau)}{\sqrt{t-\tau}}e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}d\xi d\tau$$

is the solution of the equations

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2\dfrac{\partial^2 u}{\partial x^2} + f(x,t), \\ u(x,0) = 0. \end{cases}$$

Proof. By using homogeneitisation principle, we know that:

$$u(x,t) = \int_0^t w(x,t;\tau)d\tau$$

where $w(x,t)$ is the solution of:

$$\begin{cases} \dfrac{\partial w}{\partial t} = a^2\dfrac{\partial^2 w}{\partial x^2}, & t>\tau \\ w(x,\tau) = f(x,\tau). \end{cases}$$

Then, by using Lemma 3.1, we obtain the solution of $u(x,t)$.

□

By Lemma 3.1 and 3.2, it is easy to see the following lemma holds.

Lemma 3.3. *Suppose that* $u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$, *and* $\varphi(x), f(x) : \mathbb{R} \to \mathbb{R}$ *is continuous and bounded. Then*

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} \varphi(\xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi$$

$$+ \frac{1}{2a\sqrt{\pi}} \int_0^t \int_{-\infty}^{+\infty} \frac{f(\xi,\tau)}{\sqrt{t-\tau}} e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}} d\xi d\tau$$

is the solution of Eq. (1)

The main result of this paper is given in the next theorem.

Theorem 3.4. *Let* ε *be a nonnegative number,* $0 < T < \infty$. *If* $u(x,t) : \mathbb{R} \times [0,T) \to \mathbb{R}$,$\varphi(x)$ *and* $f(x,t)$ *are continuous and bounded, and* u *satisfies the following inequality*

$$\begin{cases} ||\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} - f(x,t)|| \le \varepsilon, \\ u(x,0) = \varphi(x) \end{cases}$$

for every $(x,t) \in \mathbb{R} \times [0,T)$, *there exists a solution* v *of Eq.(1) with the property*

$$||u(x,t) - v(x,t)|| \le M\varepsilon$$

for every $(x,t) \in \mathbb{R} \times [0,T)$, *where* M *is a constant.*

Proof. Let u be a solution of inequality in theorem and put

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} - f(x,t) =: g(x,t)$$

Using Lemma 3.3, we can know that the solution of the above equation is

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} \varphi(\xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi$$

$$+ \frac{1}{2a\sqrt{\pi}} \int_0^t \int_{-\infty}^{+\infty} \frac{f(\xi,\tau) + g(\xi,\tau)}{\sqrt{t-\tau}} e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}} d\xi d\tau.$$

On the other hand, the solution of Eq.(1) is

$$u_0(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} \varphi(\xi) e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi$$

$$+ \frac{1}{2a\sqrt{\pi}} \int_0^t \int_{-\infty}^{+\infty} \frac{f(\xi,\tau)}{\sqrt{t-\tau}} e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}} d\xi d\tau.$$

So

$$\begin{aligned} ||u(x,t) - u_0(x,t)|| &= \frac{1}{2a\sqrt{\pi}} \left| \int_0^t \int_{-\infty}^{+\infty} \frac{g(\xi,\tau)}{\sqrt{t-\tau}} e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}} d\xi d\tau \right| \\ &\leq \frac{\varepsilon}{2a\sqrt{\pi}} \left| \int_0^t \int_{-\infty}^{+\infty} \frac{1}{\sqrt{t-\tau}} e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}} d\xi d\tau \right|. \end{aligned}$$

Moreover, by noticing

$$\int_{-\infty}^{+\infty} e^{-\xi^2} d\xi = \sqrt{\pi},$$

we have

$$\begin{aligned} & \frac{\varepsilon}{2a\sqrt{\pi}} \left| \int_0^t \int_{-\infty}^{+\infty} \frac{1}{\sqrt{t-\tau}} e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}} d\xi d\tau \right| \\ \leq\ & \varepsilon \left| \int_0^t \int_{-\infty}^{+\infty} \frac{1}{2a\sqrt{\pi\tau}} e^{-\frac{(x-\xi)^2}{4a^2\tau}} d\xi d\tau \right| \\ \leq\ & \varepsilon \left| \int_0^t \int_{-\infty}^{+\infty} e^{-\xi^2} d\xi d\tau \right| \\ \leq\ & \sqrt{\pi}\varepsilon t. \end{aligned}$$

Hence,

$$||u(x,t) - u_0(x,t)|| \leq T\sqrt{\pi}\varepsilon$$

for all $(x,t) \in \mathbb{R} \times [0,T)$.

□

Remark 3.5. We can know from the above theorem that: for every fixed $t \in \mathbb{R}_+$, Eq.(1) has Hyers-Ulam stability.

Corollary 3.6. *Let ε be a nonnegative number. If $u(x,t) : \mathbb{R}\times[0,+\infty) \to \mathbb{R}$, and $\varphi(x)$ and $f(x,t)$ are continuous and bounded , and u satisfies the following inequality*

$$\begin{cases} \|\frac{1}{g(x,t)}\frac{\partial u}{\partial t} - \frac{a^2}{g(x,t)}\frac{\partial^2 u}{\partial x^2} - \frac{f(x,t)}{g(x,t)}\| \le \varepsilon, \\ u(x,0) = \varphi(x) \end{cases}$$

for every $(x,t) \in \mathbb{R}\times\mathbb{R}_+$, where $g(x,t)$ satisfies

$$\left|\frac{1}{2a\sqrt{\pi}}\int_0^t\int_{-\infty}^{+\infty}\frac{g(\xi,\tau)}{\sqrt{t-\tau}}e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}d\xi d\tau\right| \le M,$$

for all $t \in \mathbb{R}$, then there exists a solution v of

$$\begin{cases} \frac{1}{g(x,t)}\frac{\partial u}{\partial t} - \frac{a^2}{g(x,t)}\frac{\partial^2 u}{\partial x^2} = \frac{f(x,t)}{g(x,t)}, \\ u(x,0) = \varphi(x) \end{cases}$$

with the property

$$\|u(x,t) - v(x,t)\| \le M\varepsilon$$

for every $(x,t) \in \mathbb{R}\times\mathbb{R}_+$.

By using the similar method, we can prove the situation of two-dimensional space.

Corollary 3.7. *Let ε be a nonnegative number, $0 < T < \infty$. If $u(x,y,t) : \mathbb{R}\times\mathbb{R}\times[0,T) \to \mathbb{R}$, $\varphi(x)$ and $f(x,y,t)$ are continuous and bounded, and u satisfies the following inequality*

$$\begin{cases} \|\frac{\partial u}{\partial t} - a^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) - f(x,y,t)\| \le \varepsilon, \\ u(x,y,0) = \varphi(x) \end{cases}$$

for every $(x,y,t) \in \mathbb{R}\times\mathbb{R}\times[0,T)$, then there exists a solution v of

$$\begin{cases} \frac{\partial u}{\partial t} - a^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) = f(x,y,t), \\ u(x,y,0) = \varphi(x) \end{cases}$$

with the property

$$\|u(x,y,t) - v(x,y,t)\| \le M\varepsilon$$

for every $(x,y,t) \in \mathbb{R}\times\mathbb{R}\times[0,T)$, where $M = \pi T$.

Corollary 3.8. *Let ε be a nonnegative number. If $u(x, y, t) : \mathbb{R} \times \mathbb{R} \times [0, +\infty) \to \mathbb{R}$, $\varphi(x)$ and $f(x, y, t)$ are continuous and bounded, and u satisfies the following inequality*

$$\begin{cases} \|\frac{1}{g(x,y,t)}\frac{\partial u}{\partial t} - \frac{a^2}{g(x,y,t)}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) - \frac{f(x,y,t)}{g(x,y,t)}\| \leq \varepsilon, \\ u(x,y,0) = \varphi(x). \end{cases}$$

for every $(x, y, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$, where $g(x, y, t)$ satisfies

$$|\frac{1}{4\pi a^2 t}\int_0^t \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{g(\xi,\tau)}{t-\tau} e^{-\frac{(x-\xi)^2+(x-\eta)^2}{4a^2(t-\tau)}} d\xi d\eta d\tau| \leq M,$$

for all $t \in \mathbb{R}$, then there exists a solution v of

$$\begin{cases} \frac{1}{g(x,y,t)}\frac{\partial u}{\partial t} - \frac{a^2}{g(x,y,t)}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) = \frac{f(x,y,t)}{g(x,y,t)}, \\ u(x,y,0) = \varphi(x). \end{cases}$$

with the property

$$\|u(x,y,t) - v(x,y,t)\| \leq M\varepsilon$$

for every $(x, y, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$.

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