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HERMITE-HADAMARD INEQUALITIES FOR $\lambda\varphi-$PREINVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstract: In this article, we install a different of Hermite-Hadamard's type inequalities for $\lambda\varphi-$preinvex functions which for twice differentiable are $\lambda\varphi-$preinvex convex functions and concave functions.

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1. Introduction

The following inequality is well known in the literature as the Hermite-Hadamard's integral inequality $(see\,[1])$:

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

Where $f:I\subset\mathbb{R}\to\mathbb{R}$ is a convex function on the integral I of real numbers and $a,b\in I$ with $a<b$. A function $f:[0,b]\subset\mathbb{R}\to\mathbb{R}$ is said to be convex. Whenever $x,y\in[0,b]$ and $t\in[0,1]$, the following inequality holds

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y) \tag{1.2}$$

The inequality (1.1) whose we can say, is the first main result for convex func-

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tions with a natural geometrical commentary and many implementations, has attracted and sustains to attract [2] .

Fractional calculus was born in 1695. In the past three hundred years, fractional calculus advanced in diverse fields from physical sciences and engineering to biological sciences and economics [4 − 11] .

2. Preliminaries

In this chapter, we will give some definitions, theorem and lemma which we use after in this article.

Definition 1. Let $f \in L[a,b]$. The Riemann- Liouville fractional integral $\mathrm{J}_{a+}^{\alpha} f$ and $\mathrm{J}_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \qquad x > \alpha \tag{2.1}$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \qquad x < \alpha, \tag{2.2}$$

where Γ is the gamma function and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

Theorem 1. [3] *Let $f : [a,b] \to \mathbb{R}$ be a nonnegative function with $0 \leq a < b$ and $f \in L[a,b]$. If f is convex function on $[a,b]$, then the following inequalities for fractional integrals arranges:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{2.3}$$

S.S. Dragomir, M.I. bhatti, M.Iqbal and M.Muddassar[12] *obtained the following Lemma 1.*

Lemma 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I^0, the interior of I. Assuming that $a.b \in I^o$ with $a < b$ and $f'' \in L[a,b]$, then the following description for fractional integral with $\alpha > 0$ arranges:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{(b)-}^{\alpha} f(a)\right]$$

$$= \frac{(b-a)^2}{2(\alpha+1)} \int_0^1 t(1-t^\alpha)\left[f''(ta+(1-t)b)+f''((1-t)a+tb)\right]dt. \quad (2.4)$$

Let $\mathbb{R}^n$ be Euclidian space and K is said to a nonempty closed in $\mathbb{R}^n$. Let $f: K \to \mathbb{R}$, $\varphi: K \to \mathbb{R}$ and $\eta: K \times K \to \mathbb{R}$ be a continuous functions.

Definition 2. [13] Let $u \in K$. The set K is said to be $\varphi-$invex at u with respect to η and φ if

$$u + te^{i\varphi}\eta(v,u) \in K, \quad (2.5)$$

for all $u, v \in K$ and $t \in [0,1]$.

Remark 1. Some special cases of Definition 2 are as follows:

(1) If $\varphi = 0$, then K is called an invex set.

(2) If $\eta(v,u) = v - u$, then K is called a $\varphi-$convex set.

(3) If $\varphi = 0$ and $\eta(v,u) = v - u$, then K is called a convex set.

Definition 3. Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function. A function f on the set $K_{\varphi\eta}$ is said to be $\lambda\varphi - preinvex$ function according to φ and bifunction η. Let $\forall u, v \in I$, $\eta(v,u) > 0$ and $t \in (0,1)$, then

$$f\left(u + te^{i\varphi}\eta(v,u)\right) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(v) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} f(u). \quad (2.6)$$

3. Main Results

We demonstrate the following Lemma whose we use in our conclusions.

Lemma 2. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I^0, the interior of I. Assuming that $a.b \in I^o$ with $a < b$, $\eta(b,a) > 0$ and $f'' \in L\left[a, a + e^{i\varphi}\eta(b,a)\right]$, then the following description for fractional integral with $\alpha > 0$ arranges:*

$$\frac{f(a) + f\left(a + e^{i\varphi}\eta(b,a)\right)}{2} - \frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^\alpha}\left[J^\alpha_{a^+} f\left(a + e^{i\varphi}\eta(b,a)\right) + J^\alpha_{(a+e^{i\varphi}\eta(b,a))^-} f(a)\right]$$

$$= \frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)} \int_0^1 t(1-t^\alpha)\left[f''\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right.$$
$$\left. + f''\left(a+te^{i\varphi}\eta(b,a)\right)\right] dt. \quad (3.1)$$

Proof. By using Definition 3, if use twice the partial integration, we possess;

$$\begin{aligned} I_1 &= \int_0^1 t(1-t^\alpha)\left(f''\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right) dt \\ &= \frac{\alpha f(a) + f\left(a+e^{i\varphi}\eta(b,a)\right)}{\left[e^{i\varphi}\eta(b,a)\right]^2} \\ &\quad - \frac{\alpha(\alpha+1)}{\left[e^{i\varphi}\eta(b,a)\right]^{\alpha+2}} \int_a^{a+e^{i\varphi}\eta(b,a)} \left(a+e^{i\varphi}\eta(b,a)-u\right)^{\alpha-1} f(u)\, du \\ &= \frac{\alpha f(a) + f\left(a+e^{i\varphi}\eta(b,a)\right)}{\left[e^{i\varphi}\eta(b,a)\right]^2} - \frac{(\alpha+1)\Gamma(\alpha+1)}{\left[e^{i\varphi}\eta(b,a)\right]^{\alpha+2}} I_{a^+}^\alpha f\left(a+e^{i\varphi}\eta(b,a)\right), \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_0^1 t\left(1-t^{\alpha+1}\right)\left(f''\left(a+te^{i\varphi}\eta(b,a)\right)\right) dt \\ &= \frac{f(a) + \alpha f\left(a+e^{i\varphi}\eta(b,a)\right)}{\left[e^{i\varphi}\eta(b,a)\right]^2} \\ &\quad - \frac{\alpha(\alpha+1)}{\left[e^{i\varphi}\eta(b,a)\right]^{\alpha+2}} \int_a^{a+e^{i\varphi}\eta(b,a)} (u-a)^{\alpha-1} f(u)\, du \\ &= \frac{f(a) + \alpha f\left(a+e^{i\varphi}\eta(b,a)\right)}{\left[e^{i\varphi}\eta(b,a)\right]^2} - \frac{(\alpha+1)\Gamma(\alpha+1)}{\left[e^{i\varphi}\eta(b,a)\right]^{\alpha+2}} I_{a+e^{i\varphi}\eta(b,a)^-}^\alpha f(a). \end{aligned}$$

By motivated from I_1 and I_2, we can write;

$$\begin{aligned} I_1 + I_2 = {} & \frac{\left(f(a) + f\left(a+e^{i\varphi}\eta(b,a)\right)\right)(\alpha+1)}{\left(e^{i\varphi}\eta(b,a)\right)^2} - \frac{(\alpha+1)\Gamma(\alpha+1)}{\left(e^{i\varphi}\eta(b,a)\right)^{\alpha+2}} \\ & \times \left[J_{a^+}^\alpha f\left(a+e^{i\varphi}\eta(b,a)\right) + J_{(a+e^{i\varphi}\eta(b,a))^-}^\alpha f(a)\right]. \quad (3.2) \end{aligned}$$

By expanding both directions of (3.2) by $\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)}$, we have

$$\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2} - \frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^{+}}^{\alpha}f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^{-}}^{\alpha}f(a)\right]$$

$$=\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{2(\alpha+1)}\int_{0}^{1}t\left(1-t^{\alpha}\right)\left[f''\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)+f''\left(a+te^{i\varphi}\eta(b,a)\right)\right]dt.$$

The proof is done. ☐

Remark 2. In the Lemma 2, If take $\varphi=0$ and $\eta(b,a)>0$, the Lemma 2 reduce to the Lemma 1:

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(b)+J_{(b)^{-}}^{\alpha}f(a)\right]$$

$$=\frac{(b-a)^{2}}{2(\alpha+1)}\int_{0}^{1}t\left(1-t^{\alpha}\right)\left[f''(ta+(1-t)b)+f''((1-t)a+tb)\right]dt.$$

Theorem 2. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on I^{o}assuming that $|f''|$ is a $\lambda\varphi-$ preinvex function on I. Presume that $a,b\in I^{o}$ with $a<b$, $\eta(b,a)>0$ and $f''\in L\left[a,a+e^{i\varphi}\eta(b,a)\right]$, for $\alpha>0$ then arranges:*

$$\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2} - \frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^{+}}^{\alpha}f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^{-}}^{\alpha}f(a)\right]\right|$$

$$\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{4(\alpha+1)}\left[\left|f''(a)\right|+\left|f''(b)\right|\right]\left(\beta\left(\frac{5}{2},\alpha+\frac{1}{2}\right)+\left(\frac{1-\lambda}{\lambda}\right)\beta\left(\frac{3}{2},\alpha+\frac{3}{2}\right)\right),$$

here β is Euler Beta function.

Proof. By using Definition 3 and Lemma 2, we have following inequality;

$$\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2} - \frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^{+}}^{\alpha}f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^{-}}^{\alpha}f(a)\right]\right|$$

$$\leq \frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)} \int_0^1 |t(1-t^\alpha)| \left[\left|f''\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right| + \left|f''\left(a+te^{i\varphi}\eta(b,a)\right)\right|\right] dt$$

$$\leq \frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)} \left[\int_0^1 [t(1-t^\alpha)] \left(\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f''(a)\right| + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f''(b)\right|\right) dt + \int_0^1 [t(1-t^\alpha)] \left(\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f''(b)\right| + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f''(a)\right|\right) dt\right]$$

$$\leq \frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)} \left[\left|f''(a)\right| + \left|f''(b)\right|\right] \int_0^1 [t(1-t)^\alpha] \left(\frac{\sqrt{t}}{2\sqrt{1-t}} + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\right) dt$$

$$\leq \frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)} \left[\frac{|f''(a)| + |f''(b)|}{2}\right] \left(\beta\left(\frac{5}{2},\alpha+\frac{1}{2}\right) + \left(\frac{1-\lambda}{\lambda}\right)\beta\left(\frac{3}{2},\alpha+\frac{3}{2}\right)\right).$$

Here we use following inequalities to demonstrate the second inequality:

$$|t_1^\alpha + t_2^\alpha| \leq |t_1+t_2|^\alpha, \qquad for\ \alpha \in [0,1]\ and\ \forall t_1, t_2 \in [0,1],$$

we achieve

$$\int_0^1 t^2(1-t^\alpha)\,dt + \int_0^1 t(1-t)(1-t^\alpha)\,dt \leq \int_0^1 t(1-t)^\alpha\,dt.$$

The proof is done. □

Theorem 3. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I^o. Assuming that $p \in \mathbb{R}$, $p > 1$ such that $|f''|^{\frac{p}{p-1}}$ is a $\lambda\varphi-$preinvex function on I. Presume that $a, b \in I^0$ with $a < b$, $\eta(b,a) > 0$ and $f'' \in L\left[a, a+e^{i\varphi}\eta(b,a)\right]$, then the following inequality for fractional integrals with $\alpha > 0$ arranges:*

$$\left|\frac{f(a) + f\left(a+e^{i\varphi}\eta(b,a)\right)}{2} - \frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^\alpha}\left[J_{a^+}^\alpha f\left(a+e^{i\varphi}\eta(b,a)\right) + J_{(a+e^{i\varphi}\eta(b,a))^-}^\alpha f(a)\right]\right|$$
$$\leq \frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)}\beta^{\frac{1}{p}}(p+1,\alpha p+1)\left(\frac{\pi}{4}\right)^{\frac{1}{q}}$$

$$\times\left[\left(\left|f''(a)\right|^q+\left(\frac{1-\lambda}{\lambda}\right)\left|f''(b)\right|^q\right)^{\frac{1}{q}}+\left(\left|f''(b)\right|^q+\left(\frac{1-\lambda}{\lambda}\right)\left|f''(a)\right|^q\right)^{\frac{1}{q}}\right],$$

here β is Euler Beta function.

Proof. By using Definition 3 and Lemma 2, we have following inequality for $\frac{1}{p}+\frac{1}{q}=1$:

$$\Bigg|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}$$
$$-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^\alpha}\left[J^\alpha_{a^+}f\left(a+e^{i\varphi}\eta(b,a)\right)+J^\alpha_{(a+e^{i\varphi}\eta(b,a))^-}f(a)\right]\Bigg|$$
$$\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)}\int_0^1\left|t\left(1-t^\alpha\right)\right|\left[\left|f''\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|\right.$$
$$\left.+\left|f''\left(a+te^{i\varphi}\eta(b,a)\right)\right|\right]dt$$
$$\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)}\left(\int_0^1\left[t(1-t)^\alpha\right]^p dt\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|f''\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|f''\left(a+te^{i\varphi}\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)}\beta^{\frac{1}{p}}(p+1,\alpha p+1)$$
$$\times\left[\left(\int_0^1\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f''(a)\right|^q+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f''(b)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f''(b)\right|^q+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f''(a)\right|^q dt\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)}\beta^{\frac{1}{p}}(p+1,\alpha p+1)\left(\frac{\pi}{4}\right)^{\frac{1}{q}}$$
$$\times\left[\left(\left|f''(a)\right|^q+\frac{(1-\lambda)}{\lambda}\left|f''(b)\right|^q\right)^{\frac{1}{q}}+\left(\left|f''(b)\right|^q+\frac{(1-\lambda)}{\lambda}\left|f''(a)\right|^q\right)^{\frac{1}{q}}\right].$$

The proof is done. ☐

Theorem 4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I^o and $a, b \in I^o$. Assuming that $q \geq 1$ such that $|f''|^q$ is a $\lambda\varphi -$ preinvex function on I. Presume that $a, b \in I^o$ with $a < b$, $\eta(b,a) > 0$ and $f'' \in L\left[a, a + e^{i\varphi}\eta(b,a)\right]$, then the following inequality for fractional integrals with $\alpha > 0$ arranges:*

$$
\begin{aligned}
&\left| \frac{f(a) + f\left(a + e^{i\varphi}\eta(b,a)\right)}{2} \right. \\
&\left. -\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}} \left[J_{a^+}^{\alpha} f\left(a + e^{i\varphi}\eta(b,a)\right) + J_{(a+e^{i\varphi}\eta(b,a))^-}^{\alpha} f(a) \right] \right| \\
&\leq \frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{(\alpha+1)} \beta^{\frac{1}{p}}(p+1, \alpha p+1)\, 2^{-\frac{q+1}{q}} \\
&\times \left[\left(\beta\left(\frac{5}{2}, \alpha + \frac{1}{2}\right) |f''(a)|^q + \frac{1-\lambda}{\lambda} \beta\left(\frac{3}{2}, \alpha + \frac{3}{2}\right) |f''(b)|^q \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\frac{1-\lambda}{\lambda} \beta\left(\frac{3}{2}, \alpha + \frac{3}{2}\right) |f''(a)|^q + \beta\left(\frac{5}{2}, \alpha + \frac{1}{2}\right) |f''(b)|^q \right)^{\frac{1}{q}} \right].
\end{aligned}
$$

Proof. By using Definition 3, Lemma 2 and power mean integral inequality, we possess

$$
\begin{aligned}
&\left| \frac{f(a) + f\left(a + e^{i\varphi}\eta(b,a)\right)}{2} \right. \\
&\left. -\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}} \left[J_{a^+}^{\alpha} f\left(a + e^{i\varphi}\eta(b,a)\right) + J_{(a+e^{i\varphi}\eta(b,a))^-}^{\alpha} f(a) \right] \right| \\
&\leq \frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)} \int_0^1 |t(1-t^{\alpha})| \left[\left| f''\left(a + (1-t)e^{i\varphi}\eta(b,a)\right) \right| + \left| f''\left(a + te^{i\varphi}\eta(b,a)\right) \right| \right] dt \\
&\leq \frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)} \left(\int_0^1 \left[t(1-t)^{\alpha}\right] dt \right)^{\frac{q-1}{q}} \\
&\times \left[\left(\int_0^1 t(1-t)^{\alpha} \left| f''\left(a + (1-t)e^{i\varphi}\eta(b,a)\right) \right|^q dt \right)^{\frac{1}{q}} \right.
\end{aligned}
$$

$$+\left(\int_0^1 t(1-t)^{\alpha}\left|f''\left(a+te^{i\varphi}\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg]$$

$$\leq \frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)}\left(\beta^{\frac{q-1}{q}}(2,\alpha+1)\right)$$

$$\times\left[\left(\int_0^1 t(1-t)^{\alpha}\left[\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f''(a)\right|^q+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 t(1-t)^{\alpha}\left[\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f''(a)\right|^q+\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{(\alpha+1)}\beta^{\frac{1}{p}}(p+1,\alpha p+1)\,2^{-\frac{q+1}{q}}$$

$$\times\left[\left(\beta\left(\frac{5}{2},\alpha+\frac{1}{2}\right)\left|f''(a)\right|^q+\frac{1-\lambda}{\lambda}\beta\left(\frac{3}{2},\alpha+\frac{3}{2}\right)\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{1-\lambda}{\lambda}\beta\left(\frac{3}{2},\alpha+\frac{3}{2}\right)\left|f''(a)\right|^q+\beta\left(\frac{5}{2},\alpha+\frac{1}{2}\right)\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right].$$

The proof is done. □

Theorem 5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I^o. Assuming that $p \in \mathbb{R}$, $p > 1$ with $q = \frac{p}{p-1}$ such that $|f''|^q$ is a $\lambda\varphi-$preinvex function on I. Presume that $a, b \in I^o$ with $a < b$, $\eta(b,a) > 0$ and $f'' \in L\left[a, a+e^{i\varphi}\eta(b,a)\right]$, then the following inequality for fractional integrals with $\alpha > 0$ arranges:*

$$\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}\right.$$

$$\left.-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J^{\alpha}_{a^+}f\left(a+e^{i\varphi}\eta(b,a)\right)+J^{\alpha}_{(a+e^{i\varphi}\eta(b,a))^-}f(a)\right]\right|$$

$$\leq \frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{(\alpha+1)}\beta^{\frac{1}{p}}(p+1,\alpha p+1)\left|f''\left(\frac{2a+e^{i\varphi}\eta(b,a)}{2}\right)\right|^q.$$

Proof. By using Definition 3, Lemma 2 and the Hölder's inequality, we possess;

$$\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2(e^{i\varphi}\eta(b,a))^{\alpha}}\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta\left(b,a\right)\right)+J_{(a+e^{i\varphi}\eta(b,a))^-}^{\alpha}f\left(a\right)\right]\right|$$
$$\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)}\int_0^1\left|t\left(1-t^{\alpha}\right)\right|\left[\left|f''\left(a+\left(1-t\right)e^{i\varphi}\eta\left(b,a\right)\right)\right|+\left|f''\left(a+te^{i\varphi}\eta\left(b,a\right)\right)\right|\right]dt$$
$$\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)}\left(\int_0^1\left[t^p\left(1-t\right)^{\alpha p}\right]dt\right)^{\frac{1}{p}}$$
$$\times\left[\left(\int_0^1\left|f''\left(a+\left(1-t\right)e^{i\varphi}\eta\left(b,a\right)\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|f''\left(a+te^{i\varphi}\eta\left(b,a\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

Because of $|f''|^q$ is concave on $[a.b]$; we can write the integral Jensen's inequality and we have;

$$\begin{aligned}\int_0^1\left|f''\left(a+\left(1-t\right)e^{i\varphi}\eta\left(b,a\right)\right)\right|^q dt &= \int_0^1\left|f''\left(a+\left(1-t\right)e^{i\varphi}\eta\left(b,a\right)\right)\right|^q dt\\ &\leq\left(\int_0^1 dt\right)\left|f''\left(\frac{\int_0^1\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)dt}{\int_0^1 dt}\right)\right|^q\\ &=\left|f''\left(\frac{2a+e^{i\varphi}\eta(b,a)}{2}\right)\right|^q,\end{aligned}$$

and

$$\int_0^1\left|f''\left(a+te^{i\varphi}\eta\left(b,a\right)\right)\right|^q dt\leq\left|f''\left(\frac{2a+e^{i\varphi}\eta(b,a)}{2}\right)\right|^q,$$

then we have:

$$\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2(e^{i\varphi}\eta(b,a))^{\alpha}}\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta\left(b,a\right)\right)+J_{(a+e^{i\varphi}\eta(b,a))^-}^{\alpha}f\left(a\right)\right]\right|$$
$$\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{(\alpha+1)}\beta^{\frac{1}{p}}\left(p+1,\alpha p+1\right)\left|f''\left(\frac{2a+e^{i\varphi}\eta(b,a)}{2}\right)\right|^q.$$

The proof is done. □

Remark 3. In Theorem 5, if $\varphi=0$ and $\eta\left(b,a\right)=b-a$. Theorem 5 reduce to $[12, Theorem\ 5]$:

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f\left(b\right)+J_{(b)^-}^{\alpha}f\left(a\right)\right]\right|$$
$$\leq\frac{(b-a)^2}{(\alpha+1)}\beta^{\frac{1}{p}}\left(1+p,1+\alpha p\right)\left|f''\left(\frac{a+b}{2}\right)\right|^q.$$

Theorem 6. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I^o. Assuming that $p \geq 1$ with $q = \frac{p}{p-1}$ such that $|f''|^q$ is a $\lambda\varphi-$ preinvex function on I. Presume that $a, b \in I^o$ with $a < b$, $\eta(b,a) > 0$ and $f'' \in L\left[a, a + e^{i\varphi}\eta(b,a)\right]$, then the following inequality for fractional integrals with $\alpha > 0$ arranges:*

$$\left|\frac{f(a) + f\left(a + e^{i\varphi}\eta(b,a)\right)}{2} - \frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J^{\alpha}_{a^+} f\left(a + e^{i\varphi}\eta(b,a)\right) + J^{\alpha}_{\left(a + e^{i\varphi}\eta(b,a)\right)^-} f(a)\right]\right|$$
$$\leq \frac{\alpha\left(e^{i\varphi}\eta(b,a)\right)^2}{4(\alpha+1)(\alpha+2)}\left[\left|f''\left(a + \frac{2(\alpha+2)e^{i\varphi}\eta(b,a)}{3(\alpha+3)}\right)\right|^q + \left|f''\left(a + \frac{(\alpha+5)e^{i\varphi}\eta(b,a)}{3(\alpha+3)}\right)\right|^q\right].$$

Proof. By using concavity of $|f''|$, $|f''|^q$and the power-mean inequality, we have following inequalities;

$$\left|f''\left(a + (1-t)e^{i\varphi}\eta(b,a)\right)\right|^q > \frac{\sqrt{t}}{2\sqrt{1-t}}\left|f''(b)\right|^q + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f''(a)\right|^q$$
$$\geq \left(\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f''(b)\right| + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f''(a)\right|\right)^q,$$

and

$$\left|f''\left(a + (1-t)e^{i\varphi}\eta(b,a)\right)\right| \geq \frac{\sqrt{t}}{2\sqrt{1-t}}\left|f''(b)\right| + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f''(a)\right|.$$

By using the Jensen integral inequality, we possess;

$$\left|\int_0^1 t\left(1-t^{\alpha}\right) f''\left(a + (1-t)e^{i\varphi}\eta(b,a)\right) dt\right|$$
$$\leq \left(\int_0^1 t\left(1-t^{\alpha}\right) dt\right)\left|f''\left(\frac{\int_0^1 t\left(1-t^{\alpha}\right)\left(a + (1-t)e^{i\varphi}\eta(b,a)\right) dt}{\int_0^1 t\left(1-t^{\alpha}\right) dt}\right)\right|^q$$
$$= \frac{\alpha}{2(\alpha+2)}\left|f''\left(a + \frac{(\alpha+5)e^{i\varphi}\eta(b,a)}{3(\alpha+3)}\right)\right|^q,$$

and

$$\left|\int_0^1 t\left(1-t^{\alpha}\right) f''\left(a+te^{i\varphi}\eta(b,a)\right) dt\right|$$

$$\leq \left(\int_0^1 t\left(1-t^{\alpha}\right) dt\right)\left|f''\left(\frac{\int_0^1 t\left(1-t^{\alpha}\right)\left(a+te^{i\varphi}\eta(b,a)\right) dt}{\int_0^1 t\left(1-t^{\alpha}\right) dt}\right)\right|^q$$

$$=\frac{\alpha}{2(\alpha+2)}\left|f''\left(a+\frac{2(\alpha+2)e^{i\varphi}\eta(b,a)}{3(\alpha+3)}\right)\right|^q.$$

As the conclusion:

$$\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a+}^{\alpha} f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^{-}}^{\alpha} f(a)\right]\right|$$
$$\leq \frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)}\left(\int_0^1 t\left(1-t^{\alpha}\right) dt\right)^{\frac{q-1}{q}}$$
$$\times\left[\left(\int_0^1 t\left(1-t^{\alpha}\right)\left|f''\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 t\left(1-t^{\alpha}\right)\left|f''\left(a+te^{i\varphi}\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{\alpha\left(e^{i\varphi}\eta(b,a)\right)^2}{4(\alpha+1)(\alpha+2)}\left[\left|f''\left(a+\frac{2(\alpha+2)e^{i\varphi}\eta(b,a)}{3(\alpha+3)}\right)\right|^q\right.$$
$$\left.+\left|f''\left(a+\frac{(\alpha+5)e^{i\varphi}\eta(b,a)}{3(\alpha+3)}\right)\right|^q\right].$$

The proof is done. □

Remark 4. In Theorem 6, if $\varphi=0$ and $\eta(b,a)=b-a$. Theorem 6 reduce to $[12, Theorem\ 6]$:

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(b)+J_{(b)^{-}}^{\alpha}f(a)\right]\right|$$
$$\leq\frac{\alpha(b-a)^{2}}{4(\alpha+1)(\alpha+2)}\left[\left|f''\left(\frac{\alpha+5}{3\alpha+9}a+\frac{2\alpha+4}{3\alpha+9}b\right)\right|^{q}\right.$$
$$\left.+\left|f''\left(\frac{2\alpha+4}{3\alpha+9}a+\frac{\alpha+5}{3\alpha+9}b\right)\right|^{q}\right].$$

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