International Journal of Pure and Applied Mathematics
Volume 103 No. 1 2015, 99-107
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v103i1.8
ijpam.eu

THE GENERATORS OF THE 2-CLASS GROUP OF SOME FIELDS $\mathbb{Q}(\sqrt{pq_1q_2}, i)$: CORRECTION TO THEOREM 3 OF [5]

Abdelmalek Azizi[1 §], Abdelkader Zekhnini[2], Mohammed Taous[3]

[1]Department of Mathematics
Faculty of Sciences
Mohammed First University
Oujda, MOROCCO

[2]Department of Mathematics
Pluridisciplinary Faculty
Mohammed First University
Nador, MOROCCO

[3]Department of Mathematics
Faculty of Science and Technology
Moulay Ismail University
Errachidia, MOROCCO

Abstract: Let p, q_1 and q_2 be different primes satisfying the condition that the 2-class group of the fiel $\Bbbk = \mathbb{Q}\left(\sqrt{pq_1q_2}, i\right)$ is of type (2, 2, 2). In this paper, we are interested to give the generators of $\mathbf{C}_{\Bbbk,2}$, the 2-class group of $\Bbbk$, which corrects the Theorem 3 of A. Azizi, A. Zekhnini and M. Taous: On the generators of the 2-class group of the field $\mathbf{k} = \mathbb{Q}(\sqrt{d}, i)$, IJPAM, Volume **81**, No. 5 (2012), 773-784.

AMS Subject Classification: 11R11, 11R29, 11R32, 11R37
Key Words: 2-class group, class group, biquadratic field, Hilbert class field

Received: March 31, 2015

© 2015 Academic Publications, Ltd.
url: www.acadpubl.eu

[§]Correspondence author

1. Introduction

Let k be an algebraic number field and let $\mathbf{C}_{k,2}$ denote its 2-class group, that is the 2-Sylow subgroup of the ideal class group $\mathbf{C}_k$ of k. The structure and the generators of $\mathbf{C}_{k,2}$ play an important role in Number Theory, in fact they can help to determine the structure and the generators of the maximal unramified pro-2 extension of k, and they also help to solve the capitulation problem of the 2-ideal classes of k in its unramified extensions see [6, 7, 8, 9, 10, 11, 12, 16, 17]. Let $\mathbb{k} = \mathbb{Q}(\sqrt{d}, i)$, where d is a square-free integer. In [5], we determined the generators of $\mathbf{C}_{\mathbb{k},2}$ whenever it is of type $(2,2,2)$, but Theorem 3 is false, the purpose of this paper is to correct this Theorem.

2. Preliminaries

Let p, q_1 and q_2 be different primes such that $p \equiv 1 \pmod 4$ and $q_1 \equiv q_2 \equiv 3 \pmod 4$. Put $d = pq_1q_2$, according to [13], $\mathbf{C}_{\mathbb{k},2}$ is of type (2, 2, 2) if and only if p, q_1, q_2 satisfy the following two conditions:

$A:$ $p \equiv -q_1 \equiv -q_2 \equiv 1 \pmod 4$ and $\left(\frac{2}{p}\right) = \left(\frac{q_1}{q_2}\right) = -\left(\frac{q_2}{q_1}\right) = 1.$

$B:$ One of the following three conditions is satisfied:

(I) $\left(\frac{p}{q_1}\right)\left(\frac{p}{q_2}\right) = -1$ and $\left(\frac{2}{q_1}\right) = \left(\frac{2}{q_2}\right) = -1.$

(II) $\left(\frac{p}{q_1}\right)\left(\frac{p}{q_2}\right) = -1$, $\left(\frac{2}{q_1}\right) = 1$ and $\left(\frac{2}{q_2}\right) = -1.$

(III) $\left(\frac{p}{q_1}\right) = \left(\frac{p}{q_2}\right) = -1$ and $\left(\frac{2}{q_1}\right)\left(\frac{2}{q_2}\right) = -1.$

Definition 1. Let p, q_1 and q_2 be different primes satisfying the condition A above.

(1) p, q_1 and q_2 are called of type $B(I)(1)$ if the following conditions hold: $\left(\frac{p}{q_1}\right) = -\left(\frac{p}{q_2}\right) = 1$ and $\left(\frac{2}{q_1}\right) = \left(\frac{2}{q_2}\right) = -1.$

(2) p, q_1 and q_2 are called of type $B(I)(2)$ if the following conditions hold: $\left(\frac{p}{q_2}\right) = -\left(\frac{p}{q_1}\right) = 1$ and $\left(\frac{2}{q_1}\right) = \left(\frac{2}{q_2}\right) = -1.$

(3) p, q_1 and q_2 are called of type $B(II)(1)$ if the following conditions hold: $\left(\frac{p}{q_1}\right) = -\left(\frac{p}{q_2}\right) = 1$ and $\left(\frac{2}{q_1}\right) = -\left(\frac{2}{q_2}\right) = 1.$

(4) p, q_1 and q_2 are called of type $B(II)(2)$ if the following conditions hold: $\left(\frac{p}{q_2}\right) = -\left(\frac{p}{q_1}\right) = 1$ and $\left(\frac{2}{q_1}\right) = -\left(\frac{2}{q_2}\right) = 1$.

Lemma 2. *Let $d \equiv 1 \pmod 4$ be a positive square free integer and $\varepsilon_d = x + y\sqrt{d}$ be the fundamental unit of $\mathbb{Q}(\sqrt{d})$. Assume $N(\varepsilon_d) = 1$, then*

1. *$x+1$ and $x-1$ are not squares in $\mathbb{N}$ i.e. $2\varepsilon_d$ is not a square in $\mathbb{Q}(\sqrt{d})$.*
2. *For all prime p dividing d, $p(x+1)$ and $p(x-1)$ are not squares in $\mathbb{N}$.*

Proof. 1. As $d \equiv 1 \pmod 4$, then by [13, Corollaire 3.2] the unit index of $\mathbb{Q}(\sqrt{d}, i)$ is equal to 1, hence by [1, Applications (ii)] we get that $2\varepsilon_d$ is not a square in $\mathbb{Q}(\sqrt{d})$, this is equivalent to $x+1$ and $x-1$ are not squares in $\mathbb{N}$.

2. Assume $p(x+1)$ or $p(x-1)$ is a square in $\mathbb{N}$, then, by the decomposition uniqueness in $\mathbb{Z}$, there exist y_1, y_2 in $\mathbb{Z}$ such that $\begin{cases} x \pm 1 = py_1^2, \\ x \mp 1 = d'y_2^2; \end{cases}$ and $\begin{cases} y = y_1y_2, \\ d = pd'; \end{cases}$ thus $p(x \pm 1) = p^2y_1^2$ and $p(x \mp 1) = p^2y_1^2 \mp 2p$. This in turn yields that $p^2(x^2-1) = p^2y_1^2(p^2y_1^2 \mp 2p)$; as $x^2-1 = y^2d$, so we get $y^2d = y_1^2(p^2y_1^2 \mp 2p)$, and $y_2^2d = p^2y_1^2 \mp 2p$. Since $d \equiv 1 \pmod 4$ and $p \equiv \pm 1 \pmod 4$, we deduce that $\mp 2 \equiv y_1^2 - y_2^2 \pmod 4$. On the other hand, we know that for all $a \in \mathbb{Z}$ $a^2 \equiv 0$ or $1 \pmod 4$, thus $\mp 2 \equiv 0, 1$ or $-1 \pmod 4$. Which is absurd. □

Let p, q_1 and q_2 be different primes satisfying the condition A above. As the norm of $\varepsilon_{pq_1q_2} = x + y\sqrt{pq_1q_2}$, the fundamental unit of $\mathbb{Q}(\sqrt{pq_1q_2})$, is 1, then by the decomposition uniqueness in $\mathbb{Z}$, each of the numbers $x \pm 1$, $2(x \pm 1)$, $p(x \pm 1)$, $q_1(x \pm 1)$, $q_2(x \pm 1)$, $2p(x \pm 1)$, $2q_1(x \pm 1)$, $2q_2(x \pm 1)$ and $2pq_1q_2(x \pm 1)$ can be a square in $\mathbb{N}$.

Lemma 3. *Let p, q_1 and q_2 be different primes satisfying the condition A. Let $\varepsilon_{pq_1q_2} = x + y\sqrt{pq_1q_2}$ be the fundamental unit of $\mathbb{Q}(\sqrt{pq_1q_2})$.*

1. *If p, q_1 and q_2 are of type $B(I)(1)$ or $B(II)(1)$, then only $2q_1(x+1)$ (i.e. $2pq_2(x-1)$) is a square in $\mathbb{N}$.*
2. *If p, q_1 and q_2 are of type $B(I)(2)$ or $B(II)(2)$, then only $2q_2(x-1)$ (i.e. $2pq_1(x+1)$) is a square in $\mathbb{N}$.*
3. *If p, q_1 and q_2 are of type $B(III)$, then only $2p(x-1)$ (i.e. $2q_1q_2(x+1)$) is a square in $\mathbb{N}$.*

Proof. As $pq_1q_2 \equiv 1 \pmod 4$ and $N(\varepsilon_{pq_1q_2}) = 1$, then Lemma 2 implies that $x \pm 1$, $p(x\pm 1)$, $q_1(x\pm 1)$ and $q_2(x\pm 1)$ are not squares in $\mathbb{N}$. On the other hand, Lemma 5 of [2] yields that $2(x\pm 1)$ and $2pq_1q_2(x\pm 1)$ are not squares in $\mathbb{N}$. Hence only $2p(x\pm 1)$, $2q_1(x\pm 1)$ and $2q_2(x\pm 1)$ can be squares in $\mathbb{N}$.

Suppose p, q_1 and q_2 are of type $B(I)(1)$. If $2p(x\pm 1)$ is a square in $\mathbb{N}$, then, by the decomposition uniqueness of $x\pm 1$ in $\mathbb{Z}$, there exist y_1, y_2 in $\mathbb{Z}$ such that $\begin{cases} x\pm 1 = 2py_1^2, \\ x\mp 1 = 2q_1q_2y_2^2, \\ y = 2y_1y_2; \end{cases}$ from which we deduce that $\left(\frac{2q_1q_2}{p}\right) = \left(\frac{2}{p}\right)$, so $\left(\frac{q_1q_2}{p}\right) = 1$, but this contradicts the fact that $\left(\frac{q_1q_2}{p}\right) = -1$.

Similarly, if we assume $2q_2(x\pm 1)$ is a square in $\mathbb{N}$, we get $\begin{cases} x\pm 1 = 2q_2y_1^2, \\ x\mp 1 = 2q_1py_2^2, \\ y = 2y_1y_2; \end{cases}$ which implies that $\left(\frac{2q_2}{p}\right) = \left(\frac{2}{p}\right)$, hence $\left(\frac{q_2}{p}\right) = 1$, which is absurd.

Finally, if $2q_1(x-1)$ is a square in $\mathbb{N}$, then proceeding similarly we get $\left(\frac{q_1}{q_2}\right) = -1$, which is absurd. So the result.

The other cases are proved similarly. □

We close this section by the following lemmas.

Lemma 4 ([19]). *Let p_1, p_2,...,p_n be distinct primes and for each j, let $e_j = \pm 1$. Then there exist infinitely many primes l such that $\left(\frac{p_j}{l}\right) = e_j$, for all j.*

Lemma 5 ([18], p. 205). *If $\mathcal{H}$ is an unramified ideal in some extension $\mathbb{K}/\Bbbk = \Bbbk(\sqrt{x})/\Bbbk$, then the quadratic residue symbol is given by the Artin symbol $\varphi = \left(\frac{\Bbbk(\sqrt{x})/\Bbbk}{\mathcal{H}}\right)$ as follows: $\left(\frac{x}{\mathcal{H}}\right) = \sqrt{x}^{\varphi - 1}$.*

3. Main Result

Let $F = \mathbb{Q}(i)$ and denote by $\mathrm{A}m(\Bbbk/F)$ the group of the ambiguous classes of $\Bbbk/F$ and by $\mathrm{A}m_s(\Bbbk/F)$ its subgroup generated by the strongly ambiguous classes. As $p \equiv 1 \pmod 4$, so there exist e and f in $\mathbb{N}$ such that $p = e^2 + 4f^2 = \pi_1\pi_2$. Put $\pi_1 = e + 2if$ and $\pi_2 = e - 2if$. Let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be the prime ideals of $\Bbbk$ above π_1, π_2, q_1 and q_2 respectively. It is easy to see that $\mathcal{H}_j^2 = (\pi_j)$

(resp. $\mathcal{Q}_j^2 = (q_j)$). Therefore $[\mathcal{H}_j]$ (resp. $[\mathcal{Q}_j]$) is in $\mathrm{A}m_s(\Bbbk/F)$ for all $j \in \{1, 2\}$. Keep the notation $\varepsilon_{pq_1q_2} = x + y\sqrt{pq_1q_2}$.

Since $\mathcal{H}_j^2 = (\pi_j)$, where $1 \leq j \leq 2$, and since also $\sqrt{e^2 + (2f)^2} = \sqrt{p} \notin \mathbb{Q}(\sqrt{pq_1q_2})$, so according to [5, Proposition 1] $\mathcal{H}_j$ is not principal in $\Bbbk$.

From Lemma 3 we get the following assertions:

- If p, q_1 and q_2 are of type $B(I)(1)$ or $B(II)(1)$, then $2q_1(x+1)$ and $2pq_2(x-1)$ are squares in $\mathbb{N}$, so Remark 1 of [5] implies that $\mathcal{Q}_1$ and $\mathcal{H}_1\mathcal{H}_2\mathcal{Q}_2$ are principal in $\Bbbk$.
- If p, q_1 and q_2 are of type $B(I)(2)$ or $B(II)(2)$, then $2q_2(x-1)$ and $2pq_1(x+1)$ are squares in $\mathbb{N}$, so Remark 1 of [5] implies that $\mathcal{Q}_2$ and $\mathcal{H}_1\mathcal{H}_2\mathcal{Q}_1$ are principal in $\Bbbk$.
- If p, q_1 and q_2 are of type $B(III)$, then $2p(x-1)$ and $2q_1q_2(x+1)$ are squares in $\mathbb{N}$, so Remark 1 of [5] implies that $\mathcal{H}_1\mathcal{H}_2$ and $\mathcal{Q}_1\mathcal{Q}_2$ are squares in $\Bbbk$ and $\mathcal{Q}_1$, $\mathcal{Q}_2$ are not. Moreover, as $(\mathcal{H}_1\mathcal{Q}_1)^2 = (\pi_1 q_1)$ and $q_1\sqrt{p} \notin \mathbb{Q}(\sqrt{d})$, so [5, Proposition 1] implies that $\mathcal{H}_1\mathcal{Q}_1$ is not principal in $\Bbbk$.

According to the ambiguous class number formula (see [14]) we have:

$$|\mathrm{A}m(\Bbbk/F)| = \frac{h(F)2^{t-1}}{[E_F : E_F \cap N_{\Bbbk/F}(\Bbbk^\times)]}, \tag{1}$$

where $h(F)$ is the class number of F and t is the number of finite and infinite primes of F ramified in $\Bbbk/F$. Moreover as the class number of F is equal to 1, so the formula (1) yields that

$$|\mathrm{A}m(\Bbbk/F)| = 2^r, \tag{2}$$

where $r = \mathrm{rank}\mathbf{C}_{\Bbbk,2} = t - e - 1$ and $2^e = [E_F : E_F \cap N_{\Bbbk/F}(\Bbbk^\times)]$ (see for example [20]). The relation between $|\mathrm{A}m(\Bbbk/F)|$ and $|\mathrm{A}m_s(\Bbbk/F)|$ is given by the following formula (see for example [15]):

$$\frac{|\mathrm{A}m(\Bbbk/F)|}{|\mathrm{A}m_s(\Bbbk/F)|} = [E_F \cap N_{\Bbbk/F}(\Bbbk^\times) : N_{\Bbbk/F}(E_\Bbbk)]. \tag{3}$$

Since $r = \mathrm{rank}\mathbf{C}_{\Bbbk,2} = 3$, so Formula (2) implies that $|Am(\Bbbk/\mathbb{Q}(i))| = 2^3 = 8$. Moreover, we know that $p \equiv 1 \pmod 8$, hence by [21] i is a norm in $\Bbbk/\mathbb{Q}(i)$, thus Formula (3) allows us to deduce that

$$\frac{|\mathrm{Am}(\Bbbk/\mathbb{Q}(i))|}{|\mathrm{Am}_f(\Bbbk/\mathbb{Q}(i))|} = [E_{\mathbb{Q}(i)} \cap N_{\Bbbk/\mathbb{Q}(i)}(\Bbbk^\times) : N_{\Bbbk/\mathbb{Q}(i)}(E_\Bbbk)] = [< i >:< -1 >] = 2$$

($E_\Bbbk = \langle i, \varepsilon_{pq_1q_2}\rangle$ since $x \pm 1$ is not a square in $\mathbb{N}$ see [3]), so $|\mathrm{Am}_s(\Bbbk/\mathbb{Q}(i))| = 4$. We conclude that:

• If p, q_1 and q_2 are of type $B(III)$, then $\mathrm{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \langle[\mathcal{H}_1], [\mathcal{Q}_1]\rangle$.
• Else, $\mathrm{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \langle[\mathcal{H}_1], [\mathcal{H}_2]\rangle$.
Therefore there exists in $\mathbb{k}$ an unambiguous ideal $\mathcal{I}$ of order 2 such that

$$\begin{cases} \mathbf{C}l_2(\mathbb{k}) = \langle[\mathcal{H}_1], [\mathcal{Q}_1], [\mathcal{I}]\rangle & \text{if p, } q_1 \text{ and } q_2 \text{ are of type } B(III), \\ \mathbf{C}l_2(\mathbb{k}) = \langle[\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{I}]\rangle & \text{otherwise.} \end{cases}$$

By Chebotarev theorem, $\mathcal{I}$ can always be chosen as a prime ideal of $\mathbb{k}$ above a prime l in $\mathbb{Q}$, which splits completely in $\mathbb{k}$. So we can determine $\mathcal{I}$ by using Lemma 4.

(a) Suppose p, q_1 and q_2 are of type $B(III)$. Let $l \equiv 1 \pmod 4$ be a prime integer such that $\left(\frac{pq_1q_2}{l}\right) = 1$ and $\left(\frac{p}{l}\right) = -1$. Thus l splits completely in $\mathbb{k}$, let $\mathcal{I}$ be a prime ideal of $\mathbb{k}$ lies above l, which is an unambiguous ideal. Let us prove that $\mathcal{I}$, $\mathcal{Q}_1\mathcal{I}$, $\mathcal{H}_1\mathcal{I}$ and $\mathcal{Q}_1\mathcal{H}_1\mathcal{I}$ are not principal in $\mathbb{k}$.

• $\mathcal{I}$ is not principal in $\mathbb{k}$, otherwise we will have $N_{\mathbb{k}/\mathbb{Q}(\sqrt{pq_1q_2})}(\mathcal{I})$ principal in $\mathbb{Q}(\sqrt{pq_1q_2})$, so there exist α_1, α_2 in $\mathbb{Q}$ such that $l = \pm(\alpha_1^2 - \alpha_2^2 pq_1q_2)$. This implies that $\left(\frac{p}{l}\right) = 1$; which is absurd.

• If $\mathcal{Q}_1\mathcal{I}$ is principal in $\mathbb{k}$, then $N_{\mathbb{k}/\mathbb{Q}(\sqrt{pq_1q_2})}(\mathcal{Q}_1\mathcal{I}) = \mathcal{Q}_1^2 N_{\mathbb{k}/\mathbb{Q}(\sqrt{pq_1q_2})}(\mathcal{I})$ is also principal in $\mathbb{Q}(\sqrt{pq_1q_2})$ (note that the ideal of $\mathbb{Q}(\sqrt{pq_1q_2})$ above q_1 remains inert in $\mathbb{k}$). On the other hand, under our conditions $\mathbb{Q}(\sqrt{pq_1q_2})$ is cyclic of order 2 (see [13]), thus $N_{\mathbb{k}/\mathbb{Q}(\sqrt{pq_1q_2})}(\mathcal{I})$ is principal in $\mathbb{Q}(\sqrt{pq_1q_2})$, this implies the contradiction $\left(\frac{p}{l}\right) = 1$.

• If $\mathcal{H}_1\mathcal{I}$ is principal in $\mathbb{k}$, then $N_{\mathbb{k}/\mathbb{Q}(\sqrt{pq_1q_2})}(\mathcal{H}_1\mathcal{I})$ is principal in $\mathbb{Q}(\sqrt{pq_1q_2})$. So there exist α_1, α_2 in $\mathbb{Q}$ such that $pl = \pm(\alpha_1^2 - \alpha_2^2 pq_1q_2)$; which yields that p divides α_1, hence there exists β in $\mathbb{Q}$ such that $l = \pm(\beta^2 p - \alpha_2^2 q_1q_2)$; this in turn implies that $\left(\frac{l}{p}\right) = \left(\frac{q_1q_2}{p}\right) = -1$; but this contradicts the condition $B(III)$.

• If $\mathcal{Q}_1\mathcal{H}_1\mathcal{I}$ is principal in $\mathbb{k}$, then

$$N_{\mathbb{k}/\mathbb{Q}(\sqrt{pq_1q_2})}(\mathcal{Q}_1\mathcal{H}_1\mathcal{I}) = \mathcal{Q}_1^2 N_{\mathbb{k}/\mathbb{Q}(\sqrt{pq_1q_2})}(\mathcal{H}_1\mathcal{I})$$

is principal in $\mathbb{Q}(\sqrt{pq_1q_2})$. So $N_{\mathbb{k}/\mathbb{Q}(\sqrt{pq_1q_2})}(\mathcal{H}_1\mathcal{I})$ is principal in $\mathbb{Q}(\sqrt{pq_1q_2})$; which implies that $\left(\frac{l}{p}\right) = \left(\frac{q_1q_2}{p}\right) = -1$, but this contradicts the condition $B(III)$.

(b) Suppose p, q_1 and q_2 are of type $B(I)$ or $B(II)$. As in the assertion (a), we choose a prime $l \equiv 1 \pmod 4$ satisfying the conditions $\left(\frac{pq_1q_2}{l}\right) = 1$, $\left(\frac{p}{l}\right) = -1$ and $\left(\frac{q_1}{l}\right)\left(\frac{q_1}{p}\right) = -1$. Thus l splits completely in $\mathbb{k}$. Let $\mathcal{I}$ be a prime ideal $\mathbb{k}$ above l, so $\mathcal{I}$ is unambiguous ideal and the ideals $\mathcal{I}$, $\mathcal{H}_1\mathcal{I}$, $\mathcal{H}_2\mathcal{I}$ and $\mathcal{H}_1\mathcal{H}_2\mathcal{I}$ are not principal in $\mathbb{k}$. Indeed:
• $\mathcal{I}$ is not principal in $\mathbb{k}$, the same proof as in (a).

• Note that $\mathbb{K}_2 = \mathbb{Q}(\sqrt{q_1}, \sqrt{pq_2}, i)$ is an unramified quadratic extension of $\Bbbk$ (see [4]), hence Lemma 5 implies that $\varphi_{\mathbb{K}_2/\Bbbk}(\mathcal{H}_j\mathcal{I}) \neq 1$, where $j \in \{1, 2\}$ and $\varphi_{\mathbb{K}_2/\Bbbk}$ is the d'Artin map of $\mathbb{K}_2/\Bbbk$. Consequently $\mathcal{H}_j\mathcal{I}$ is not principal in $\Bbbk$.
• If $\mathcal{H}_1\mathcal{H}_2\mathcal{I}$ is principal in $\Bbbk$, then $N_{\Bbbk/\mathbb{Q}(\sqrt{pq_1q_2})}(\mathcal{H}_1\mathcal{H}_2\mathcal{I}) = P^2 N_{\Bbbk/\mathbb{Q}(\sqrt{pq_1q_2})}(\mathcal{I})$ is principal in $\mathbb{Q}(\sqrt{pq_1q_2})$, where P is the ideal of $\mathbb{Q}(\sqrt{pq_1q_2})$ above p; consequently $N_{\Bbbk/\mathbb{Q}(\sqrt{pq_1q_2})}(\mathcal{I})$ is principal in $\mathbb{Q}(\sqrt{pq_1q_2})$, which is absurd. This completes the proof of the main theorem.

Theorem 6. *Let $\Bbbk = \mathbb{Q}(\sqrt{d}, i)$, where $d = pq_1q_2$ with p, q_1 and q_2 are primes satisfying the conditions A et B. Let $\mathbf{C}_{\Bbbk,2}$ be the 2-class group of $\Bbbk$. Put $p = \pi_1\pi_2$, where π_1 and π_2 are in $\mathbb{Z}[i]$, denote by $\mathcal{H}_1$ (resp. $\mathcal{H}_2$ and $\mathcal{Q}_1$) the prime ideal of $\Bbbk$ lies above π_1 (resp. π_2 and q_1). Then there exists an unambiguous ideal $\mathcal{I}$ of $\Bbbk$ of order 2 such that*

1. *If p, q_1 and q_2 are of type $B(III)$, then $\mathbf{C}_{\Bbbk,2} = \langle[\mathcal{H}_1], [\mathcal{Q}_1], [\mathcal{I}]\rangle$.*

2. *Else, $\mathbf{C}_{\Bbbk,2} = \langle[\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{I}]\rangle$.*

References

[1] A. Azizi, *Unités de certains corps de nombres imaginaires et abéliens sur* $\mathbb{Q}$, Ann. Sci. Math. Qubec **23** (1999), no 1, 15-21.

[2] A.Azizi, *Sur la capitulation des 2-classes d'idéaux de $k = \mathbb{Q}(\sqrt{2pq}, i)$, oú* $p \equiv -q \equiv 1 \pmod 4$ Acta Arith. **94** (2000), 383-399.

[3] A. Azizi, *Sur les unités de certains corps de nombres de degré 8 sur $\mathbb{Q}$,* Ann. Sci. Math. Qubec **29** (2005), no. 2, 111-129.

[4] A. Azizi, A. Zekhnini and M. Taous, *On the unramified quadratic and biquadratic extensions of the field* $\mathbb{Q}(\sqrt{d}, i)$, IJA, Vol. 6 **No. 24** (2012), 1169-1173.

[5] A. Azizi, A. Zekhnini and M. Taous, *On the generators of the 2-class group of the field* $\mathbf{k} = \mathbb{Q}(\sqrt{d}, i)$, IJPAM, Volume **81**, No. 5 (2012), 773-784.

[6] A. Azizi, A. Zekhnini and M. Taous, *On the 2-class field tower of $\mathbb{Q}(\sqrt{2p_1p_2}, i)$ and the Galois group of its second Hilbert 2-class field,* Collect. Math. Volume 65, **1** (2014), 131-141.

[7] A. Azizi, A. Zekhnini and M. Taous, *Structure of* $Gal(\mathbb{k}_2^{(2)}/\mathbb{k})$ *for some fields* $\mathbb{k}=\mathbb{Q}(\sqrt{2p_1p_2},i)$ *with* $\mathbf{C}l_2(\mathbb{k})\simeq(2,2,2)$, Abh. Math. Sem. Univ. Hamburg, Volume 84, **2** (2014), 203-231.

[8] A. Azizi, A. Zekhnini et M. Taous, *Sur un problème de capitulation du corps* $\mathbb{Q}(\sqrt{p_1p_2},i)$ *dont le* 2*-groupe de classes est élémentaire*, Czechoslovak. Math. J. Volume 64, **1**, (2014) 11-29.

[9] A. Azizi, A. Zekhnini, M. Taous and Daniel C. Mayer, *Principalization of* 2*-class groups of type* $(2,2,2)$ *of biquadratic fields* $\mathbb{Q}(\sqrt{p_1p_2q},i)$, Int. J. Number Theory. Vol. 11, **4**, (2015) 1177-1215.

[10] A. Azizi, A. Zekhnini and M. Taous, *Coclass of* $\mathrm{Gal}(\mathbb{k}_2^{(2)}/\mathbb{k})$ *for some fields* $\mathbb{k}=\mathbb{Q}\left(\sqrt{p_1p_2q},i\right)$ *with* 2*-class groups of type* $(2,2,2)$, J. Algebra Appl, DOI: 10.1142/S0219498816500274.

[11] A. Azizi, A. Zekhnini and M. Taous, *On the strongly ambiguous classes of some biquadratic number fields,* To appear in Mathematica Bohemica.

[12] A. Azizi, A. Zekhnini and M. Taous, *On the strongly ambiguous classes of* $\mathbb{k}/\mathbb{Q}(i)$ *where* $\mathbf{k}=\mathbb{Q}(\sqrt{2p_1p_2},i)$, Asian-Eur. J. Math. Volume 7, No. **1** , (2014).

[13] A. Azizi and M. Taous, *Détermination des corps* $\mathbf{k}=\mathbb{Q}(\sqrt{d},\sqrt{-1})$ *dont les* 2*-groupes de classes sont de type* $(2,4)$ *ou* $(2,2,2)$, Rend. Istit. Mat. Univ. Trieste. **40** (2009), 93-116.

[14] C. Chevalley, *Sur la théorie du corps de classes dans les corps finis et les corps locaux*, J. Fac. Sc. Tokyo, Sect. 1, t.**2**, (1933), 365-476.

[15] F. Lemmermeyer, *The ambiguous class number formula revisited,* J. of the Ramanujan Math. Soc. Volume 28, **4**, 415-421.

[16] F. Lemmermeyer, *On* 2*-class field towers of imaginary quadratic number fields,* J. Théor. Nombres Bordeaux **6** (1994), 261-272.

[17] F. Lemmermeyer, *On* 2*-class field towers of some imaginary quadratic number fields,* Abh. Math. Sem. Hamburg **67** (1997), 205-214.

[18] G.Gras, *Class field theory, from theory to practice,* Springer Verlag (2003).

[19] P. J. Sime, *On the ideal class group of real biquadratic fields,* Trans. Am. Math. Soc. **347** No. 12 (1995), 4855-4876.

[20] T. M. McCall, C. J. Parry and R. R. Ranalli, *On imaginary bicyclic biquadratic fields with cyclic 2-class group*, J. Number Theory **53**, (1995), 88-99.

[21] T. M. McCall, C. J. Parry and R. R. Ranalli, *The 2-rank of the class group of imaginary bicyclic biquadratic fields,* Can. J. Math. **49** (2) (1997), 283-300.