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CONTINUED FRACTION FOR TRIBONACCI RATIO

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Abstract: We study generalized continued fractions for the expression of tribonacci ratios $\frac{T_{n+k}}{T_n}$ when n is large, and find nth convergents of the continued fraction explicitly for each n.

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1. Introduction

The tribonacci sequence $\{T_n\}$ is a generalized fibonacci sequence defined by $T_n = T_{n-1}+T_{n-2}+T_{n-3}$ with $T_0 = 0$ and $T_1 = T_2 = 1$, so $\{T_n\} = \{0, 1, 1, 3, 4, 7, 13, \cdots\}$. It is well known that the fibonacci ratio $\frac{F_{n+1}}{F_n}$ with large n satisfies the fibonacci polynomial $x^2 = x + 1$. Due to continued fraction (CF for short) expansion for quadratic polynomial, the ratio is exhibited by $\langle 1; 1, 1, \cdots \rangle$. Similar to this, the tribonacci ratio shows

$$\frac{T_{n+1}}{T_n} = \frac{T_n + T_{n-1} + T_{n-2}}{T_n} = 1 + \frac{1}{\frac{T_n}{T_{n-1}}} + \frac{1}{\frac{T_n}{T_{n-1}}\frac{T_{n-1}}{T_{n-2}}},$$

so when n is large, $\frac{T_{n+1}}{T_n}$ holds the tribonacci polynomial $x^3 = x^2 + x + 1$.

The CF has been used to solve many problems not only in mathematics but

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in engineering, physics and computer field, since it provides good approximations of roots of quadratic polynomials. However real roots of cubic polynomials may not be naturally explained by means of classical CF. A question of generalization of CF to higher degree polynomial has been raised by Hermite in 1850 [7]. However, as is mentioned in [6] the CF expression even for cubic polynomial has not been studied extensively. Among few researches, [4] and [5] made use of bifurcating CF for cubic polynomial. It was called as a ternary CF in [3].

This work is devoted to investigating tribonacci ratios with respect to CF expressions. We will study generalized CFs for the expression of $\frac{T_{n+k}}{T_n}$ when n is large, and also find nth convergent of CF explicitly for each n.

2. Continued Fraction for Cubic Polynomial

We shall denote a finite CF

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{\ddots + \frac{b_n}{a_n}}} = a_0 + \frac{b_1}{a_1+}\frac{b_2}{a_2+}\frac{}{\cdots+}\frac{b_n}{a_n}$$

by a double CF $\langle a_0; \frac{b_1}{a_1}, \frac{b_2}{a_2}, \cdots, \frac{b_n}{a_n}\rangle$. If all $b_i = 1$ then $\langle a_0; \frac{1}{a_1}, \cdots, \frac{1}{a_n}\rangle$ represent a simple CF $\langle a_0; a_1, \cdots, a_n\rangle$. Moreover an infinite double CF is defined by $\langle a_0; \frac{b_1}{a_1}, \frac{b_2}{a_2}, \cdots\rangle = \lim_{n\to\infty} \langle a_0; \frac{b_1}{a_1}, \cdots, \frac{b_n}{a_n}\rangle$

Let α be a real root of $x^3 = x^2 + x + 1$. Then

$$\alpha - 1 = \frac{\alpha^2(\alpha-1)}{\alpha^2} = \frac{\alpha+1}{\alpha^2} = \frac{1}{(\alpha-1) + \frac{1}{\alpha+1}}$$

shows a simple CF $\alpha = \langle 1; \alpha-1, \alpha+1\rangle$. And we also have

$$\alpha = 1 + \frac{1}{\frac{\alpha^2}{\alpha+1}} = 1 + \frac{1}{\frac{\alpha(\alpha+1)-\alpha}{\alpha+1}} = 1 + \frac{1}{\alpha - \frac{1}{1+\frac{1}{\alpha}}} = \langle 1; \alpha, -1, -\alpha\rangle.$$

On the other hand since $\alpha + 1 = 2 + \frac{1}{\alpha - \frac{1}{1+\frac{1}{\alpha}}}$, it follows that

$$\alpha = 1 + \frac{1}{(\alpha-1) + \frac{1}{\alpha+1}} = 1 + \frac{1}{\frac{1}{\alpha - \frac{1}{1+\frac{1}{\alpha}}} + \frac{1}{2 + \frac{1}{\alpha - \frac{1}{1+\frac{1}{\alpha}}}}}.$$

So if we let $\frac{1}{\alpha-\frac{1}{1+\frac{1}{\alpha}}}=\beta$ then $\alpha=1+\frac{1}{\beta+}\frac{1}{\beta+2}=\langle 1;\beta,\beta+2\rangle$ where $\beta=\frac{1}{\alpha+}\frac{1}{-1+}\frac{1}{-\alpha}=\langle 0;\alpha,-1,-\alpha\rangle$.

Besides these, there could be various simple CF expressions for the root of $x^3=x^2+x+1$. But they do not seem like satisfactory forms to find real root explicitly. We now shall deal with double CF expansion.

Theorem 1. *For a real root α of $x^3=x^2+x+1$, a double CF expansion is $\alpha=\langle 1;\frac{\beta}{1},\frac{\beta}{1},\frac{\beta}{1},\ldots\rangle$ with $\beta=\langle 1;\frac{1}{1},\frac{\beta}{1},\frac{\beta}{1},\ldots\rangle$*

Proof. The polynomial $x^3=x^2+x+1$ yields

$$\alpha=1+\frac{\alpha+1}{\alpha^2}=1+\frac{\alpha+1}{\alpha}\,\frac{1}{\alpha}.$$

Let $\beta=\frac{\alpha+1}{\alpha}$. Then

$$\alpha=1+\frac{\beta}{\alpha}=1+\frac{\beta}{1+\frac{\beta}{\alpha}}=1+\frac{\beta}{1+\frac{\beta}{1+\frac{\beta}{\alpha}}}=\cdots=\langle 1;\frac{\beta}{1},\frac{\beta}{1},\frac{\beta}{1},\cdots\rangle$$

while $\beta=1+\frac{1}{\alpha}=1+\frac{1}{1+\frac{\beta}{\alpha}}=\langle 1;\frac{1}{1},\frac{\beta}{\alpha}\rangle=\langle 1;\frac{1}{1},\frac{\beta}{1},\frac{\beta}{1},\cdots\rangle$. $\square$

Let a_i, b_i $(i\geq 0)$ be integers and α_i, β_i $(i\geq 1)$ be nonzero integers. Suppose that a real number $\alpha=\alpha_0$ bifurcates into a_0 and $\frac{\beta_1}{\alpha_1}$, i.e., $\alpha_0=a_0+\frac{\beta_1}{\alpha_1}$. And assume that $\alpha_i=a_i+\frac{\beta_{i+1}}{\alpha_{i+1}}$ and $\beta_i=b_i+\frac{1}{\alpha_{i+1}}$ for all $i\geq 1$. Then the first few level of bifurcations of α and β are as follows.

$$\alpha=a_0+\frac{\beta_1}{\alpha_1}=a_0+\frac{b_1+\frac{1}{\alpha_2}}{a_1+\frac{\beta_2}{\alpha_2}}=a_0+\frac{b_1+\frac{1}{a_2+\frac{\beta_3}{\alpha_3}}}{a_1+\frac{b_2+\frac{1}{\alpha_3}}{a_2+\frac{\beta_3}{\alpha_3}}}=\cdots$$

$$\beta=\beta_0=b_0+\frac{1}{\alpha_1}=b_0+\frac{1}{a_1+\frac{\beta_2}{\alpha_2}}=b_0+\frac{1}{a_1+\frac{b_2+\frac{1}{\alpha_3}}{a_2+\frac{\beta_3}{\alpha_3}}}=\cdots \qquad (\ddagger)$$

The pair of objects representing α and β is called a bifurcating CF and denoted by $(\alpha,\beta)=[\{a_0,a_1,\cdots\},\{b_0,b_1,\cdots\}]$ in [1]. Hence the real root α of $x^3=x^2+x+1$ in Theorem 1 is expressed by $(\alpha,\beta)=[\{1,1,\cdots\},\{1,1,\cdots\}]$.

3. Continued Fraction for Tribonacci Ratio

Let $\alpha_i = a_i + \frac{\beta_{i+1}}{\alpha_{i+1}}$ and $\beta_i = b_i + \frac{1}{\alpha_{i+1}}$ for every $i > 0$. Then due to (‡), the ith bifurcation $\alpha^{[i]}$ of $\alpha = \alpha_0$ can be written by

$$\alpha = \alpha^{[i]} = a_0 + \frac{N_i(i)}{D_i(i)}$$

where $\alpha^{[1]} = a_0 + \frac{N_1(1)}{D_1(1)}$ with $N_1(1) = \beta_1$, $D_1(1) = \alpha_1$,
$\alpha^{[2]} = a_0 + \frac{N_2(2)}{D_2(2)}$ with $N_2(2) = b_1 + \frac{1}{\alpha_2}$, $D_2(2) = a_1 + \frac{\beta_2}{\alpha_2}$,
$\alpha^{[3]} = a_0 + \frac{N_3(3)}{D_3(3)}$ with $N_3(3) = b_1 + \frac{1}{a_2+}\frac{\beta_3}{\alpha_3}$, $D_3(3) = a_1 + \frac{b_2+\frac{1}{\alpha_3}}{a_2+}\frac{\beta_3}{\alpha_3}$,
and so on.

Now for $i \le j$, let $N_i(j)$ denote the same formula of $N_i(i)$ where all values a_s, b_s, α_s and β_s in $N_i(i)$ are replaced by $a_{s+(j-i)}$, $b_{s+(j-i)}$, $\alpha_{s+(j-i)}$ and $\beta_{s+(j-i)}$ in $N_i(j)$. We also define $D_i(j)$ analogously from $D_i(i)$. For instance $N_3(6) = b_4 + \frac{1}{a_5+}\frac{\beta_6}{\alpha_6}$ and $D_3(5) = a_3 + \frac{b_4+\frac{1}{\alpha_5}}{a_4+}\frac{\beta_5}{\alpha_5}$.

Theorem 2. *With the same context above, the ith bifurcation of $\alpha = \alpha_0$ forms $\alpha^{[i]} = a_0 + \frac{N_i(i)}{D_i(i)}$, where $N_i(i)$, $D_i(i)$ are written by double CF that*

$$N_1(1) = N_2(2) = \langle b_1; D_1(2)\rangle, \;\; D_1(1) = D_2(2) = \langle a_1; \frac{N_1(2)}{D_1(2)}\rangle$$

and for all $i > 2$,

$$N_i(i) = \langle b_1; \frac{1}{a_2}, \frac{N_{i-2}(i)}{a_3}, \cdots, \frac{N_2(i)}{a_{i-1}}, \frac{N_1(i)}{D_1(i)}\rangle = \langle b_1; D_{i-1}(i)\rangle,$$

$$D_i(i) = \langle a_1; \frac{N_{i-1}(i)}{a_2}, \frac{N_{i-2}(i)}{a_3}, \cdots, \frac{N_2(i)}{a_{i-1}}, \frac{N_1(i)}{D_1(i)}\rangle = \langle a_1; \frac{N_{i-1}(i)}{D_{i-1}(i)}\rangle.$$

Proof. The $N_i(i)$, $D_i(i)$ in $\alpha^{[i]}$ follow inductively from (‡):
(i) $N_1(1) = \beta_1$ and $D_1(1) = \alpha_1$.
(ii) $N_2(2) = b_1 + \frac{1}{\alpha_2} = b_1 + \frac{1}{D_1(2)} = \langle b_1; D_1(2)\rangle$,

$$D_2(2) = a_1 + \frac{\beta_2}{\alpha_2} = \langle a_1; \frac{N_1(2)}{D_1(2)}\rangle.$$

(iii)
$$N_3(3) = b_1 + \frac{1}{a_2+}\frac{\beta_3}{\alpha_3} = b_1 + \frac{1}{a_2+}\frac{N_1(3)}{D_1(3)} = \langle b_1; \frac{1}{a_2}, \frac{N_1(3)}{D_1(3)}\rangle$$
$$= b_1 + \frac{1}{D_2(3)} = \langle b_1; D_2(3)\rangle$$
$$D_3(3) = a_1 + \frac{b_2+\frac{1}{\alpha_3}}{a_2+}\frac{\beta_3}{\alpha_3} = a_1 + \frac{N_2(3)}{a_2+}\frac{N_1(3)}{D_1(3)} = \langle a_1; \frac{N_2(3)}{a_2}, \frac{N_1(3)}{D_1(3)}\rangle$$
$$= a_1 + \frac{N_2(3)}{D_2(3)} = \langle a_1; \frac{N_2(3)}{D_2(3)}\rangle.$$

(iv) $N_4(4) = b_1 + \frac{1}{a_2+}\frac{b_3+\frac{1}{\alpha_4}}{a_3+}\frac{\beta_4}{\alpha_4} = b_1 + \frac{1}{a_2+}\frac{N_2(4)}{a_3+}\frac{N_1(4)}{D_1(4)}$

$$= \langle b_1; \frac{1}{a_2}, \frac{N_2(4)}{a_3}, \frac{N_1(4)}{D_1(4)}\rangle = b_1 + \frac{1}{D_3(4)} = \langle b_1; D_3(4)\rangle$$

$$D_4(4) = a_1 + \frac{b_2+\frac{1}{a_3+\frac{\beta_4}{\alpha_4}}}{a_2+}\frac{b_3+\frac{1}{\alpha_4}}{a_3+}\frac{\beta_4}{\alpha_4} = a_1 + \frac{N_3(4)}{a_2+}\frac{N_2(4)}{a_3+}\frac{N_1(4)}{D_1(4)}$$

$$= \langle b_1; \frac{N_3(4)}{a_2}, \frac{N_2(4)}{a_3}, \frac{N_1(4)}{D_1(4)}\rangle = a_1 + \frac{N_3(4)}{D_3(4)} = \langle b_1; \frac{N_3(4)}{D_3(4)}\rangle.$$

Continuing this process, $N_i(i)$ and $D_i(i)$ follow that

$$b_1 + \frac{1}{a_2+}\frac{N_{i-2}(i)}{a_3+}\cdots\frac{N_2(i)}{a_{i-1}+}\frac{N_1(i)}{D_1(i)} = \langle b_1; \frac{1}{a_2}, \frac{N_{i-2}(i)}{a_3}, \cdots, \frac{N_2(i)}{a_{i-1}}, \frac{N_1(i)}{D_1(i)}\rangle,$$

$$a_1 + \frac{N_{i-1}(i)}{a_2+}\frac{}{a_3+}\cdots\frac{N_2(i)}{a_{i-1}+}\frac{N_1(i)}{D_1(i)} = \langle a_1; \frac{N_{i-1}(i)}{a_2}, \cdots, \frac{N_2(i)}{a_{i-1}}, \frac{N_1(i)}{D_1(i)}\rangle$$

respectively. □

Moreover we define $\hat{N}_i(i)$ and $\hat{D}_i(i)$ by the same $N_i(i)$ and $D_i(i)$ respectively, in which the last fractions $\frac{\beta_i}{\alpha_i}$ and $\frac{1}{\alpha_i}$ are omitted.

Theorem 3. *$\hat{N}_i(i)$ and $\hat{D}_i(i)$ satisfy the followings.*
(1) For $i = 1, 2$, $\hat{N}_i(i) = b_1$ and $\hat{D}_i(i) = a_1$.
(2) For $i > 2$, $\hat{N}_i(i) = \langle b_1; \hat{D}_{i-1}(i)\rangle$ and $\hat{D}_i(i) = \langle a_1; \frac{\hat{N}_{i-1}(i)}{\hat{D}_{i-1}(i)}\rangle$. In particular $\hat{N}_i(i) = \langle 1; \hat{D}_{i-1}(i-1)\rangle$ and $\hat{D}_i(i) = \langle 1; \frac{\hat{N}_{i-1}(i-1)}{\hat{D}_{i-1}(i-1)}\rangle$, if $a_i = b_i = 1$ for all i.
(3) $\lim\limits_{i\to\infty} \frac{\hat{N}_i(i)}{\hat{D}_i(i)} = \lim\limits_{i\to\infty} \frac{N_i(i)}{D_i(i)}$.

Proof. It is clear that $\hat{N}_1(1) = \hat{N}_2(2) = b_1$ and $\hat{D}_1(1) = \hat{D}_2(2) = a_1$. And $\hat{N}_3(3) = b_1 + \frac{1}{a_2} = \langle b_1; \hat{D}_2(3)\rangle$ and $\hat{D}_3(3) = a_1 + \frac{b_2}{a_2} = \langle a_1; \frac{\hat{N}_2(3)}{\hat{D}_2(3)}\rangle$. Similarly $\hat{N}_4(4) = b_1 + \frac{1}{a_2+\frac{b_3}{a_3}} = \langle b_1; \hat{D}_3(4)\rangle$ and $\hat{D}_4(4) = a_1 + \frac{b_2+\frac{1}{a_3}}{a_2+\frac{b_3}{a_3}} = \langle a_1; \frac{\hat{N}_3(4)}{\hat{D}_3(4)}\rangle$. So we have $\hat{N}_i(i) = \langle b_1; \hat{D}_{i-1}(i)\rangle$ and $\hat{D}_i(i) = \langle a_1; \frac{\hat{N}_{i-1}(i)}{\hat{D}_{i-1}(i)}\rangle$ for all $i > 2$.

Obviously $\hat{N}_2(2)$ and $\hat{D}_2(2)$ are the 1st convergents of $N_2(2)$ and $D_2(2)$ respectively. Moreover $\hat{N}_3(3) = \langle b_1; \frac{1}{a_2}\rangle$ and $\hat{D}_3(3) = \langle a_1; \frac{b_2}{a_2}\rangle = \langle a_1; \frac{\hat{N}_2(3)}{a_2}\rangle$ are the 2nd convergents of $N_3(3) = \langle b_1; \frac{1}{a_2}, \frac{\beta_3}{\alpha_3}\rangle$ and $D_3(3) = \langle a_1; \frac{N_2(3)}{a_2}, \frac{N_1(3)}{D_1(3)}\rangle$ respectively. Continuing this consideration we may say $\hat{N}_i(i)$ and $\hat{D}_i(i)$ are the $(i-1)$th convergents of $N_i(i)$ and $D_i(i)$ respectively. Hence it follows that $\lim_{i\to\infty}\frac{\hat{N}_i(i)}{\hat{D}_i(i)} = \lim_{i\to\infty}\frac{N_i(i)}{D_i(i)}$.

In particular if $a_i = b_i = 1$ then the bifurcation

$$\alpha = \alpha_0 = 1 + \frac{\beta_1}{\alpha_1} = 1 + \frac{1+\frac{1}{\alpha_2}}{1+\frac{\beta_2}{\alpha_2}} = 1 + \frac{1+\frac{1}{1+\frac{\beta_3}{\alpha_3}}}{1+\frac{1+\frac{1}{\alpha_3}}{1+\frac{\beta_3}{\alpha_3}}} = \cdots$$

implies that

$\hat{N}_2(2) = \hat{D}_2(2) = 1$ and $\hat{N}_3(3) = \hat{D}_3(3) = 1 + \frac{1}{1} = 2$,

$\hat{N}_4(4) = 1 + \frac{1}{1+\frac{1}{1}} = 1 + \frac{1}{\hat{D}_3(3)}$, $\hat{D}_4(4) = 1 + \frac{1+\frac{1}{1}}{1+\frac{1}{1}} = 1 + \frac{\hat{N}_3(3)}{\hat{D}_3(3)}$,

and so on. Therefore we have

$\hat{N}_i(i) = 1 + \frac{1}{\hat{D}_{i-1}(i-1)} = \langle 1; \hat{D}_{i-1}(i-1)\rangle$

$\hat{D}_i(i) = 1 + \frac{\hat{N}_{i-1}(i-1)}{\hat{D}_{i-1}(i-1)} = \langle 1; \frac{\hat{N}_{i-1}(i-1)}{\hat{D}_{i-1}(i-1)}\rangle$. $\square$

Theorem 4. *Let $\alpha = \lim\limits_{n\to\infty}\frac{T_{n+1}}{T_n}$. Then we have the followings.*

(1) The double CF α is $\alpha = \langle 1; \frac{\beta}{1}, \frac{\beta}{1}, \cdots\rangle$ with $\beta = \langle 1; \frac{1}{1}, \frac{\beta}{1}, \cdots\rangle$.

(2) The ith bifurcation is $\alpha^{[i]} = 1 + \frac{N_i(i)}{D_i(i)}$, where $N_1(1) = \beta$, $D_1(1) = \alpha$ and $N_i(i) = 1 + \frac{1}{D_{i-1}(i-1)}$, $D_i(i) = 1 + \frac{N_{i-1}(i-1)}{\hat{D}_{i-1}(i-1)}$ for all $i > 1$.

(3) $\lim\limits_{i\to\infty}\frac{N_i(i)}{D_i(i)} = \lim\limits_{i\to\infty}\frac{\hat{N}_i(i)}{\hat{D}_i(i)}$, and $\lim\limits_{n\to\infty}\frac{T_{n+1}}{T_n} = 1.8392\cdots$.

Proof. When n is large, $\alpha = \frac{T_{n+1}}{T_n}$ satisfies $x^3 = x^2 + x + 1$, so (1) is due to Theorem 1. Moreover by letting $\beta = \frac{\alpha+1}{\alpha}$,

$$\alpha = 1 + \frac{\beta}{\alpha} = 1 + \frac{\beta}{1+\frac{\beta}{\alpha}} = 1 + \frac{\beta}{1+\frac{\beta}{1+\frac{\beta}{\alpha}}} = \cdots$$

implies α bifurcates into 1 and $\frac{\beta}{\alpha}$, while β into 1 and $\frac{1}{\alpha}$. Thus

$$\alpha = 1 + \frac{\beta}{\alpha} = 1 + \frac{1+\frac{1}{\alpha}}{1+\frac{\beta}{\alpha}} = 1 + \frac{1+\frac{1}{1+\frac{\beta}{\alpha}}}{1+\frac{1+\frac{1}{\alpha}}{1+\frac{\beta}{\alpha}}} = \cdots$$

Now owing to Theorem 2 and 3, we have $\alpha^{[i]} = 1 + \frac{N_i(i)}{D_i(i)}$ where
$N_1(1) = \beta$, $D_1(1) = \alpha$

$N_2(2) = 1 + \frac{1}{\alpha} = 1 + \frac{1}{D_1(1)}$, $D_2(2) = 1 + \frac{\beta}{\alpha} = 1 + \frac{N_1(1)}{D_1(1)}$
$N_3(3) = 1 + \frac{1}{1+\frac{\beta}{\alpha}} = 1 + \frac{1}{D_2(2)}$, $D_3(3) = 1 + \frac{1+\frac{1}{\alpha}}{1+\frac{\beta}{\alpha}} = 1 + \frac{N_2(2)}{D_2(2)}$
$N_4(4) = 1 + \frac{1}{1+\frac{1+\frac{1}{\alpha}}{1+\frac{\beta}{\alpha}}} = 1 + \frac{1}{D_3(3)}$, $D_4(4) = 1 + \frac{N_3(3)}{D_3(3)}, \cdots$

thus $N_i(i) = 1 + \frac{1}{D_{i-1}(i-1)}$, $D_i(i) = 1 + \frac{N_{i-1}(i-1)}{D_{i-1}(i-1)}$ for all $i > 1$. Moreover for $\hat{N}_i(i)$ and $\hat{D}_i(i)$, we have $\hat{N}_2(2) = 1$, $\hat{D}_2(2) = 1$, and
$\hat{N}_3(3) = 1 + \frac{1}{1} = 1 + \frac{1}{\hat{D}_2(2)}$, $\hat{D}_3(3) = 1 + \frac{1}{1} = 1 + \frac{\hat{N}_2(2)}{\hat{D}_2(2)}$,
$\hat{N}_4(4) = 1 + \frac{1}{1+\frac{1}{1}} = 1 + \frac{1}{\hat{D}_3(3)}$, $\hat{D}_4(4) = 1 + \frac{1}{1+\frac{1}{1}} = 1 + \frac{\hat{N}_3(3)}{\hat{D}_3(3)}$,
so $\hat{N}_i(i) = 1+\frac{1}{\hat{D}_{i-1}(i-1)}$ and $\hat{D}_i(i) = 1+\frac{\hat{N}_{i-1}(i-1)}{\hat{D}_{i-1}(i-1)}$ for all $i > 2$. So by computing the first some $\hat{N}_i(i)$ and $\hat{D}_i(i)$ explicitly, we have

i	2	3	4	5	6	7	8	9	10	11
$\hat{N}_i(i)$	1	2	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{11}{7}$	$\frac{20}{13}$	$\frac{37}{24}$	$\frac{68}{44}$	$\frac{125}{81}$	$\frac{230}{149}$
$\hat{D}_i(i)$	1	2	2	$\frac{7}{4}$	$\frac{13}{7}$	$\frac{24}{13}$	$\frac{44}{24}$	$\frac{81}{44}$	$\frac{149}{81}$	$\frac{274}{149}$

So the table shows $\frac{\hat{N}_i(i)}{\hat{D}_i(i)}$ $(2 \le i \le 13)$ approximates to $0.8392\cdots$.

i	$\hat{N}_i(i)/\hat{D}_i(i)$	i	$\hat{N}_i(i)/\hat{D}_i(i)$	i	$\hat{N}_i(i)/\hat{D}_i(i)$
2	1	3	$\frac{2}{2} = 1$	4	$\frac{3}{4} = 0.75$
5	$\frac{6}{7} = 0.85714$	6	$\frac{11}{13} = 0.84615$	7	$\frac{20}{24} = 0.83333$
8	$\frac{37}{44} = 0.84090$	9	$\frac{68}{81} = 0.83950$	10	$\frac{125}{149} = 0.83892$
11	$\frac{230}{274} = 0.83941$	12	$\frac{423}{504} = 0.83928$	13	$\frac{778}{927} = 0.83926$

Hence we have $\lim_{n\to\infty} \frac{T_{n+1}}{T_n} = \alpha = \lim_{i\to\infty} \alpha^{[i]} = 1 + \lim_{i\to\infty} \frac{\hat{N}_i(i)}{\hat{D}_i(i)} = 1.8392\cdots$ by Theorem 3. $\square$

Compare it to the real root $\frac{1}{3}(1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}})$ of $x^3 = x^2 + x + 1$ which approximates to $1.8392\cdots$.

Corollary 5. *Let* $\frac{\hat{N}_n(n)}{\hat{D}_n(n)} = \frac{p_n}{q_n}$ *with* $p_n, q_n \in \mathbb{N}$. *Then* $\{q_n\}$ *is the (original) tribonacci sequence having initials* $1, 2, 4$, *while* $\{p_n\}$ *is a tribonacci type sequence having initials* $1, 2, 3$. *In particular,* $1 + \frac{\hat{N}_n(n)}{\hat{D}_n(n)} = \frac{T_{n+1}}{T_n}$ *for every* n.

Proof. It is easy to see that $\{p_n + q_n\}$ is a tribonacci sequence with initials $2, 4, 7$. Thus $\{p_n + q_n\} = \{T_{n+1}\}$ and $1 + \frac{\hat{N}_n(n)}{\hat{D}_n(n)} = 1 + \frac{p_n}{q_n} = \frac{T_{n+1}}{T_n}$. $\square$

For instance $1+\frac{\hat{N}_{16}(16)}{\hat{D}_{16}(16)}=1+\frac{4841}{5768}=\frac{10609}{5768}=\frac{T_{17}}{T_{16}}$.

4. Continued Fraction for k Step Tribonacci Ratio

According to [8], ratios that appear in phyllotaxis are fractions of fibonacci numbers spaced 2 steps such as 8/3, 13/5, 21/8, $\cdots$ rather than consecutive terms. The 2 step fibonacci ratio is expressed by CF that $\frac{F_{n+2}}{F_n}=1+\frac{F_{n+1}}{F_n}=1+\langle 1;\underbrace{1,\cdots,1}_{n-1}\rangle=\langle 2;\underbrace{1,\cdots,1}_{n-1}\rangle$. However when n is large, any k step fibonacci ratio $\frac{F_{n+k}}{F_n}=\frac{F_{n+k}}{F_{n+k-1}}\cdots\frac{F_{n+1}}{F_n}=\langle 1;1,\cdots\rangle^k$ may not be simply represented by CF since arithmetic algorithms of CFs are not trivial. In this section we shall study k step tribonacci ratios $\frac{T_{n+k}}{T_n}$ with CF expansions. Let us begin with $k=4$.

Theorem 6. *For 4 step tribonacci ratio* $\frac{T_{n+4}}{T_n}$, $\lim_{n\to\infty}\frac{T_{n+4}}{T_n}=\langle 11;\frac{\beta}{11},\frac{\beta}{11},\cdots\rangle$ *with* $\beta=\langle 5;\frac{1}{11},\frac{\beta}{11},\cdots\rangle$.

Proof. T_n satisfies the 4 step recurrence rule $T_{n+4}=11T_n+5T_{n-4}+T_{n-8}$ (see [2]), hence we have

$$\frac{T_{n+4}}{T_n}=11+\frac{5}{\frac{T_n}{T_{n-4}}}+\frac{1}{\frac{T_n}{T_{n-4}}\frac{T_{n-4}}{T_{n-8}}}.$$

Then $\alpha=\lim_{n\to\infty}\frac{T_{n+4}}{T_n}$ is a root of $x^3=11x^2+5x+1$. So we have $\alpha=11+\frac{5\alpha+1}{\alpha}\ \frac{1}{\alpha}=11+\frac{\beta}{\alpha}$ by letting $\beta=\frac{5\alpha+1}{\alpha}$. Hence α bifurcates into 11 and $\frac{\beta}{\alpha}$ that

$$\alpha=11+\frac{\beta}{11+\frac{\beta}{\alpha}}=11+\frac{\beta}{11+\frac{\beta}{11+\frac{\beta}{\alpha}}}=\cdots=\langle 11;\frac{\beta}{11},\frac{\beta}{11},\cdots\rangle$$

while β bifurcates into 5 and $\frac{1}{\alpha}$ that

$$\beta=5+\frac{1}{11+\frac{y}{\alpha}}=5+\frac{1}{11+\frac{\beta}{11+\frac{\beta}{\alpha}}}=\cdots=\langle 5;\frac{1}{11},\frac{\beta}{11},\cdots\rangle.$$

□

We now shall compute $\frac{T_{n+4}}{T_n}$ explicitly.

Theorem 7. *Let $n = 4t + r$ ($t, r \in \mathbb{N}$, $1 \le r \le 4$) and $\alpha_t = \frac{T_{n+4}}{T_n}$ be the 4 step tribonacci ratio. Then we have the followings.*
(1) The double CF expansion of α_t is $\alpha_t = \langle 11; \frac{\beta_{t-1}}{11}, \cdots, \frac{\beta_2}{11}, \frac{\beta_1}{\alpha_1}\rangle$ for $t \ge 2$, where $\beta_i = \langle 5; \frac{1}{11}, \frac{\beta_{i-2}}{11}, \cdots, \frac{\beta_2}{11}, \frac{\beta_1}{\alpha_1}\rangle$ for $3 \le i < t$. In particular $\alpha_0 = \frac{T_{4+r}}{T_r}$, $\alpha_1 = \frac{T_{8+r}}{T_{4+r}}$ and $\beta_i = \langle 5; \alpha_{i-1}\rangle$ if $i = 1, 2$.
(2) The ith bifurcation of α_t is $\alpha_t{}^{[i]} = 11 + \frac{N_i(i)}{D_i(i)}$, where double CFs of $N_i(i)$ and $D_i(i)$ are $N_1(1) = \beta_{t-1}$ and $D_1(1) = \alpha_{t-1}$ and

$$\begin{cases} N_2(2) = \langle 5; D_1(2)\rangle \\ D_2(2) = \langle 11; \frac{N_1(2)}{D_1(2)}\rangle \end{cases}, \quad \begin{cases} N_{\mathsf{i}}(i) = \langle 5; \frac{1}{11}, \frac{N_{\mathsf{i}-2}(i)}{11}, \cdots, \frac{N_2(i)}{11}, \frac{N_1(i)}{D_1(i)}\rangle \\ D_{\mathsf{i}}(i) = \langle 11; \frac{N_{\mathsf{i}-1}(i)}{11}, \cdots, \frac{N_2(i)}{11}, \frac{N_1(i)}{D_1(i)}\rangle \end{cases}$$

for $3 \le i < t$. Here $N_i(j)$ [resp. $D_i(j)$] for $i \le j$ is defined from $N_i(i)$ [resp. $D_i(i)$] where α_{t-i} and β_{t-i} in $N_i(i)$ are replaced by α_{t-j} and β_{t-j} in $N_i(j)$.
(3) Moreover $\lim_{n\to\infty} \frac{T_{n+4}}{T_n} = 11.44452\cdots$.

Proof. The $T_{4(t+1)+r} = 11T_{4t+r} + 5T_{4(t-1)+r} + T_{4(t-2)+r}$ shows

$$\begin{aligned} \alpha_t = \frac{T_{4(t+1)+r}}{T_{4t+r}} &= 11 + \frac{5}{\frac{T_{4t+r}}{T_{4(t-1)+r}}} + \frac{1}{\frac{T_{4t+r}}{T_{4(t-1)+r}} \frac{T_{4(t-1)+r}}{T_{4(t-2)+r}}} \\ &= 11 + \frac{5}{\alpha_{t-1}} + \frac{1}{\alpha_{t-1}\alpha_{t-2}} = 11 + (5 + \frac{1}{\alpha_{t-2}})\frac{1}{\alpha_{t-1}}, \end{aligned}$$

so we have $\alpha_t = 11 + \frac{\beta_{t-1}}{\alpha_{t-1}}$ with $\beta_{t-1} = 5 + \frac{1}{\alpha_{t-2}}$. Thus

$$\alpha_t = 11 + \cfrac{\beta_{t-1}}{11 + \cfrac{\beta_{t-2}}{11 + \frac{\beta_{t-3}}{\alpha_{t-3}}}} = \cdots = \langle 11; \frac{\beta_{t-1}}{11}, \frac{\beta_{t-2}}{11}, \cdots, \frac{\beta_2}{11}, \frac{\beta_1}{\alpha_1}\rangle$$

for $t \ge 2$, while for $3 \le i \le t-1$,

$$\beta_i = 5 + \cfrac{1}{11 + \cfrac{\beta_{i-2}}{11 + \frac{\beta_{i-3}}{\alpha_{i-3}}}} = \cdots = \langle 5; \frac{1}{11}, \frac{\beta_{i-2}}{11}, \cdots, \frac{\beta_2}{11}, \frac{\beta_1}{\alpha_1}\rangle.$$

In particular $\alpha_0 = \frac{T_{4+r}}{T_r}$, $\alpha_1 = \frac{T_{4\cdot 2+r}}{T_{4+r}}$ and $\beta_2 = 5 + \frac{1}{\alpha_1} = \langle 5; \alpha_1\rangle$, $\beta_1 = 5 + \frac{1}{\alpha_0} =$

$\langle 5; \alpha_0 \rangle$. On the other hand the bifurcating

$$\alpha_t - 11 = \frac{\beta_{t-1}}{\alpha_{t-1}} = \frac{5 + \frac{1}{\alpha_{t-2}}}{11 + \frac{\beta_{t-2}}{\alpha_{t-2}}} = \frac{5 + \frac{1}{11 + \frac{\beta_{t-3}}{\alpha_{t-3}}}}{11 + \frac{5 + \frac{1}{\alpha_{t-3}}}{11 + \frac{\beta_{t-3}}{\alpha_{t-3}}}} = \cdots$$

shows the ith level bifurcation ${\alpha_t}^{[i]} = 11 + \frac{N_i(i)}{D_i(i)}$ as follows.
$N_1(1) = \beta_{t-1}$, $D_1(1) = \alpha_{t-1}$
$N_2(2) = \langle 5; \alpha_{t-2} \rangle = \langle 5; D_1(2) \rangle, \quad D_2(2) = \langle 11; \frac{\beta_{t-2}}{\alpha_{t-2}} \rangle = \langle 11; \frac{N_1(2)}{D_1(2)} \rangle$
$N_3(3) = \langle 5; \frac{1}{11}, \frac{\beta_{t-3}}{\alpha_{t-3}} \rangle = \langle 5; \frac{1}{11}, \frac{N_1(3)}{D_1(3)} \rangle$, $D_3(3) = \langle 11; \frac{N_2(3)}{11}, \frac{N_1(3)}{D_1(3)} \rangle$
$N_4(4) = \langle 5; \frac{1}{11}, \frac{N_2(4)}{11}, \frac{N_1(4)}{D_1(4)} \rangle, \quad D_4(4) = \langle 11; \frac{N_3(4)}{11}, \frac{N_2(4)}{11}, \frac{N_1(4)}{D_1(4)} \rangle$
and so on, where for $i \le j$, the $N_i(j)$ and $D_i(j)$ come from $N_i(i)$ and $D_i(i)$ in which α_{t-i}, β_{t-i} are substituted by α_{t-j}, β_{t-j}. Thus for all $3 \le i < t$, we have

$$N_i(i) = \langle 5; \frac{1}{11}, \frac{N_{i-2}(i)}{11}, \cdots, \frac{N_2(i)}{11}, \frac{N_1(i)}{D_1(i)} \rangle,$$

$$D_i(i) = \langle 11; \frac{N_{i-1}(i)}{11}, \cdots, \frac{N_2(i)}{11}, \frac{N_1(i)}{D_1(i)} \rangle.$$

Furthermore by considering $\hat{N}_i(i)$ and $\hat{D}_i(i)$ that are the $(i-1)$th convergent of $N_i(i)$ and $D_i(i)$ respectively, we have
$\hat{N}_2(2) = 5$, $\hat{D}_2(2) = 11$, so $\frac{\hat{N}_2(2)}{\hat{D}_2(2)} = \frac{5}{11} \approx 0.454545$,
$\hat{N}_3(3) = 5 + \frac{1}{11} = \frac{56}{11}$, $\hat{D}_3(3) = 11 + \frac{5}{11} = \frac{126}{11}$, so $\frac{\hat{N}_3(3)}{\hat{D}_3(3)} = \frac{56}{126} \approx 0.444444$,
$\hat{N}_4(4) = \frac{641}{126}$, $\hat{D}_4(4) = \frac{1442}{126}$, so $\frac{\hat{N}_4(4)}{\hat{D}_4(4)} = \frac{641}{1442} \approx 0.444521$,
and so on. Therefore it follows that
$\lim_{n\to\infty} \frac{T_{n+4}}{T_n} = \lim_{t\to\infty} \alpha_t = 11 + \lim_{t\to\infty} \frac{\hat{N}_{t-1}(t-1)}{\hat{D}_{t-1}(t-1)} = 11.44452\cdots.$ $\square$

Recall the real root $\frac{1}{3}(11 + 2\sqrt[3]{199 + 3\sqrt{33}} + 2\sqrt[3]{199 - 3\sqrt{33}})$ of $x^3 = 11x^2 + 5x + 1$ that approximates to $11.44452\cdots$. And it equals $(1.8392\cdots)^4$.

Example. For $\frac{T_{n+4}}{T_n}$, $\hat{N}_1(1) = \hat{N}_2(2) = 5$ and $\hat{D}_1(1) = \hat{D}_2(2) = 11$. And $\hat{N}_i(i) = 5 + \frac{1}{\hat{D}_{i-1}(i-1)}$, $\hat{D}_i(i) = 11 + \frac{\hat{N}_{i-1}(i-1)}{\hat{D}_{i-1}(i-1)}$ yield the table:

i	2	3	4	5	6	7
$\hat{N}_i(i)$	5	$\frac{56}{11}$	$\frac{641}{126}$	$\frac{7336}{1442}$	$\frac{83957}{16503}$	$\frac{960848}{188869}$
$\hat{D}_i(i)$	11	$\frac{126}{11}$	$\frac{1442}{126}$	$\frac{16503}{1442}$	$\frac{188869}{16503}$	$\frac{2161516}{188869}$
$\frac{\hat{N}_i(i)}{\hat{D}_i(i)}$		$\frac{56}{126}$	$\frac{641}{1442}$	$\frac{7336}{16503}$	$\frac{83957}{188869}$	$\frac{960848}{2161516}$
$\approx$		.444	.44452	.44452	.44452	.44452

Hence, for instance, the ratio $\frac{T_{38}}{T_{34}} = \frac{T_{4(9)+2}}{T_{4(8)+2}} = \alpha_8$ is obtained by the 7th bifurcation $\alpha_8^{[7]}$ that $\alpha_8 = \alpha_8^{[7]} = 11 + \frac{\hat{N}_7(7)}{\hat{D}_7(7)} \approx 11.44452\cdots$.

Moreover all $4t$ subscripted tribonacci numbers can be retrieved in this way.

Corollary 8. *For any $t > 1$, $11 + \frac{\hat{N}_t(t)}{\hat{D}_t(t)}$ retrieves $\frac{T_{4(t+1)}}{T_{4t}}$.*

Proof. Write $11 + \frac{\hat{N}_t(t)}{\hat{D}_t(t)} = \frac{p_t}{q_t}$. Then the above table shows

t	2	3	4	5	6	7
$\frac{p_t}{q_t}$	$\frac{126}{11}$	$\frac{1442}{126}$	$\frac{16503}{1442}$	$\frac{188869}{16503}$	$\frac{2161516}{188869}$	$\frac{24737524}{2161516}$

So we have the set of fractions $\frac{4p_t}{4q_t}$ for $2 \leq t \leq 7$ that

$\{\frac{504}{44}, \frac{5768}{504}, \frac{66012}{5768}, \frac{755476}{66012}, \frac{8646064}{755476}, \frac{98950096}{8646064}\}$

which equals $\{\frac{T_{4(t+1)}}{T_{4t}}\}$. Moreover since $p_t = q_{t+1}$ and the set $\{q_t\} = \{11, 126, 1442, 16503, 188869, 2161516\}$ satisfy the recurrence $q_{t+1} = 11q_t + 5q_{t-1} + q_{t-2}$, it follows that $\frac{4p_t}{4q_t} = \frac{T_{4(t+1)}}{T_{4t}}$ for all $t > 2$. □

Corollary 9. *If n is divisible by 4 then so is T_n.*

Proof. $T_4 = 4$, $T_8 = 44$ and $T_{12} = 504$ are all zeros by mod 4. So if $T_{4i} \equiv 0(mod4)$ for $1 \leq i \leq t$ then $T_{4(t+1)} = 11T_{4t} + 5T_{4(t-1)} + T_{4(t-2)} \equiv 0(mod4)$. □

It is now obvious by Theorem 4 and 7 that any k step tribonacci ratio $\frac{T_{n+k}}{T_n}$ depends on the k step recurrence rule for T_n.

Lemma 1. *[2] Any k step recurrence satisfies $T_{n+k} = a_kT_n + b_kT_{n-k} + T_{n-2k}$ with $a_k = 3T_k - T_{k-6}$ and $b_k = -a_{-k}$. The $\{a_k\}$ and $\{b_k\}$ satisfy $a_{k+3} = a_{k+2} + a_{k+1} + a_k$ and $b_k = b_{k+1} + b_{k+2} + b_{k+3}$ with initial $1, 3, 7$ and $1, 1, -5$, respectively.*

In particular the pairs (a_k, b_k) for $1 \leq k \leq 10$ are $(1, 1)$, $(3, 1)$, $(7, -5)$, $(11, 5)$, $(21, 1)$, $(39, -11)$, $(71, 15)$, $(131, -3)$, $(241, -23)$ and $(443, 41)$. So for example, $T_{n+3} = 7T_n - 5T_{n-3} + T_{n-6}$ and $T_{n+10} = 443T_n + 41T_{n-10} + T_{n-20}$. And the next theorem follows immediately.

Theorem 10. *Let a_k, b_k be integers satisfying $T_{n+k} = a_k T_n + b_k T_{n-k} + T_{n-2k}$. When n is large, we have the followings.*
(1) The k step tribonacci ratio $\frac{T_{n+k}}{T_n}$ holds $x^3 = a_k x^2 + b_k x + 1$.
(2) If $n = kt + r$ $(1 \leq r \leq k)$ then $\frac{T_{n+k}}{T_n} = \langle a_k; \frac{\beta_{t-1}}{a_k}, \cdots, \frac{\beta_2}{a_k}, \frac{\beta_1}{\alpha_1} \rangle$ for $t \geq 2$, with $\beta_i = \langle b_k; \frac{1}{a_k}, \frac{\beta_{i-2}}{a_k}, \cdots, \frac{\beta_2}{a_k}, \frac{\beta_1}{\alpha_1} \rangle$ for $3 \leq i < t$. In particular $\alpha_0 = \frac{T_{k+r}}{T_r}$, $\alpha_1 = \frac{T_{2k+r}}{T_{k+r}}$, and $\beta_i = \langle b_k; \alpha_{i-1} \rangle$ if $i = 1, 2$.
(3) $\lim\limits_{n\to\infty} \frac{T_{n+k}}{T_n} = \lim\limits_{t\to\infty} {\alpha_t}^{[i]} = a_k + \lim\limits_{t\to\infty} \frac{N_{t-1}(t-1)}{D_{t-1}(t-1)}$.

For instance, $\frac{T_{80}}{T_{70}} = \frac{T_{10(8)}}{T_{10(7)}} = \alpha_7 = \alpha_7^{[6]} = 443 + \frac{\hat{N}_6(6)}{\hat{D}_6(6)}$, where $\hat{N}_2(2) = 41$, $\hat{D}_2(2) = 443$, $\hat{N}_3(3) = 41 + \frac{1}{443} = \frac{18164}{443}$, $\hat{D}_3(3) = 443 + \frac{41}{443} = \frac{196290}{443}$, $\hat{N}_4(4) = 41 + \frac{443}{196290} = \frac{8048333}{196290}$, $\hat{D}_4(4) = 443 + \frac{18164}{196290} = \frac{86974634}{196290}$, $\hat{N}_5(5) = \frac{3566156284}{86974634}$, $\hat{D}_5(5) = \frac{38537811195}{86974634}$, $\hat{N}_6(6) = \frac{1580137233629}{38537811195}$, and $\hat{D}_6(6) = \frac{17075816515669}{38537811195}$. These fractions converge to 0.09253, so $\frac{T_{80}}{T_{70}} \approx 443.09253$. It can be compared to the root $\frac{1}{3}(443 + \sqrt[3]{87022054 + 1914\sqrt{33}} + \sqrt[3]{87022054 - 1914\sqrt{33}}) \approx 443.09253$ of $x^3 = 443x^2 + 41x + 1$ as well as to $1.839286755^{10} \approx 443.09253$.

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