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# EXISTENCE OF FIXED POINT FOR WEAKLY HARDY-ROGERS CONTRACTIONS IN PARTIALLY ORDERED METRIC SPACES

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**Abstract:** The aim of this paper is to present some fixed point theorems for the class of weakly Hardy-Rogers contractions in a partially ordered complete metric space. Moreover, we show that this class possesses the P property. Finally, some results for contractions of integral type are given.

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**Key Words:** fixed point, ordered metric space, complete metric space

## 1. Introduction and Mathematical Preliminaries

Fixed point theory is an interesting field of mathematics which is started in 1922, when Banach [2] proved his famous result that today is well known as Banach's contraction principle. This essential theorem is concerning with the existence and uniqueness of fixed point for contractive mappings, defined on a complete metric space. In these years, extension of the Banach's contraction principle has been considered by many authors in different metric spaces such

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as partially ordered spaces, cone metric spaces, etc.

In this paper, we establish some fixed point theorems for a class of mappings that we introduce it as weakly Hardy-Rogers contraction in a partially ordered complete metric space. The motivation of our definition is [3], [4] and [6]. In fact, in this papers the concepts of C-contraction, weak $C$-contraction and T-Hardy-Rogers contraction have been introduced by Chatterjea, Choudhury and Filipović et al, respectively.

**Definition 1.1.** [3] A mapping $T : X \to X$, where $(X, d)$ is a metric space is said to be a $C$-contraction if there exists $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$ the following inequality holds:

$$d(Tx, Ty) \leq \alpha(d(x, Ty) + d(y, Tx)).$$

**Definition 1.2.** ([4], Definition 1.3) A mapping $T : X \to X$, where $(X, d)$ is a metric space is said to be weakly $C$-contraction (or a weak $C$-contraction) if for all $x, y \in X$,

$$d(Tx, Ty) \leq \frac{1}{2}(d(x, Ty) + d(y, Tx)) - \varphi(d(x, Ty), d(y, Tx)),$$

where $\varphi : [0, \infty)^2 \to [0, \infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only if $x = y = 0$.

Harjani *et al.* have presented the following theorem in a partially ordered complete metric space.

**Theorem 1.3.** *([7], Theorem 2.1) Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ in $X$ such that $(X, d)$ is a complete metric space. Let $T : X \longrightarrow X$ be a continuous and nondecreasing mapping such that*

$$d(Tx, Ty) \leq \frac{1}{2}(d(x, Ty) + d(y, Tx)) - \varphi(d(x, Ty), d(y, Tx)),$$

*for $x \succeq y$, where $\varphi : [0, \infty)^2 \to [0, \infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only if $x = y = 0$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then $T$ has a fixed point.*

In what follows, we need the following definitions.

**Definition 1.4.** [1] Let $(X, \preceq)$ be a partially ordered set. A mapping $T : X \longrightarrow X$ is called dominating, if $x \preceq Tx$ for each $x \in X$.

**Definition 1.5.** [13] Let $(X, d)$ be a metric space and $T : X \to X$ a mapping. T is said to be a Hardy-Rogers contraction, if there exist $\beta_i > 0$, $i = 1, \cdots, 5$ with $\sum_{i=1}^{5} \beta_i < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \beta_1 d(x, y) + \beta_2 d(x, Tx) + \beta_3 d(x, Ty) + \beta_4 d(y, Tx) + \beta_5 d(y, Ty).$$

**Definition 1.6.** A mapping $T : X \to X$, where $(X, d)$ is a metric space is said to be a weakly Hardy-Rogers contraction, if there exist $\alpha_i > 0$, $i = 1, \cdots, 5$ with $\sum_{i=1}^{5} \alpha_i = 1$ such that for all $x, y \in X$,

$$\begin{aligned} d(Tx, Ty) \quad &\leq \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(x, Ty) + \alpha_4 d(y, Tx) + \alpha_5 d(y, Ty) \\ &-\varphi(d(x, y), d(x, Tx), d(x, Ty), d(y, Tx), d(y, Ty)), \end{aligned} \tag{1.1}$$

where $\varphi : [0, \infty)^5 \to [0, \infty)$ is a continuous function such that $\varphi(x, y, z, t, u) = 0$ if and only if $x = y = z = t = u = 0$.

## 2. Main Results

Our first result is the following.

**Theorem 2.1.** *Let $(X, \preceq, d)$ be a partially ordered complete metric space and let $E$ be a nonempty closed subset of $X$. Let $T : E \to E$ be a continuous dominating ordered weakly Hardy-Rogers contraction, that is, 1.1 holds, for every comparable elements $x, y \in X$. Assume that $\alpha_3 \leq \alpha_4$. Then $T$ has a fixed point.*

*Proof.* Let $x_0 \in X$ be arbitrary. We know that $x_0 \preceq Tx_0$. If $Tx_0 = x_0$, then the proof is finished. Suppose that $x_0 \prec Tx_0$. We obtain by induction

$$x_0 \prec Tx_0 \preceq T^2 x_0 \preceq T^3 x_0 \preceq \cdots \preceq T^n x_0 \preceq T^{n+1} x_0 \preceq \cdots .$$

We define $x_{n+1} = Tx_n = T^n x_0$. First, we show that $\{x_n\}$ is a Cauchy sequence in $E$.

If $x_n = x_{n+1}$ for some $n$, then $Tx_{n-1} = Tx_n = T(Tx_{n-1})$ and the proof is completed.

Now, let $x_n \neq x_{n+1}$ for every positive integer $n$. So, we obtain that

$$\begin{aligned} d(x_{n+1}, x_{n+2}) = \;& d(Tx_n, Tx_{n+1}) \\ \leq \;& \alpha_1 d(x_n, x_{n+1}) + \alpha_2 d(x_n, Tx_n) + \alpha_3 d(x_n, Tx_{n+1}) \\ & +\alpha_4 d(x_{n+1}, Tx_n) + \alpha_5 d(x_{n+1}, Tx_{n+1}) \\ & -\varphi(d(x_n, x_{n+1}), d(x_n, Tx_n), \\ & d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_n), d(x_{n+1}, Tx_{n+1})) \\ = \;& \alpha_1 d(x_n, x_{n+1}) + \alpha_2 d(x_n, x_{n+1}) + \alpha_3 d(x_n, x_{n+2}) \\ & +\alpha_4 d(x_{n+1}, x_{n+1}) + \alpha_5 d(x_{n+1}, x_{n+2}) \\ & -\varphi(d(x_n, x_{n+1}), d(x_n, x_{n+1}), \\ & d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1}), d(x_{n+1}, x_{n+2})) \\ \leq \;& \alpha_1 d(x_n, x_{n+1}) + \alpha_2 d(x_n, x_{n+1}) + \alpha_3 d(x_n, x_{n+2}) \\ & +\alpha_5 d(x_{n+1}, x_{n+2}) \\ \leq \;& \alpha_1 d(x_n, x_{n+1}) + \alpha_2 d(x_n, x_{n+1}) + \alpha_3 d(x_n, x_{n+1}) \\ & +\alpha_3 d(x_{n+1}, x_{n+2}) + \alpha_5 d(x_{n+1}, x_{n+2}) \\ = \;& (\alpha_1 + \alpha_2 + \alpha_3) d(x_n, x_{n+1}) \\ & +(\alpha_3 + \alpha_5) d(x_{n+1}, x_{n+2}). \end{aligned} \tag{2.1}$$

Therefore,

$$(1 - (\alpha_3 + \alpha_5)) d(x_{n+1}, x_{n+2}) \leq (\alpha_1 + \alpha_2 + \alpha_3) d(x_n, x_{n+1}).$$

Thus, $\{d(x_{n+1}, x_n)\}$ is a decreasing sequence of nonnegative real numbers and hence it should be convergent.

Assume that, $\lim_{n\to\infty} d(x_{n+1}, x_n) = r$.

From the above argument we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) \leq & \alpha_1 d(x_n, x_{n+1}) + \alpha_2 d(x_n, x_{n+1}) \\ & + \alpha_3 d(x_n, x_{n+2}) + \alpha_5 d(x_{n+1}, x_{n+2}) \\ \leq & (\alpha_1 + \alpha_2 + \alpha_3) d(x_n, x_{n+1}) + (\alpha_3 + \alpha_5) d(x_{n+1}, x_{n+2}). \end{aligned}$$

If $n \to \infty$, we have

$$\begin{aligned} r &\leq (\alpha_1 + \alpha_2 + \alpha_5) r + \alpha_3 \lim_{n\to\infty} [d(x_n, x_{n+2})] \\ &\leq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_3 + \alpha_5) r \\ &\leq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) r \leq (1) r. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = \frac{(1 - (\alpha_1 + \alpha_2 + \alpha_5))}{\alpha_3} r.$$

We have proved in 2.1 that,

$$\begin{aligned} d(x_{n+1}, x_{n+2}) \leq\ & \alpha_1 d(x_n, x_{n+1}) + \alpha_2 d(x_n, x_{n+1}) \\ & + \alpha_3 d(x_n, x_{n+2}) + \alpha_5 d(x_{n+1}, x_{n+2}) \\ & - \varphi(d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_n, x_{n+2}), \\ & d(x_{n+1}, x_{n+1}), d(x_{n+1}, x_{n+2})) \\ \leq\ & (\alpha_1 + \alpha_2 + \alpha_3) d(x_n, x_{n+1}) \\ & + (\alpha_3 + \alpha_5) d(x_{n+1}, x_{n+2}). \end{aligned} \tag{2.2}$$

Now, if $n \to \infty$ and from the continuity of $\varphi$, we obtain

$$\begin{aligned} r \leq & (\alpha_1 + \alpha_2 + \alpha_5) r + \alpha_3 \Big( \frac{(1 - (\alpha_1 + \alpha_2 + \alpha_5))}{\alpha_3} r \Big) \\ & - \varphi\Big(r, r, \frac{(1 - (\alpha_1 + \alpha_2 + \alpha_5))}{\alpha_3} r, 0, r\Big), \end{aligned}$$

and consequently, $\varphi(r, r, \frac{(1-(\alpha_1+\alpha_2+\alpha_5))}{\alpha_3} r, 0, r) = 0$. This gives us that

$$r = \lim_{n \to \infty} d(x_n, x_{n+1}) = 0, \tag{2.3}$$

by our assumption about $\varphi$.

Now, we show that $\{x_n\}$ is a Cauchy sequence in $E$.

If not, then there is $\varepsilon > 0$ and there exist subsequences $\{m(k)\}$ and $\{n(k)\}$ of integers with $m(k) > n(k) > k$ such that $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$ and

$$d(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{2.4}$$

From 2.3 and triangle inequality

$$\begin{aligned} \varepsilon \leq d(x_{m(k)}, x_{n(k)}) & \leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \\ & \leq \varepsilon + d(x_{n(k)-1}, x_{n(k)}). \end{aligned}$$

Letting $k \to \infty$, from 2.3 we can conclude that

$$\lim_{k \to \infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon. \tag{2.5}$$

Also 2.5 and inequality

$$d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)}),$$

yield that

$$\varepsilon \leq \lim_{k \to \infty} d(x_{m(k)+1}, x_{n(k)}),$$

while 2.4, 2.5 and inequality

$$d(x_{m(k)+1}, x_{n(k)}) \leq d(x_{m(k)+1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}),$$

yield that

$$\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}) \leq \varepsilon,$$

and hence

$$\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}) = \varepsilon. \tag{2.6}$$

Moreover, from 2.3, 2.6 and inequality

$$d(x_{m(k)+1}, x_{n(k)-1}) \leq d(x_{m(k)+1}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)-1})$$

we have

$$\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)-1}) \leq \varepsilon.$$

On the other hand, 2.3, 2.6 and relation

$$d(x_{m(k)+1}, x_{n(k)}) \leq d(x_{m(k)+1}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}),$$

yield that

$$\varepsilon \leq \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)-1}),$$

and hence

$$\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)-1}) = \varepsilon. \tag{2.7}$$

Furthermore, 2.3, 2.5 and

$$d(x_{n(k)-1}, x_{m(k)}) \leq d(x_{n(k)-1}, x_{n(k)}) + d(x_{n(k)}, x_{m(k)}),$$

imply that

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) \leq \varepsilon.$$

Also, from 2.3, 2.4 and inequality

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}),$$

we conclude that

$$\varepsilon \leq \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}),$$

and hence

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) = \varepsilon.$$

Consequently, we have

$$\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}) = \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)+1}) = \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) = \varepsilon. \tag{2.8}$$

As $x_{m(k)}$ and $x_{n(k)-1}$ are comparable, using 1.1 we have

$$\begin{aligned} d(x_{m(k)+1}, x_{n(k)}) = \; & d(Tx_{m(k)}, Tx_{n(k)-1}) \\ \leq \; & \alpha_1 d(x_{m(k)}, x_{n(k)-1}) + \alpha_2 d(x_{m(k)}, Tx_{m(k)}) \\ & +\alpha_3 d(x_{m(k)}, Tx_{n(k)-1}) \\ & +\alpha_4 d(x_{n(k)-1}, Tx_{m(k)}) + \alpha_5 d(x_{n(k)-1}, Tx_{n(k)-1}) \\ & -\varphi(d(x_{m(k)}, x_{n(k)-1}), d(x_{m(k)}, Tx_{m(k)}), \\ & d(x_{m(k)}, Tx_{n(k)-1}), d(x_{n(k)-1}, Tx_{m(k)}), d(x_{n(k)-1}, Tx_{n(k)-1})) \\ = \; & \alpha_1 d(x_{m(k)}, x_{n(k)-1}) + \alpha_2 d(x_{m(k)}, x_{m(k)+1}) \\ & +\alpha_3 d(x_{m(k)}, x_{n(k)}) \\ & +\alpha_4 d(x_{n(k)-1}, x_{m(k)+1}) + \alpha_5 d(x_{n(k)-1}, x_{n(k)}) \\ & -\varphi(d(x_{m(k)}, x_{n(k)-1}), d(x_{m(k)}, x_{m(k)+1}), \\ & d(x_{m(k)}, x_{n(k)}), d(x_{n(k)-1}, x_{m(k)+1}), d(x_{n(k)-1}, x_{n(k)})). \end{aligned} \tag{2.9}$$

Making $k \to \infty$, from 2.3, 2.4, 2.8 and the continuity of $\varphi$, we have

$$\varepsilon \leq \alpha_1\varepsilon + 0 + \alpha_3\varepsilon + \alpha_4\varepsilon + 0 - \varphi(\varepsilon, 0, \varepsilon, \varepsilon, 0)$$

and so we have $\varphi(\varepsilon, 0, \varepsilon, \varepsilon, 0) \leq (\alpha_1+\alpha_3+\alpha_4-1)\varepsilon \leq 0$. Therefore, $\varphi(\varepsilon, 0, \varepsilon, \varepsilon, 0) = 0$. By our assumption about $\varphi$, we have $\varepsilon = 0$, which is a contradiction and it follows that $\{x_n\}$ is a Cauchy sequence in $E$.

Now, we show that $T$ has a fixed point.

Since $(X, d)$ is complete and $\{x_n\}$ is Cauchy, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Since $E$ is closed and $\{x_n\} \subseteq E$, we have $z \in E$. Moreover, the continuity of $T$ implies that

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tz,$$

and this proves that $z$ is a fixed point for $T$. □

**Remark 2.2.** In the above theorem, we have proved that, for every $x_0 \in X$, the sequence $T^n x_0$ is Cauchy and there is $z_{x_0} \in X$ such that $\lim_{n\to\infty} T^n x_0 = z_{x_0}$.

Now, our plan is to omit the continuity assumption of T. For this goal, we consider the concept of regularity.

**Definition 2.3.** Let $(X, \preceq, d)$ be a partially ordered metric space. We say that $X$ is regular if and only if the following hypothesis holds:

if $\{z_n\}$ is a non-decreasing sequence in $X$ with respect to $\preceq$ such that $z_n \to z \in X$ as $n \to \infty$, then $z_n \preceq z$, for all $n \in \mathbb{N}$.

In the next theorem, we assume that $T$ is only dominating and $X$ is regular.

**Theorem 2.4.** *Let $(X, \preceq, d)$ be a partially ordered complete metric space and let $E$ be a nonempty closed subset of $X$. Let $T : E \to E$ be a dominating weakly Hardy-Rogers contraction (not necessarily continuous). Then $T$ has a fixed point.*

*Proof.* As $\{x_n\}$ is a nondecreasing sequence in $X$ and $x_n \to z$, then, regularity of $X$ gives us that $x_n \preceq z$ for every $n \in \mathbb{N}$, *i.e.*, $x_n$, $z$ are comparable for any $n \in \mathbb{N}$ and, consequently, 1.1 yields that,

$$\begin{aligned} d(x_{n+1}, Tz) = \;& d(Tx_n, Tz) \\ & \leq \alpha_1 d(x_n, z) + \alpha_2 d(x_n, Tx_n) + \alpha_3 d(x_n, Tz) \\ & +\alpha_4 d(z, Tx_n) + \alpha_5 d(z, Tz) \\ & -\varphi(d(x_n, z), d(x_n, Tx_n), d(x_n, Tz), d(z, Tx_n), d(z, Tz)) \\ = \;& \alpha_1 d(x_n, z) + \alpha_2 d(x_n, x_{n+1}) + \alpha_3 d(x_n, Tz) \\ & +\alpha_4 d(z, x_{n+1}) + \alpha_5 d(z, Tz) \\ & -\varphi(d(x_n, z), d(x_n, x_{n+1}), d(x_n, Tz), d(z, x_{n+1}), d(z, Tz)). \end{aligned} \tag{2.10}$$

If $n \to \infty$,

$$d(z, Tz) \leq \alpha_3 d(z, Tz) + \alpha_5 d(z, Tz) - \varphi(0, 0, d(z, Tz), 0, d(z, Tz)),$$

and hence,

$$\varphi(0, 0, d(z, Tz), 0, d(z, Tz)) \leq (\alpha_3 + \alpha_5 - 1)d(z, Tz) \leq 0.$$

Therefore, $d(z, Tz) = 0$. So, we have $Tz = z$. □

The following simple example (inspired by ([8], Example 4.1.)) shows that the conditions of theorems 2.1 and 2.4 can not guarantee the uniqueness of fixed point.

**Example 2.5.** Let $X = \{(0, 1), (1, 0)\} \subset \mathbb{R}^2$ with the euclidean distance $d$. $(X, d)$ is, obviously, a complete metric space. Moreover, we consider the order $\preceq$ in X given by $(x_1, y_1) \preceq (x_2, y_2) \Leftrightarrow x_1 \leq x_2$ and $y_1 \leq y_2$. Define $\varphi : [0, \infty)^5 \to [0, \infty)$ by

$$\varphi(x, y, z, t, u) = \frac{\alpha_1 x + \alpha_2 y + \alpha_3 z + \alpha_4 t + \alpha_5 u}{2},$$

where $\alpha_i > 0$ for all $i = 1, \cdots, 5$ and $\sum_{i=1}^{5} \alpha_i = 1$.

Also, let $T : X \to X$ be the identity map.

As the only comparable pairs of points in $X$ are $(x, x)$ for $x \in X$, we see that condition 1.1, appearing in Theorem 2.1 and 2.4, holds. Finally, Theorems 2.1 and 2.4 give us the existence of two fixed points for $T$ (*i.e.*, the points $(1, 0)$ and $(0, 1)$).

In the following theorem, we give a sufficient condition that guarantees the uniqueness of the fixed point.

**Theorem 2.6.** *Let all the conditions of theorems 2.1 and 2.4 be fulfilled, $\alpha_2 \geq \alpha_5$ and let the following condition also holds:*

*For arbitrary two points $x, y \in X$ there exists $w \in X$ which is comparable with both $x$ and $y$.*

*Then the fixed point of $T$ is unique.*

*Proof.* Let $u$ and $v$ be two fixed points of $T$, *i.e.*, $Tu = u$ and $Tv = v$. Consider the following two cases.

1. $u$ and $v$ are comparable. Then we can apply condition 1.1 and obtain

$$\begin{aligned} d(u,v) = \ & d(Tu,Tv) \\ & \leq \alpha_1 d(u,v) + \alpha_2 d(u,Tu) + \alpha_3 d(u,Tv) \\ & +\alpha_4 d(v,Tu) + \alpha_5 d(v,Tv) \\ & -\varphi(d(u,v), d(u,Tu), d(u,Tv), d(v,Tu), d(v,Tv)) \\ \leq \ & (\alpha_1 + \alpha_3 + \alpha_4) d(u,v) - \varphi(d(u,v), 0, d(u,v), d(v,u), 0). \end{aligned} \tag{2.11}$$

So, $\varphi(d(u,v), 0, d(u,v), d(v,u), 0) \leq (\alpha_1 + \alpha_3 + \alpha_4 - 1) d(u,v) \leq 0$. Therefore, $u = v$.

2. Suppose that $u$ and $v$ are not comparable. Choose an element $w \in X$ which is comparable with both $u$ and $v$. Then, since $T$ is dominating, $u = T^n u$ is comparable with $T^n w$ for each $n$.

Therefore, condition 1.1 gives us

$$\begin{aligned}
&d(u,T^n w)\\
&= d(T^n u,T^n w)\\
&= d(TT^{n-1}u,TT^{n-1}w)\\
&\le \alpha_1 d(T^{n-1}u,T^{n-1}w)+\alpha_2 d(T^{n-1}u,TT^{n-1}u)+\alpha_3 d(T^{n-1}u,TT^{n-1}w)\\
&+\alpha_4 d(T^{n-1}w,TT^{n-1}u)+\alpha_5 d(T^{n-1}w,TT^{n-1}w)\\
&-\varphi(d(T^{n-1}u,T^{n-1}w),d(T^{n-1}u,TT^{n-1}u),d(T^{n-1}u,TT^{n-1}w),\\
&d(T^{n-1}w,TT^{n-1}u),d(T^{n-1}w,TT^{n-1}w))\\
&= \alpha_1 d(u,T^{n-1}w)+\alpha_2 d(u,u)+\alpha_3 d(u,T^n w)\\
&+\alpha_4 d(T^{n-1}w,u)+\alpha_5 d(T^{n-1}w,T^n w)\\
&-\varphi(d(u,T^{n-1}w),d(u,u),d(u,T^n w),d(T^{n-1}w,u),d(T^{n-1}w,T^n w))\\
&= \alpha_1 d(u,T^{n-1}w)+\alpha_3 d(u,T^n w)\\
&+\alpha_4 d(T^{n-1}w,u)+\alpha_5 d(T^{n-1}w,T^n w)\\
&-\varphi(d(u,T^{n-1}w),0,d(u,T^n w),d(T^{n-1}w,u),d(T^{n-1}w,T^n w))\\
&\le \alpha_1 d(u,T^{n-1}w)+\alpha_3 d(u,T^n w)\\
&+\alpha_4 d(T^{n-1}w,u)+\alpha_5 d(T^{n-1}w,u)+\alpha_5 d(u,,T^n w).
\end{aligned} \tag{2.12}$$

Hence,

$$(1-\alpha_3-\alpha_5)d(u,T^n w)\le(\alpha_1+\alpha_4+\alpha_5)d(T^{n-1}w,u),$$

or, equivalently,

$$d(u,T^n w)\le\frac{(\alpha_1+\alpha_4+\alpha_5)}{(1-\alpha_3-\alpha_5)}d(u,T^{n-1}w)\le d(u,T^{n-1}w).$$

Note that, as $\alpha_2\ge\alpha_5$, we have

$$\frac{(\alpha_1+\alpha_4+\alpha_5)}{(1-\alpha_3-\alpha_5)}\le 1.$$

This proves that the nonnegative decreasing sequence $d(u,T^n w)$ is convergent. Let

$$\lim_{n\to\infty} d(u,T^n w)=r.$$

As we saw in Theorem 2.1, we can prove that $d(T^{n-1}w,T^n w)\to 0$ when $n\to\infty$ (Remark 2.8.).

Letting $n \to \infty$ in 2.12, from the continuity of $\varphi$ we obtain

$$r \leq (\alpha_1 + \alpha_3 + \alpha_4)r - \varphi(r, 0, r, r, 0).$$

This gives us that $\varphi(r, 0, r, r, 0) = 0$ and, by our assumption about $\varphi$, $r = 0$. Consequently, $\lim_{n\to\infty} d(u, T^n w) = 0$. Analogously, it can be proved that $\lim_{n\to\infty} d(v, T^n w) = 0$. Finally, since the limit is unique, we obtain that $u = v$. This finishes the proof. □

In what follows, we will denote the set of all fixed points of a self-mapping T by $F(T)$, *i.e.*, $F(T) = \{z \in X : Tz = z\}$.

It is well-known that, if u is a fixed point of $T$, then u is also a fixed point of $T^n$ for every $n \in \mathbb{N}$. However, the converse is not true. For example, consider, $X = \mathbb{R}$ and T defined by $T(x) = 1 - x$. Then $T$ has a unique fixed point at $\frac{1}{2}$, but for every $n > 1$, $T^n(x) = x$ and every $x \in \mathbb{R}$ is a fixed point of $T^n$. If a map $T$ satisfies $F(T) = F(T^n)$ for each $n \in \mathbb{N}$, then we say that $T$ has property $P$, or, we say that $T$ has no periodic points.

**Theorem 2.7.** *Let $T$ be a mapping satisfying all the conditions of Theorem 2.1. Then $T$ has the property P.*

*Proof.* We only have to prove that $F(T^n) \subset F(T)$.

Let $u \in F(T^n)$ for some fixed $n > 1$. In order to prove that $u \in F(f)$, we will show that $u = f(u)$. We suppose that $u \neq f(u)$, *i.e.*, $d(u, Tu) > 0$. Hence, we have

$$\begin{aligned} d(u,Tu) =& d(T^n u, Tu) \\ =& d(TT^{n-1}u, Tu) \\ \leq& \alpha_1 d(T^{n-1}u, u) + \alpha_2 d(T^{n-1}u, T^n u) + \alpha_3 d(T^{n-1}u, Tu) \\ & + \alpha_4 d(u, T^n u) + \alpha_5 d(u, Tu) \\ & - \varphi(d(T^{n-1}u, u), d(T^{n-1}u, T^n u), d(T^{n-1}u, Tu), d(u, T^n u), d(u, Tu)) \\ & \leq \alpha_1 d(T^{n-1}u, u) + \alpha_2 d(T^{n-1}u, T^n u) + \alpha_3 d(T^{n-1}u, Tu) \\ & + \alpha_4 d(u, T^n u) + \alpha_5 d(u, Tu). \end{aligned} \tag{2.13}$$

Therefore,

$$\begin{aligned} d(u,Tu) \quad & \leq \alpha_1 d(T^{n-1}u, T^n u) + \alpha_2 d(T^{n-1}u, T^n u) + \alpha_3 d(T^{n-1}u, T^n u) \\ & + \alpha_3 d(u, Tu) + \alpha_5 d(u, Tu). \end{aligned} \tag{2.14}$$

Equivalently,

$$(1-\alpha_3-\alpha_5)d(u,Tu) \leq (\alpha_1+\alpha_2+\alpha_3)d(T^{n-1}u,T^n u). \tag{2.15}$$

So, we have

$$d(u,Tu) \leq \frac{(\alpha_1+\alpha_2+\alpha_3)}{(\alpha_1+\alpha_2+\alpha_4)} d(T^{n-1}u,T^n u) \leq d(T^{n-1}u,T^n u).$$

As $d(T^{n-1}u,T^n u) \to 0$ (Remark 2.8) and we suppose that $d(u,Tu) > 0$, we have a contradiction. □

We conclude this paper with two corollaries of Theorems 2.1 and 2.4.

**Corollary 2.8.** *Let $T$ satisfies the conditions of Theorems 2.1 and 2.4, except that condition 1.1 is replaced by the following:*

*There exists a positive Lebesque integrable function $\phi$ on $\mathbb{R}$ such that $\int_0^\varepsilon \phi(t)dt > 0$ for each $\varepsilon > 0$ and*

$$\begin{aligned} d(Tx,Ty) \ &\leq \alpha_1 d(x,y)+\alpha_2 d(x,Tx)+\alpha_3 d(x,Ty)+\alpha_4 d(y,Tx)+\alpha_5 d(y,Ty) \\ &\quad - \int_0^{\varphi(d(x,y),d(x,Tx),d(x,Ty),d(y,Tx),d(y,Ty))} \phi(t)dt. \end{aligned} \tag{2.16}$$

*Then $T$ has a fixed point.*

*Proof.* Consider the function $\Phi(x) = \int_0^x \phi(t)dt$. Then 2.16 will be

$$\begin{aligned} d(Tx,Ty) \ &\leq \alpha_1 d(x,y)+\alpha_2 d(x,Tx)+\alpha_3 d(x,Ty)+\alpha_4 d(y,Tx)+\alpha_5 d(y,Ty) \\ &\quad -\Phi(\varphi(d(x,y),d(x,Tx),d(x,Ty),d(y,Tx),d(y,Ty))). \end{aligned} \tag{2.17}$$

Taking $\Psi = \Phi o \varphi$ and applying Theorems 2.1 and 2.4, we obtain the proof (it is easy to verify that $\Psi$ is a continuous function such that $\Psi(x,y,z,t,u) = 0$ if and only if $x=y=z=t=u=0$.). □

**Corollary 2.9.** *If in the above corollary, 2.16 be replaced by the following:*

$$\begin{aligned} &d(Tx,Ty) \\ &\leq \alpha_1 d(x,y)+\alpha_2 d(x,Tx)+\alpha_3 d(x,Ty)+\alpha_4 d(y,Tx)+\alpha_5 d(y,Ty) \\ &-\varphi\Bigg( \int_0^{d(x,y)} \phi(t)dt, \int_0^{d(x,Tx)} \phi(t)dt, \int_0^{d(x,Ty)} \phi(t)dt, \\ &\int_0^{d(y,Tx)} \phi(t)dt, \int_0^{d(y,Ty)} \phi(t)dt \Bigg), \end{aligned} \tag{2.18}$$

*then the result of corollary 2.8 also holds.*

*Proof.* Consider the function $\Phi(x) = \int_0^x \phi(t)dt$. Then 2.18 will be

$$\begin{aligned} d(Tx, Ty) &\leq \alpha_1 d(x,y) + \alpha_2 d(x,Tx) + \alpha_3 d(x,Ty) + \alpha_4 d(y,Tx) + \alpha_5 d(y,Ty) \\ &-\varphi(\Phi(d(x,Tx)), \Phi(d(x,Ty)), \Phi(d(y,Tx)), \Phi(d(y,Ty))). \end{aligned} \tag{2.19}$$

Taking

$$\Psi(x,y,z,t,u) = \varphi(\Phi(x), \Phi(y), \Phi(z), \Phi(t), \Phi(u))$$

and applying Theorems 2.1 and 2.4, we obtain the proof (it is obvious that $\Psi$ is a continuous function such that $\Psi(x,y,z,t,u) = 0$ if and only if $x = y = z = t = u = 0$.). □

**Example 2.10.** Let $X = \{(0,-1), (0,0), (0,1)\} \subset \mathbb{R}^2$ with the order $\preceq$ defined as:

$$(x_1, y_1) \preceq (x_2, y_2) \Leftrightarrow x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

Let $d$ be given as

$$d(x,y) = \max\{|x_1 - x_2|, |y_1 - y_2|\},$$

where $x = (x_1, y_1)$ and $y = (x_2, y_2)$. $(X, d)$ is, obviously, a complete metric space.

Let $T : X \to X$ be defined as follows:

$$T((0,-1)) = (0,0) \text{ and } T((0,0)) = T((0,1)) = (0,1),$$

and let $\varphi : [0,\infty)^5 \to [0,\infty)$ be the following:

$$\varphi(x,y,z,t,u) = \frac{\alpha_1 x + \alpha_2 y + \alpha_3 z + \alpha_4 t + \alpha_5 u}{4},$$

where $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{16} = \alpha_4$, $\alpha_3 = \frac{1}{2}$, and $\alpha_5 = \frac{1}{8}$.

By a careful computation it is easy to see that all conditions that appear in Theorem 2.6 hold. Finally, this Theorem guarantees the existence of a unique fixed point for $T$ (*i.e.*, the point $(0,0)$).

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