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SOLUTION OF THE DIOPHANTINE EQUATION $(2^k - 1)^x + (2^k)^y = z^2$

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Abstract: In this paper, we will apply the Catalan's conjecture to find the solution of the Diophantine equation $(2^k - 1)^x + (2^k)^y = z^2$ where k is an odd integer at least 3.

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1. Introduction

In 1844, Catalan [1] posed a conjecture that $(a, b, x, y) = (3, 2, 2, 3)$ is a unique solution of the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with $min\{a, b, x, y\} > 1$. Then Mihailescu [2] proved the Catalan's conjecture in 2004.

In 2011, Suvarnamani, Singta and Chotchaisthit [6] proved that two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution.

In 2012, Suvarnamani [5] found that the Diophantine equation $A^x + B^y = C^z$ has some non-negative integer solutions. Then Sroysang [4] proved that $(0, 1, 3)$ is a unique non-negative integer solution of the Diophantine equation $7^x + 8^y = z^2$.

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In 2014, Simtrakankul [3] fonnd that the Diophantine equation $(2^k-1)^x + (2^k)^y = z^2$ has a unique solution $(1, 0, 2^{k/2})$ where k is an even integer at least 4. Then Suvarnamani [7] found the solution of the Diophantine equation $p^x + q^y = z^2$ where p is an odd prime number which $q - p = 2$ and x, y and z are non-negative integers. After that Suvarnamani studied in [8] about the Diophantine equation $p^x + (p+1)^y = z^2$ where p is an odd prime number and x, y and z are non-negative integers.

In this paper, we will use the Catalan's conjecture to solve the Diophantine equation $(2^k-1)^x + (2^k)^y = z^2$ where k is an odd integer at least 3.

2. Preliminaries

Lemma 2.1. $(a, b, x, y) = (3, 2, 2, 3)$ *is a unique solution of the Diophantine equation* $a^x - b^y = 1$ *where* a, b, x *and* y *are integers with* $min\{a, b, x, y\} > 1$.

Proof. See in [4]. □

Lemma 2.2. *Let* k *is an odd integer at least 3 and* y, z *are non-negative integers. Then the Diophantine equation* $1 + (2^k)^y = z^2$ *has a unique solution* $(y, z, k) = (1, 3, 3)$.

Proof. Let k is an odd integer at least 3 and y, z are non-negative integers such that $1 + (2^k)^y = z^2$. We consider in 2 cases.

Case 1: $y = 0$. Then $z^2 = 2$ which is impossible.

Case 2: $y \geq 1$. Then $z^2 - 2^{ky} = 1$. We have $z > 2$, then $min\{z, 2, ky\} > 1$. By Lemma 2.1, we get $z = 3$ and $ky = 3$. Hence $k = 3$ and $y = 1$.

Then the Diophantine equation $1+(2^k)^y = z^2$ has a unique solution $(y, z, k) = (1, 3, 3)$. □

Lemma 2.3. *Let* k *is an odd integer at least 3 and* x, z *are non-negative integers. Then the Diophantine equation* $(2^k-1)^x + 1 = z^2$ *has no solution.*

Proof. Let k is an odd integer at least 3 and x, z are non-negative integers such that $(2^k-1)^x + 1 = z^2$. We consider in 3 cases.

Case 1: $x = 0$. Then $z^2 = 2$ which is impossible.

Case 2: $x = 1$. Then $z^2 = 2^k$ which is impossible.

Case 3: $x > 1$. We have $(2^k-1)^x \geq (2^k-1)^2$. We have $z \geq 2$ and $2^k-1 \geq 4$, then $min\{z, 2^k-1, 2, x\} > 1$. By Lemma 2.1, we get $z = 3$, $2^k - 1 = 2$ and $x = 3$. But it is impossible.

Then the Diophantine equation $(2^k-1)^x + 1 = z^2$ has no solution. □

3. Main Theorem

Main Theorem 3.1. $(0, 1, 3, 3)$ *and* $(2, 2, 5, 2)$ *are only two solutions (x,y,z,k) of the Diophantine equation* $(2^k-1)^x + (2^k)^y = z^2$ *where* k *is an odd integer at least 3 and* x*,* y *and* z *are non-negative integers.*

Proof. Let k is an odd integer at least 3 and x, y and z are non-negative integers such that $(2^k-1)^x + (2^k)^y = z^2$. By Lemma 2.3, we have $y \geq 1$. Then we consider in 2 cases.

Case 1: $x = 0$. Then $1 + (2^k)^y = z^2$. By Lemma 2.2, we get $(x, y, z, k) = (0, 1, 3, 3)$ is a solution of the Diophantine equation $(2^k-1)^x + (2^k)^y = z^2$.

Case 2: $x \geq 1$. We have z as an odd integer. So, $z^2 \equiv 1(mod4)$. Then $z^2 - 2^{ky} \equiv 1(mod4)$. So, we get x is an even integer, i.e., $x = 2n$ where n is a positive integer. Then $2^{ky} = z^2 - (2^k-1)^{2n} = [z-(2^k-1)^n][z+(2^k-1)^n]$. So, $2(2^k-1)^n = 2^u(2^{ky-2u}-1)$ where $z-(2^k-1)^n = 2^u$ and $z+(2^k-1)^n = 2^{ky-u}$, for $ky > 2u$ and u is a non-negative integer. Hence $u = 1$, then $2^{ky-2}-(2^k-1)^n = 1$.

Next, we will consider in 3 subcases.

Subcase 1: $n = 1$. We have $2^{ky-2} - (2^k-1) = 1$. Then $k = \frac{2}{y-1}$. We get $k = 1$ or $k = 2$. Contradiction.

Subcase 2: $ky - 2 = 1$. That is $k = 3$ and $y = 1$. We get $2 - 7^n = 1$. It is impossible.

Subcase 3: $min\{2, ky-2, 2^k-1, n\} > 1$. By Lemma 2.1, we get the Diophantine equation $(2^k-1)^x + (2^k)^y = z^2$ has no solution. □

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