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ON DOUBLE CONVERGENCE IN MEASURE

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Abstract: In this paper, we define some modes of convergence in the Pringsheim's sense for a double sequence (f_{mn}) of measurable functions. We also give the relations between modes of these convergences.

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It is quite common in analysis, both for theoretical an practical purposes, to form infinite sequences of functions and to try to understand the limiting behavior of such sequences. Thus it is also quite meaningful to discuss convergence concepts. There are several possible modes of convergence for sequences of functions. The most important of these are the notions of pointwise and uniform convergence, but these convergence modes are too restrictive. In many cases, some of the function sequences do not convergence for every element in the do-

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main. Therefore, the convergence modes in the weaker sense has been define;. for example , almost everywhere convergence and convergence in measure and so on. In this part, we will summarize the knowledge given in some article. Krız and Stepan [6] show the conditions under which the almost everywhere convergence and the convergence in measure coincide. Komisarski [5] defined pointwise I-convergence and I-convergence in measure of sequences of measurable functions defined on a measure space with finite measure. Futhermore, He gived the relationship between these two convergences. Djurcic and Kocinac [2] proved that some classes of sequences of measurable functions satisfy certain selection principles related to special modes of convergence.

The notion of convergence for double sequences was first introduced by Pringsheim [8]. A double sequence $x = (x_{mn})_{m,n=1}^{\infty}$ is said to be convergent in the Pringsheim's sense if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$ whenever $j, k \geq N$, L is called the Pringsheim limit (or double limit) of x and denoted by $P - lim x_{mn} = L$.

Double sequences that the elements are functions are defined in the same way as single sequences of real-valued functions.

Throughout this note (Ω, Γ, μ), or shortly Ω, denotes a measure space with a complete measure $\mu : \Gamma \to \mathbb{R}$ (and Γ is a $\sigma-$algebra of subsets of Ω measurable with respect to μ). E is always an element in Γ such that $\mu(E) < \infty$ [4], [9], [10].

A function $f : E \to \mathbb{R}$ is a measurable function if for each $c \in \mathbb{R}$,

$$\{x \in E : f(x) > c\} \in \Gamma$$

or ,equivalently, $f^{-1}(B) \in \Gamma$ for each $B \in B_{\mathbb{R}}$,where $B_{\mathbb{R}}$ is a $\sigma-$algebra of Borel sets in $\mathbb{R}$.

It is well known,in probability and statistics, a probability measure is a measure with total measure one, i.e. $\mu(\Omega) = 1$. A probability space is a measure space with a probability measure. A random variable is a measurable real function whose domain is the probability space [1], [3], [7]

Throughout this paper, as we mentioned in Introduction, (Ω, Γ, μ) will be always a measure space, and E is an element in Γ such that $\mu(E) < \infty$.

In this paper, we deal with some modes of convergence for a double sequence (f_{mn}) of measurable functions on a set E.

As real-valued functions, the convergence of a double sequence of measurable functions (f_{mn}) is defined as follows:

For every $x_0 \in E, (f_{mn}(x_0))$ converges to $f(x_0)$, i.e. $P - \lim_{m,n\to\infty} f_{mn}(x_0) = f(x_0)$. Then we say that (f_{mn}) converges (pointwise) everywhere on E. Now, we begin with the following definition being weaker than pointwise convergence.

Definition 1. Let (f_{mn}) be a double sequence of measurable functions on a set E, and let f be a measurable function on E. Then (f_{mn}) is said to converge almost everywhere, shortly a.e., to f iff there exists a null set M such that$\forall x \in E - M : P - \lim_{m,n\to\infty} f_{mn}(x) = f(x)$ finite.

This definition can be expressed as follows:

$$\mu\left\{x \in E : P - \lim_{m,n\to\infty} f_{mn}(x) \neq f(x)\right\} = 0$$

We denote this symbolically by writting

$$a.e.P - \lim_{m,n\to\infty} f_{mn}(x) = f(x) \text{ or } f_{mn} \overset{P-a.e.}{\to} f \text{ on } E.$$

In measure theory, one talks about a.e. convergence of a sequence of measurable functions defined on a measure space.That means pointwise a.e. convergence .

Example 2. Let $\Omega = [0, 1]$, we define $f_{mn}(x)$ by

$$f_{mn}(x) = \begin{cases} 1, & x \leq \frac{1}{mn} \\ 0, & x > \frac{1}{mn} \end{cases}$$

and for every pair a,b $(0 \leq a, b \leq 1)$, we will require that $\mu(a \leq x \leq b) = b - a$.

This sequence $(f_{mn}(x))$ converges a.e. to $f(x) = 0$ on Ω. For every $x \neq 0$ and every $\varepsilon > 0$,there is an $k(\varepsilon)$ sufficiently large that for $m, n > k(\varepsilon)$,

$$|f_{mn}(x) - f(x)| < \varepsilon.$$

Thus, for $x \neq 0$, $P - \lim_{m,n\to\infty} f_{mn}(x) = f(x)$, and

$$\mu\left\{x \in E : P - \lim_{m,n\to\infty} f_{mn}(x) = f(x)\right\} = \mu\{\Omega - \{0\}\} = 1,$$

i.e. $a.e.P - \lim_{m,n\to\infty} f_{mn}(x) = 0$ on Ω.

Definition 3. Let (f_{mn}) be a double sequence of measurable functions on a set E, and let f be a measurable function on E. Then the double sequence (f_{mn}) is said to converge in measure on E to f iff for every $\varepsilon > 0$, we have

$$\lim_{m,n\to\infty} \mu\{x \in E : |f_{mn}(x) - f(x)| \geq \varepsilon\} = 0.$$

We will indicate convergence in measure by writting

$$\mu P - \lim_{m,n\to\infty} f_{mn}(x) = f(x) \text{ or } f_{mn} \overset{P\mu}{\to} f \text{ on } E.$$

Notice that if $\mu(\Omega) = 1$ then the convergence in measure is probabilty convergence on E.

Example 4. Let Ω, μ and (f_{mn}) be as given in Example 2. Then, we obtain $\mu P - \lim\limits_{m,n\to\infty} f_{mn}(x) = f(x) = 0$ on Ω. Indeed, let

$$A_{mn}(\varepsilon) = \{x \in \Omega : |f_{mn}(x) - f(x)| < \varepsilon\}.$$

For $\varepsilon > 1$, we have $A_{mn}(\varepsilon) = \Omega$ and $\mu\{A_{mn}(\varepsilon)\} = 1$ for all m, n. For $0 < \varepsilon \leq 1$,we have $x \in A_{mn}(\varepsilon)$ if and only if $f_{mn}(x) = 0$.Hence $\mu\{A_{mn}(\varepsilon)\} = 1 - \frac{1}{mn}$. Thus $\mu P - \lim\limits_{m,n\to\infty} \mu\{A_{mn}(\varepsilon)\} = 1$.

Theorem 5. *Let (f_{mn}) and (g_{mn}) be a double sequence of measurable functions on a set E, and let f and g be two measurable function on E.*

i) If $f_{mn} \overset{P\mu}{\to} f$ and $f_{mn} \overset{P\mu}{\to} g$ then $f = g$ a.e.
ii) If $f_{mn} \overset{P\mu}{\to} f$ and $g_{mn} \overset{P\mu}{\to} g$ then If $f_{mn} \pm g_{mn} \overset{P\mu}{\to} f \pm g$ on E.

Proof. It is easy to prove that. □

Theorem 6. *If (f_{mn}) is a double sequence of measurable functions which converges in measure to f on E, then there is a subsequence $(f_{m_k n_k})$ which converges to f a.e..*

Proof. Choose $m_1, n_1 \in \mathbb{N}$ such that

$$\mu\{x \in E : |f_{m_1 n_1}(x) - f(x)| \geq 1\} < \frac{1}{2}.$$

Suppose that $n_1, n_2..., n_k$ and $m_1, m_2, ..., m_l$ have been chosen. Then pick n_{k+1} and m_{l+1} such that $n_{k+1} > n_k$ and $m_{l+1} > m_l$ and

$$\mu\left\{x \in E : \left|f_{m_{l+1}, n_{k+1}}(x) - f(x)\right| \geq \frac{1}{k+l+1}\right\} < \frac{1}{2^{k+l+1}}.$$

Let

$$A_j = \bigcup_{k,l=j}^{\infty} \left\{x \in E : |f_{m_l, n_k}(x) - f(x)| \geq \frac{1}{k+l}\right\}$$

for each $j \in \mathbb{N}$. Clearly we have $A_1 \supset A_2 \supset A_3 \supset \ldots$ .Next let $B = \bigcap_{j=1}^{\infty} A_j$. Since

$$\mu(A_1) < \sum_{k,l=1}^{\infty} \frac{1}{2^{k+l}} < \infty.$$

It follows that

$$\mu(B) = \lim_{j\to\infty} \mu(A_j) \le \lim_{j\to\infty} \sum_{k,l=j}^{\infty} \frac{1}{2^{k+l}} = 0,$$

that is, $\mu(B) = 0$. Next, let

$$x \in E - B = \bigcup_{j=1}^{\infty} (E - A_j).$$

Then there is a j_x such that

$$x \in E - A_{j_x} = \bigcap_{k,l=j_x}^{\infty} \left\{ t \in E : |f_{m_l,n_k}(t) - f(t)| < \frac{1}{k+l} \right\}.$$

Given $\varepsilon > 0$, choose k_0, l_0 such that $k_0, l_0 \ge j_x$ and $\frac{1}{k_0+l_0} \le \varepsilon$. Then $k \ge k_0$ and $l \ge l_0$ implies that

$$|f_{m_l,n_k}(x) - f(x)| < \frac{1}{k+l} \le \varepsilon$$

and this proves that $f_{m_l,n_k}(x) \to f(x)$ for all $x \in E - B$. The proof is finished. □

Definition 7. Let (f_{mn}) be a double sequence of measurable functions on a set E, and let f be a measurable function on E. We say that (f_{mn}) converges almost uniformly to f if for every $\varepsilon > 0$ there is a measurable set E such that $\mu(E) < \varepsilon$ and $f_{mn} \to f$ uniformly on E^c. We will indicate convergence in almost uniformly by writing $f_{mn} \overset{P-a.u.}{\to} f$ on E.

Theorem 8. *If (f_{mn}) converges to f a.u. then (f_{mn}) converges to f a.e. (without assumption $\mu(F) < \infty$).*

Proof. Let $E_{mn} \subseteq \Omega$ be a measurable set with the property that $\mu(E_{mn}) < \frac{1}{2^{m+n}}$ and $f_{mn} \to f$ uniformly on E^c_{mn}. Let $F_{kl} = \bigcup_{m=k,\ n=l}^{\infty} E_{mn}$ and let $F = \bigcap_{k,l=1}^{\infty} F_{kl}$. By countable subaddivity of measure, $\mu(F_{kl}) \le \sum_{m=k,n=l}^{\infty} \frac{1}{2^{m+n}} = \frac{1}{2^{k+l-2}}$

and thus by continuity of measure from above we have $\mu(F) = 0$. Moreover $F^c = \bigcup_{k,l=1}^{\infty} \bigcap_{m=k,\ n=l}^{\infty} E_{mn}^c$; since f_{mn} converges uniformly and therefore pointwise to f on each E_{mn}^c it follows that f_{mn} converges pointwise on F^c, i.e. $f_{mn} \overset{P-a.e.}{\to} f$ on F. □

Theorem 9. *Assume that $\mu(E) < \infty$ and let (f_{mn}) be a double sequence of measurable functions on a set E and let f be a measurable function on E such that $f_{mn} \overset{P-a.e.}{\to} f$. Then f_{mn} converges to f almost uniformly .*

Proof. Possibly adjusting on a set of measure 0, we may assume that f_{mn} converges to f everywhere on E. Consider the sets

$$E_{kl}(m,n) = \bigcup_{p=k,\ q=l}^{\infty} \left\{ x \in E : |f_{pq}(x) - f(x)| \geq \frac{1}{m+n} \right\}.$$

The assumption that f_{mn} converges pointwise to f implies that $\bigcap_{k,l=1}^{\infty} E_{kl}(m,n)$ is empty for each fixed m and n, so since $\mu(E) < \infty$ we may apply continuity from above to deduce that $\mu(E_{kl}(m,n)) \to 0$ as $k,l \to \infty$. Now, fix $\varepsilon > 0$ and choose $m,\ n \in \mathbb{N}$. We may choose k_m and l_n so that $\mu(E_{k_m,l_n}(m,n)) < \varepsilon \frac{1}{2^{m+n}}$. The set we need is $E = \bigcup_{m,n=1}^{\infty} E_{k_m,l_n}(m,n)$ which by countable subadditivity has $\mu(E) < \varepsilon \sum_{m,n=1}^{\infty} \frac{1}{2^{m+n}} = \varepsilon$. If $x \in E^c$ then $x \in [E_{k_m,l_n}(m,n)]^c$ for every m,n, which means that $|f_{mn}(x) - f(x)| < \frac{1}{m+n}$ for every $m > k_m$ and $n > l_n$.By definition, this means f_{mn} converges uniformly to f. □

Corollary 10. *If $\mu(E) < \infty$ and let $f_{mn}, f : E \to \mathbb{R}$ be a measurable functions for each $m,n \in \mathbb{N}$ on E, then the following holds: a.e. double convergence $\Leftrightarrow$ a.u. double convergence.*

Theorem 11. *Let (f_{mn}) be a double sequence of measurable functions on a set E, and let f be a measurable function on E. If (f_{mn}) converges to f almost uniformly then (f_{mn}) converges to f in measure.*

Proof. For each $m,n \in \mathbb{N}$,let E_{mn} be a measurable set such that $\mu(E_{mn}) < \frac{1}{m+n}$ and $f_{mn} \to f$ uniformly on E_{mn}^c. The statement that $f_{mn} \to f$ uniformly means that for every $\varepsilon > 0$ there exists N_{mn} such that $|f_{kl}(x) - f(x)| < \varepsilon$ whenever $k,l > N_{mn}$ for any $x \in E_{mn}^c$. This means that for $k,l > N_{mn}$,

$\{x \in E : |f_{kl}(x) - f(x)| \geq \varepsilon\} \subseteq E_{mn}$ and hence

$$\mu \{x \in E : |f_{kl}(x) - f(x)| \geq \varepsilon\} < \frac{1}{m+n}.$$

Passing to the limit as $m, n \to \infty$, we conclude that

$$\mu \{x \in E : |f_{kl}(x) - f(x)| \geq \varepsilon\} \to 0$$

as $k, l \to \infty$.

Corollary 12. *Let be $\mu(E) < \infty$ and $f_{mn}, f : E \to \mathbb{R}$ for each $m, n \in \mathbb{N}$ be measurable functions on a set E. If $f_{mn} \overset{P-a.e.}{\to} f$ (or equivalently $f_{mn} \overset{P-a.u.}{\to} f$) on E, then $f_{mn} \overset{P\mu}{\to} f$ on E.*

□

We note that the converse is not true.There are sequences of functions that converge in measure and do not converge almost everywhere.

For example, consider the sequence $f_{mn} : [0, 1] \to \mathbb{R}$ defined by

$$f_{mn}(x) = \begin{cases} 1, & \frac{j}{2^k} \leq x \leq \frac{j+1}{2^k}, n = 2^k + j, 0 \leq j < 2^k, m = 1, 2, ... \\ 0, & \qquad\qquad otherwise \end{cases}$$

where $\mu(a \leq x \leq b) = b - a$. Define the $f(x)$ (real-valued) by $f(x) = 0$. Then for $\varepsilon > 1$, trivially, $\mu \{x \in [0, 1] : |f_{mn}(x)| \geq \varepsilon\} = 0$ and for $0 < \varepsilon \leq 1$, $\mu \{x \in [0, 1] : |f_{mn}(x)| \geq \varepsilon\} \leq \frac{1}{2^k} \to 0$ as $m, n \to \infty$, where k is the largest integer with the property that $2^k \leq n$. Thus (f_{mn}) converges to 0 in measure.However $f_{mn}(x)$ does not converge almost everywhere to $f(x)$.To see this , observe that for any $x \in [0, 1]$ and any positive integer N, there are two $m, n > N$, such that $f_{mn}(x) = 1$. In fact $f_{mn}(x)$ converges to $f(x)$ at no point of $[0, 1]$.

In the above examples, the double limit $f(x)$ was the constant function $f(x) = 0$. There is no loss in generality in using a constant function, since if $f(x)$ is not constant, and $f_{mn}(x) \to f(x)$, we can define $g_{mn}(x) = f_{mn}(x) - f(x)$, and then $g_{mn}(x) \to 0$.

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