International Journal of Pure and Applied Mathematics
Volume 104 No. 1 2015, 19-28
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v104i1.2

ijpam.eu

SOME CONSTRUCTION OF GROUP DIVISIBLE DESIGNS $\mathsf{GDD}(m, n; 1, 3)$

Chariya Uiyyasathian[1 §], Narong Punnim[2]

[1]Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Payathai Rd., Bangkok 10330, THAILAND

[2]Department of Mathematics
Srinakharinwirot University
Sukhumvit 23, Bangkok 10110, THAILAND

Abstract: A group divisible design $\mathsf{GDD}(m, n; 1, 3)$ is an ordered pair $(V, \mathcal{B})$ where V is an $(m+n)$-set of symbols and $\mathcal{B}$ is a collection of 3-subsets (called blocks) of V satisfying the following properties: the $(m+n)$-set is divided into two groups of size m and n; each pair of symbols from the same group occurs in exactly one block in $\mathcal{B}$; and each pair of symbols from different groups occurs in exactly three blocks in $\mathcal{B}$. Given positive integers m and n, two necessary conditions on m and n for the existence of a $\mathsf{GDD}(m, n; 1, 3)$ are $6 \mid [m(m-1) + n(n-1)]$ and $m \not\equiv n \pmod 2$. We show that these conditions are sufficient for the most cases.

Key Words: group divisible design, difference triple, graph decomposition

1. Introduction

A *Steiner triple system* of order v, denoted by $\mathsf{STS}(v)$, is an ordered pair $(V, \mathcal{B})$ where V is a v-set of symbols and $\mathcal{B}$ is a collection of 3-subsets of V (called *blocks*), such that each pair of distinct elements of V occurs together in exactly one block of $\mathcal{B}$. A *group divisible design* $\mathsf{GDD}(v = v_1 + v_2 + \cdots + v_g, g, k; \lambda_1, \lambda_2)$ is an ordered pair $(V, \mathcal{B})$ where V is a v-set of symbols and $\mathcal{B}$ is a collection of

Received: March 23, 2015

© 2015 Academic Publications, Ltd.
url: www.acadpubl.eu

§Correspondence author

k-subsets (called *blocks*) of V satisfying the following properties: the v-set is divided into g groups of sizes $v_1, v_2, \dots, v_g$; each pair of symbols from the same group occurs in exactly λ_1 blocks in $\mathcal{B}$; and each pair of symbols from different groups occurs in exactly λ_2 blocks in $\mathcal{B}$. Symbols occurring in the same group are known to statisticians as *first associates*, and symbols occurring in different groups are called *second associates*. The existence of such GDDs has been of interest over the years, going back to at least the work of Bose and Shimamoto in 1952 who began classifying such designs [12]. More recently, much work has been done on the existence of such designs when $\lambda_1 = 0$ (see [2] for a summary).

Most interestingly, if the number of groups is less than the block size, then the construction of such GDDs is notoriously difficult. For $k = 3$ this existence problem was completely solved by Sarvate, Fu and Rodger [5, 4] in the case where all groups have the same size. In this paper we focus on an existence of a $\mathsf{GDD}(v = m + n, 2, 3; 1, 3)$ for any m and n. Since we are dealing with GDDs with two groups and block size 3, we will use $\mathsf{GDD}(m, n; \lambda_1, \lambda_2)$ for $\mathsf{GDD}(v = m + n, 2, 3; \lambda_1, \lambda_2)$ from now on, and we refer to the blocks as *triples*. We denote $(X, Y; \mathcal{B})$ for a $\mathsf{GDD}(m, n; \lambda_1, \lambda_2)$ if X and Y are m-set and n-set, respectively. Chaiyasena et al. [1] have written the first paper in this direction. In particular they have completely determined all pairs of integers (n, λ) for which a $\mathsf{GDD}(1, n; 1, \lambda)$ exists. Other work on the existence problem of a $\mathsf{GDD}(m, n; \lambda_1, \lambda_2)$ for possible m, n, λ_1 and λ_2 includes work on a $\mathsf{GDD}(m, n; \lambda, 1)$ [9], a $\mathsf{GDD}(m, n; \lambda, 2)$ when $\lambda \neq 1$ [14], a $\mathsf{GDD}(m, n; \lambda, 3)$ when $\lambda \geq 3$ [10] and a $\mathsf{GDD}(m, n; \lambda, 4)$ when $\lambda \geq 4$ [15]. When $\lambda_1 < \lambda_2$, the sufficient conditions for its existence seem to be complicated. Recently, the existence of a $\mathsf{GDD}(m, n; 1, 2)$ was studied in [3] and [11]. In this paper we investigate the existence of a $\mathsf{GDD}(m, n; 1, 3)$. The necessary conditions can be easily obtained by describing it graphically as follows.

Let ${}^{\lambda}K_v$ denote the graph on v vertices in which each pair of distinct vertices is joined by λ edges. Let G_1 and G_2 be vertex disjoint graphs. The graph $G_1 \vee_\lambda G_2$ is formed from the union of G_1 and G_2 by joining each vertex in G_1 to each vertex in G_2 with λ edges. Let G and H be graphs with G is a subgraph of H. A *G-decomposition* of a graph H is a partition of the edge set of H such that each element of the partition induces a copy of G. Thus an existence of a $\mathsf{GDD}(m, n; \lambda_1, \lambda_2)$ is easily seen to be equivalent to an existence of a K_3-decomposition of ${}^{\lambda_1}K_m \vee_{\lambda_2} {}^{\lambda_1}K_n$. In particular the existence of a $\mathsf{GDD}(m, n; 1, 3)$ is equivalent to a K_3-decomposition of $K_m \vee_3 K_n$, and triples in the design are equivalent to K_3's, called *triangles*, in the decomposition.

The following notations will be used for our constructions.

1. Let $T = \{x, y, z\}$ be a triple and $a \notin T$. We use $a * T$ for three triples of the form $\{a, x, y\}, \{a, x, z\}$, $\{a, y, z\}$. If $\mathcal{T}$ is a set of triples, then $a * \mathcal{T}$ is defined as $\{a * T : T \in \mathcal{T}\}$.

2. Let V be a v-set. Then there may be many different $\mathsf{STS}(v)$s that can be constructed on the set V. Let $\mathsf{STS}(V)$ be defined as

$$\mathsf{STS}(V) = \{\mathcal{B} : (V, \mathcal{B}) \text{ is an } \mathsf{STS}(v)\}.$$

3. When we say that $\mathcal{B}$ is a *collection* of subsets (blocks) of a v-set V, $\mathcal{B}$ may contain repeated blocks. Thus " $\cup$ " in our construction will be used for the union of multi-sets.

The notation $K_n(V)$ stands for the complete graph on the vertex set V of size n. It is easy to see that the graph ${}^{\lambda_1}K_m(V_1) \vee_{\lambda_2} {}^{\lambda_1}K_n(V_2)$ has the following properties.

1. It has order $m + n$ and size $\lambda_1\binom{m}{2} + \lambda_1\binom{n}{2} + \lambda_2 mn$.

2. It contains m vertices of degree $\lambda_1(m-1) + \lambda_2 n$ and n vertices of degree $\lambda_1(n-1) + \lambda_2 m$.

Thus an existence of K_3-decomposition of ${}^{\lambda_1}K_m(V_1) \vee_{\lambda_2} {}^{\lambda_1}K_n(V_2)$ implies the following relations.

1. $6 \mid \lambda_1[m(m-1) + n(n-1)] + 2\lambda_2 mn$.

2. $2 \mid \lambda_1(m-1) + \lambda_2 n$ and $2 \mid \lambda_1(n-1) + \lambda_2 m$.

In particular, if $\lambda_1 = 1$ and $\lambda_2 = 3$, then necessary conditions on an existence of $\mathsf{GDD}(m, n; 1, 3)$ can be obtained as follows.

1. $6 \mid [m(m-1) + n(n-1)]$.

2. $2 \mid m-1+3n$ and $2 \mid n-1+3m$ and that m and n are in different parity.

Hence, the necessary conditions on m and n to obtain a GDD $(m, n; 1, 3)$ are summarized in the next theorem.

Theorem 1.1. *Let m and n be positive integers. If there exists a* $\mathsf{GDD}(m, n; 1, 3)$ *then* $6 \mid [m(m-1) + n(n-1)]$, *and* $m \not\equiv n (\mathrm{mod}\ 2)$.

By solving a system of congruences we easily obtain more precise necessary conditions on an existence of GDD $(m, n; 1, 3)$.

Theorem 1.2. *Let m and n be positive integers. If there exists a* GDD $(m,n;1,3)$ *then there exist non-negative integers h and k such that* $\{m,n\} \in \{\{6k+1,6h\},\{6k+1,6h+4\},\{6k+3,6h\},\{6k+3,6h+4\}\}$.

Proof. Since $2 \mid m-1+3n$, $2 \mid n-1+3m$ and $6 \mid [m(m-1)+n(n-1)]$, we have that $m \not\equiv n(\text{mod } 2)$ satisfying $m(m-1)+n(n-1) \equiv 0(\text{mod } 6)$. Each of the possible cases is verified. If $m \equiv 0$ or $4(\text{mod } 6)$, then, $n(n-1) \equiv 0$ or $4(\text{mod } 6)$; thus $n \equiv 1$ or $3(\text{mod } 6)$. If $m \equiv 1$ or $3(\text{mod } 6)$, then $n(n-1) \equiv 0(\text{mod } 6)$; thus $n \equiv 0$ or $4(\text{mod } 6)$. However, if $m \equiv 2$ or $5(\text{mod } 6)$, then $2+n(n-1) \equiv 0(\text{mod } 6)$, there is no n satisfying the congruence. Therefore, the only possible values of m and n are $\{m,n\} \in \{\{6k+1,6h\},\{6k+1,6h+4\},\{6k+3,6h\},\{6k+3,6h+4\}\}$. □

2. Preliminary

A *factor* of a graph G is a spanning subgraph of G. A k-factor of a graph is a spanning k-regular subgraph, and a k-factorization partitions the edge set of the graph into disjoint k-factors. In particular, a 1-factor is a perfect matching and a 2-factor is a collection of cycles that spans all vertices of the graph. If all cycles in the collection is K_3, then it is called *triangle-factor* or Δ*-factor*. It is well-known that K_{2n} has a 1-factorization while K_{2n+1} has a 2-factorization. Since a union of k 1-factors of K_{2n} is a k-factor of K_{2n}, it follows that K_{2n} has a k-factorization if and only if $k \mid 2n-1$. An existence of a 3-factorization of K_{6k+4} can be used to construct a $\mathsf{GDD}(t,3t+1;1,3)$ as shown in Theorem 3.3.

Theorem 1.2 gives us necessary conditions on m (mod 6) and n (mod 6) on the existence of $\mathsf{GDD}(m,n;1,3)$. We prove the necessary conditions become sufficient by constructing a $\mathsf{GDD}(m,n;1,3)$ for corresponding (m,n) given in Theorem 1.2. The following results will be useful for our construction. Details can be found in [8].

For any integer v, a *difference triple* is a subset of three distinct elements $\{x,y,z\}$ of $\{1,2,\dots,v-1\}$ such that $x+y \equiv \pm z(\text{mod } v)$, and its corresponding *base block* is the triple $\{0,x,x+y\}$. In 1896, Heffter [6] posted a problem called Heffter's Difference Problem. The solution of this problem can be used to construct cyclic Steiner triple systems of order v when $v \equiv 1$ or $3(\text{mod } 6)$. The following result on the existence of Steiner triple systems is classic (see e.g. [8]).

Theorem 2.1. *For a positive integer v, an* $\mathsf{STS}(v)$ *exists if and only if* $v \equiv 1$ *or* $3(\text{mod } 6)$.

When $v \equiv 4 (\mathrm{mod}\, 6)$, an $\mathsf{STS}(v)$ does not exist. However, Lapchinda et al. use similar idea of Heffter's Difference Problem resulted in Theorem 2.2 to decompose K_v into one 3-factor and a collection of triangles concluded in Theorem 2.3 [7].

Theorem 2.2. [7] *Let k be a positive integer and $v = 6k+4$. Then there exists $t \in \{1, 2, \ldots, 3k+1\}$ such that $\{1, 2, \ldots, 3k+1\} \setminus \{t\}$ can be partitioned into k difference triples of the set of size v.*

Let K_v be the complete graph of order v with $\mathbb{Z}_v = \{0, 1, 2, \ldots,\ v-1\}$ as its vertex set. If we place the elements of $V(K_v) = \mathbb{Z}_v$ around an v-gon, then the *length* of an edge xy, denoted by $\ell(x, y)$, is the shortest distance from x to y around the polygon. Thus $\ell(x, y) = \min\{|x-y|, n-|x-y|\}$. It should be noted that if $v = 2k+1$ is an odd integer, then $\ell(x, y) \in \{1, 2, \ldots, k\}$ for any pair of distinct integers $x, y \in \mathbb{Z}_v$ and for each $i = 1, 2, \ldots, k$, there are exactly v pairs of $x, y \in \mathbb{Z}_v$ such that $\ell(x, y) = i$. If $v = 2k$ is an even integer, then $\ell(x, y) \in \{1, 2, \ldots, k\}$ for any pair of distinct integers $x, y \in \mathbb{Z}_v$ and for each $i = 1, 2, \ldots, k-1$, there are exactly v pairs of $x, y \in \mathbb{Z}_v$ such that $\ell(x, y) = i$ and $\frac{v}{2}$ pairs of distance k.

The following observations are useful.

1. $\ell(x, y) = \ell(y, x)$ and for each integer i $\ell(x+i, y+i) = \ell(x, y)$, where the sum is taken in mod v.

2. Let i be an integer with $1 \leq i < \frac{v}{2}$. Then the set of edges of K_v of length i forms a 2-factor of K_v.

3. If $v = 2m$ is an even integer, then the set of edges of K_v of length m forms a 1-factor of K_v.

Let $v = 6k+4$ where $k \in \mathbb{N}$, and $V = \{0, 1, 2, \ldots, v-1\}$ and $D(v)$ be a set of difference triples obtained from Theorem 2.2 for a set of size v. The set of difference triples $D(v)$ yields the collection of base blocks. For any difference triple $\{x, y, z\}$ of $\{0, 1, \ldots, v-1\}$, by adding modulo v we define $\{0, x, x+y\}$ be the corresponding base block and consequently, this base block generates v triples in the collection $\mathcal{B} = \{\{0+i, x+i, x+y+i\} : i = 0, 1, 2, \ldots, v-1\}$. Each difference triple in $D(v)$ yields a collection of triples, therefore, Theorem 2.3 is concluded.

Theorem 2.3. [7] *Let k be a positive integer. If $v = 6k+4$. Then there exist a 1-factor M and a 2-factor H of K_v such that $K_3 \mid (K_v \setminus (M \cup H))$.*

3. GDD$(m, n; 1, 3)$

Chaiyasena et al. [1] studied the existence of the designs when one group is a singleton. Their result is in Theorem 3.1, our desired designs are immediately followed in Corollary 3.2.

Theorem 3.1. [1] *For an integer $n \geq 3$, there exists a* GDD$(1, n; 1, \lambda)$ *if and only if* $2 \mid (n - 1 - \lambda)$ *and* $6 \mid n(n - 1 - \lambda)$.

Corollary 3.2. *A* GDD$(1, n; 1, 3)$ *exists if and only if* $n \equiv 0$ *or* $4 (\mathrm{mod}\ 6)$.

The next theorem gives infinitely many pairs of integers (m, n)s for existence of GDD$(m, n; 1, 3)$s.

Theorem 3.3. *Let t be a positive integer with $t \equiv 1$ or $3 (\mathrm{mod}\ 6)$. Then there exists a* GDD$(t, 3t + 1; 1, 3)$.

Proof. Put $X = \{a_1, a_2, \dots, a_t\}$ and $Y = \mathbb{Z}_{3t+1}$. Let $\{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_t\}$ be a 3-factorization of $K_{3t+1}(Y)$. Choose $\mathcal{B}_1 \in$ STS(X). Thus

$$\mathcal{B} = \mathcal{B}_1 \cup \bigcup_{i=1}^{t} (a_i * \mathcal{F}_i).$$

forms a GDD $(t, 3t + 1; 1, 3)$. □

Theorem 3.3 shows a very special case of an existence of GDD $(m, n; 1, 3)$, namely $n = 3m + 1$ or $m = \frac{n-1}{3}$. We next extend this idea to construct GDD$(m, n; 1, 3)$s for $m < \frac{n-1}{3}$ when $m \equiv 1$ or $3 (\mathrm{mod}\ 6)$. Lemmas 3.4 and 3.6 are the main tools.

Lemma 3.4. *Let $n \equiv 4 (\mathrm{mod}\ 6)$ be a positive integer. Then K_n can be decomposed into p 3-factors and a collection of triangles, where $1 \leq p \leq \frac{n-1}{3}$ is an odd integer.*

Proof. Since K_4 is a 3-factor of itself, we may assume $n \geq 10$. Write $n = 6k + 4$ for a positive integer k. By Theorem 2.3, K_n can be decomposed into one 1-factor M, one 2-factor H and a collection triangles. The union of M and H is a 3-factor. It remains to show that the remaining edges can be decomposed into an even number of 3-factors and a collection of triangles. First note the following three observations.

1. Let $x \in \{1, 2, \dots, 3k + 1\}$ and denote $G[x]$ be the subgraph of K_{6k+4} induced by all edges of length x. Then $G[x]$ is a union of $\frac{6k+4}{p}$ cycles of

order $p = \frac{6k+4}{(x,\,6k+4)}$ and hence $G[x]$ is a 2-factor of K_{6k+4}. Furthermore, if p is even, then $G[x]$ can be decomposed into two 1-factors.

2. Let $\{x, y, z\}$ be a difference triple and denote $G[x, y, z]$ be the subgraph of K_{6k+4} induced by all edges of length x, y and z. Then the 6-factor $G[x, y, z]$ can be decomposed into two 3-factors if there exists $t \in \{x, y, z\}$ such that $\frac{6k+4}{(t,\,6k+4)}$ is even. In particular, if there exists an odd $t \in \{x, y, z\}$, $G[x, y, z]$ can be decomposed into two 3-factors.

3. Note further that any difference triples $\{x, y, z\}$ of a set of size $n = 6k+4$ has exactly 0 or 2 odd lengths. If x and y are odd, we can pair the difference triple $\{x, y, z\}$ up with another difference triple $\{a, b, c\}$, then $G[x] \cup G[y]$ gives four 1-factors, and the other four lengths give four 2-factors. Hence we can decompose $G[x, y, z] \cup G[a, b, c]$ into four 3-factors.

There are $3k$ lengths of K_{6k+4} in all difference triples from Theorem 2.2 in which half of them are odd. Besides each odd length can give two 3-factors, so in an appropriate way we can construct any possible even number of 3-factors while the rest edges form a collection of triangles. □

In 1987, Rees studied resolvable designs [13]. He found a decomposition of K_n for $n \equiv 0 (\text{mod } 6)$ into 1-factors and Δ-factors as in the following theorem.

Theorem 3.5. [13] *Let $n \equiv 0 (\text{mod } 6)$ be a positive integer. Then K_n can be decomposed into t 1-factors and some Δ-factors, where $t \in \{3, 5, 7, \dots \lfloor \frac{n-1}{3} \rfloor\}$.*

Lemma 3.6. *Let $n \equiv 0 (\text{mod } 6)$ be a positive integer. Then K_n can be decomposed into p 1-factors and a collection of triangles, where $1 \le p \le \frac{n-1}{3}$ is an odd integer.*

Proof. Let p be an odd integer. If $3 \le p \le \frac{n-1}{3}$, it follows directly from Theorem 3.5 as one 1-factor and one Δ-factor form a 3-factor of K_n. Since there are an odd number of 1-factors, the number of 3-factors is also odd and the maximum number is $\lfloor \frac{n-1}{3} \rfloor$. When $p = 1$, consider a Steiner triple system of order $n+1$. The existence of $\mathsf{STS}(n+1)$ is guaranteed by Theorem 2.1. Deleting a vertex of this design yields a decomposition of K_n into a 1-factor and a collection of triangles. □

Theorem 3.7. *Let m, n be positive integers. If $m \equiv 1$ or $3 (\text{mod } 6)$, $n \equiv 0$ or $4 (\text{mod } 6)$ and $m \le \frac{n-1}{3}$, then there exists a $\mathsf{GDD}(m, n; 1, 3)$.*

Proof. Let $X = \{a_1, a_2, \dots, a_m\}$ and Y be sets of size m and n, respectively. By Lemmas 3.4 and 3.6, $K_n(Y)$ can be decomposed into m 3-factors,

$\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$, and a collection of triangles $\mathcal{T}$. Moreover, since $m \equiv 1$ or $3 (\mathrm{mod}\ 6)$, by Theorem 2.1, $\mathsf{STS}(X)$ is not empty, let $\mathcal{B}' \in \mathsf{STS}(X)$. Finally, put

$$\mathcal{B} = \mathcal{B}' \cup \mathcal{T} \cup \bigcup_{i=1}^{m} (a_i * \mathcal{F}_i).$$

Therefore $(X, Y; \mathcal{B})$ is a $\mathsf{GDD}(m, n; 1, 3)$. □

Theorem 3.8. *Let m, n be positive integers. If $m \equiv 0$ or $4 (\mathrm{mod}\ 6)$, $n \equiv 1 (\mathrm{mod}\ 6)$ and $m \leq \frac{n-3}{3}$, then there exists a $\mathsf{GDD}(m, n; 1, 3)$.*

Proof. Let $X = \{a_1, \ldots, a_m\}$ and Y be sets of size m and n, respectively. Let $a \in Y$. Since $m+1$ is odd, by Theorem 3.5, $K_n(Y \setminus \{a\})$ can be decomposed into $m+1$ 1-factors and s Δ-factors. Since $m \leq \frac{n-3}{3}$, $s \geq m$. Combine those m 1-factors with m Δ-factors to get m 3-factors. Hence $K_n(Y \setminus \{a\})$ can be decomposed into one 1-factor M, m 3-factors $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ and a collection of triangles $\mathcal{T}$. Now since $m \equiv 0$ or $4 (\mathrm{mod}\ 6)$, by Lemmas 3.4 and 3.6, $K_m(X)$ can be decomposed into one 3-factor $\mathcal{F}'$ and a collection of triangles $\mathcal{T}'$. Therefore we have a $\mathsf{GDD}(m, n; 1, 3)$, namely, $(X, Y; \mathcal{B})$ where

$$\mathcal{B} = \mathcal{T} \cup \mathcal{T}' \cup (a * \mathcal{F}') \cup (a * M) \cup \bigcup_{i=1}^{m} (a_i * \mathcal{F}_i).$$ □

Theorem 3.9. *Let m, n be positive integers. If $m \equiv 0$ or $4 (\mathrm{mod}\ 6)$, $n \equiv 3 (\mathrm{mod}\ 6)$ and $m \leq \frac{n-7}{3}$, then there exists a $\mathsf{GDD}(m, n; 1, 3)$.*

Proof. Let $X = \{a_1, \ldots, a_m\}$ and Y be sets of size m and n, respectively. Let $b_1, b_2, b_3 \in Y$. Since $m+3$ is odd, by Theorem 3.5, $K_n(Y \setminus \{b_1, b_2, b_3\})$ can be decomposed into $m+3$ 1-factors and s Δ-factors. Since $m \leq \frac{n-7}{3}$, $s \geq m$. Combine those m 1-factors with m Δ-factors to get m 3-factors. Therefore, $K_n(Y \setminus \{b_1, b_2, b_3\})$ can be decomposed into three 1-factors M_1, M_2, M_3, m 3-factors $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ and a collection of triangles $\mathcal{T}$. Now since $m \equiv 0, 4 (\mathrm{mod}\ 6)$, by Lemmas 3.4 and 3.6, $K_m(X)$ can be partitioned into three 3-factor $\mathcal{F}'_1, \mathcal{F}'_2, \mathcal{F}'_3$ and a set of triangles $\mathcal{T}'$. Hence we have $\mathsf{GDD}(m, n; 1, 3)$, namely, $(X, Y; \mathcal{B})$ where

$$\mathcal{B} = \mathcal{T} \cup \mathcal{T}' \cup \bigcup_{i=1}^{3} (b_i * \mathcal{F}'_i) \cup \bigcup_{i=1}^{3} (b_i * M_i) \cup \bigcup_{i=1}^{m} (a_i * \mathcal{F}_i).$$ □

Let $A = \{(m,n) : 6|m(m-1) + n(n-1), m \not\equiv n (\text{mod } 2)\}$. Theorem 1.1 states that if a $\mathsf{GDD}(m,n;1,3)$ exists then $(m,n) \in A$. However the set A can be rewritten as $\{(m,n) : m,n \equiv 0,1,3$ or $4 (\text{mod } 6), m \not\equiv n (\text{mod } 2)$ and $m \leq \frac{n}{3} + \epsilon$ for some small $\epsilon.\}$ Therefore in general our construction gives the target designs for the most part but not a complete solution. Examples 3.10 and 3.11 show the existence of $\mathsf{GDD}(4,13;1,3)$ and $\mathsf{GDD}(6,19;1,3)$ which cannot be constructed by our technique. It is still an open problem to complete the set of solutions.

Example 3.10. The existence of a $\mathsf{GDD}(4,13;1,3)$.

Let $X = \{a,b,c,d\}$ and $Y = \{0,1,2,\ldots,12\}$. Let $\mathcal{B}$ contain the following 80 triples. It can be verified that $(X,Y;\mathcal{B})$ is a $\mathsf{GDD}(4,13;1,3)$. In each column, the set of three elements in the same row are triples.

a 1 2	b 2 5	c 4 6	d 6 11	c 8 12	d 4 8	a 10 4	b 4 11	
a 2 3	b 3 6	c 5 7	d 7 12	c 3 7	d 12 3	a 11 5	b 5 12	1 3 4
a 3 4	b 4 7	c 6 8	d 8 0	c 11 2	d 7 11	a 12 6	b 6 0	2 5 6
a 4 5	b 5 8	c 7 9	d 9 1	c 6 10	d 2 6	a 0 7	b 7 1	
a 5 6	b 6 9	c 8 10	d 10 2	c 1 5	d 10 1	a 1 8	b 8 2	
a 6 7	b 7 10	c 9 11	d 11 3	c 9 0	d 5 9	a 2 9	b 9 3	b c 4
a 7 8	b 8 11	c 10 12	d 12 4					a b 1
a 8 9	b 9 12	c 11 0	d 0 5					a d 0
a 9 10	b 10 0	c 12 1	d 1 6					a c 3
a 10 11	b 11 1	c 0 2	d 2 7					c d 5
a 11 12	b 12 2	c 1 3	d 3 8					b d 10
a 12 0	b 0 3	c 2 4	d 4 9					

Example 3.11. The existence of a $\mathsf{GDD}(6,19;1,3)$.

Let $X = \{a,b,c,d,e,f\}$ and $Y = \{0,1,2,\ldots,18\}$. Let $\mathcal{B}$ contain the following 176 triples. It can be verified that $(X,Y;\mathcal{B})$ is a $\mathsf{GDD}(6,19;1,3)$.

d 1 2	b 2 5	a 4 6	f 6 13	c 3 8	e 8 14	a 4 8	e 12 2	d 13 5	
d 2 3	b 3 6	a 5 7	f 7 14	c 4 9	e 9 15	a 12 16	e 11 1	d 16 8	a b c
d 3 4	b 4 7	a 6 8	f 8 15	c 5 10	e 10 16	a 1 5	e 10 0	d 0 11	d e f
d 4 5	b 5 8	a 7 9	f 9 16	c 6 11	e 11 17	a 9 13	e 9 18	d 3 14	
d 5 6	b 6 9	a 8 10	f 10 17	c 7 12	e 12 18	a 17 2	e 8 17	d 6 17	0 1 4
d 6 7	b 7 10	a 9 11	f 11 18	c 8 13	e 13 0	a 6 10	e 7 16	d 9 1	3 5 12
d 7 8	b 8 11	a 10 12	f 12 0	c 9 14	e 14 1	a 14 18	e 6 15	d 12 4	2 7 13
d 8 9	b 9 12	a 11 13	f 13 1	c 10 15	e 15 2	a 3 7	e 5 14	d 15 7	
d 9 10	b 10 13	a 12 14	f 14 2	c 11 16	e 16 3	a 11 15	e 4 13	d 18 10	a d 0
d 10 11	b 11 14	a 13 15	f 15 3	c 12 17	e 17 4	f 8 12	c 2 11	b 5 16	b d 1
d 11 12	b 12 15	a 14 16	f 16 4	c 13 18	e 18 5	f 16 1	c 1 10	b 8 0	b f 4
d 12 13	b 13 16	a 15 17	f 17 5	c 14 0	e 0 6	f 5 9	c 0 9	b 11 3	a e 3
d 13 14	b 14 17	a 16 18	f 18 6	c 15 1	e 17	f 13 17	c 18 8	b 14 6	a f 5
d 14 15	b 15 18	a 17 0	f 0 7	c 16 2	e 2 8	f 2 6	c 17 7	b 17 9	c f 12
d 15 16	b 16 0	a 18 1	f 1 8	c 17 3	e 3 9	f 10 14	c 16 6	b 1 12	c d 2
d 16 17	b 17 1	a 0 2	f 2 9	c 18 4	e 4 10	f 18 3	c 15 5	b 4 15	c e 7
d 17 18	b 18 2	a 1 3	f 3 10	c 0 5	e 5 11	f 7 11	c 14 4	b 7 18	b e 13
d 18 0	b 0 3	a 2 4	f 4 11	c 1 6	e 6 12	f 15 0	c 13 3	b 10 2	

References

[1] A. Chaiyasena, S.P. Hurd, N. Punnim and D. G. Sarvate, Group Divisible Designs with Two Association Classes, *J. Combin. Math. Combin. Comput.*, **82**, No.1 (2012), 179–198.

[2] C. J. Colbourn and D. H. Dinitz (eds), *Handbook of Combinatorial Designs(*2^{nd} *Edition)*, CRC Press, Boca Raton (2007).

[3] S. I. El-Zanati, N. Punnim and C. A. Rodger, Gregarious GDDs with Two Associate Classes, *Graphs Combin.*, **26**, No.6 (2010), 775–780.

[4] H. L. Fu and C. A. Rodger, Group Divisible Designs with Two Associate Classes: $n = 2$ or $m = 2$, *J. Combin. Theory (A)*, **83** (1998), 94–117.

[5] H. L. Fu, C. A. Rodger and D.G. Sarvate, The Existence of Group Divisible Designs with First and Second Associates, Having Block Size 3, *Ars Combin.*, **54** (2000), 33–50.

[6] L. Heffter, Über Nachbarconfigurationen, Tripesysteme and metacyklische Grupen, *Deutsche Mathem. Vereining. Jahresber.*, **5** (1896), 67–69.

[7] W. Lapchinda, N. Punnim and N. Pabhapote, GDDs with two associate classes and with group of sizes $1, n$ and n, *Australas. J. Combin.*, **58**, No. 2 (2014), 292–303.

[8] C. C. Lindner and C. A. Rodger, *Design Theory*, CRC Press, Boca Raton (1997).

[9] N. Pabhapote and N. Punnim, Group Divisible Designs with Two Associate Classes and $\lambda_2 = 1$, *Int. J. Math. Math. Sci.* (2011), Article ID 148580, 10 pages, doi:10.1155/2011/148580.

[10] N. Pabhapote and A. Chaiyasena, Group Divisible Designs with Two Associate Classes and $\lambda_2 = 3$, *Int. J. Pure Appl. Math.*, **71**, No. 3 (2011), 455–463.

[11] N. Punnim and C. Uiyyasathian, Group Divisible Designs with Two Associate Classes and $(\lambda_1, \lambda_2) = (1, 2)$, *J. Combin. Math. Combin. Comput.*, **82**, No. 1 (2012), 117–130.

[12] D. Raghavarao, *Construction and Combinatorial Problems in Design of Experiments*, Dover, New York (1988).

[13] R. Rees, Uniformly resolvable pairwise balanced designs with block sizes two and three, *Journal of Combinatorial theory, Series A*, **45**, No. 4 (1987), 207–225.

[14] C. Uiyyasathian and W. Lapchinda, Group Divisible Designs with Two Associate Classes and $\lambda_2 = 2$, *Int. J. Pure Appl. Math.*, **55**, No. 4 (2009), 561–568.

[15] C. Uiyyasathian and N. Pabhapote, Group Divisible Designs with Two Associate Classes and $\lambda_2 = 4$, *Int. J. Pure Appl. Math.*, **73**, No. 3 (2011), 289–298.