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THE STRUCTURE OF THE LIE ALGEBRA $\gamma_n(F)/[\gamma_n(F), F^{'}]$

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Abstract: Let F be a free Lie algebra of finite rank.

We show that $\gamma_n(F/F^{''})$ is free abelian of infinite rank and the algebra $\gamma_n(F) \cap F^{''}/[\gamma_n(F), F^{'}]$ is finitely generated and then we prove that the algebra $\gamma_n(F)/[\gamma_n(F), F^{'}]$ is infinitely generated.

Moreover, for sufficiently small values of n we show that $\gamma_n(F)/[\gamma_n(F), F^{'}]$ is free abelian.

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1. Introduction

Let F be a free Lie algebra freely generated by a set $X = \{x_1, x_2, ..., x_n\}$ over a field K of characteristic zero. One can construct a Hall set which is determined by the ordering given to the free generating set X (for details see [1]).

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$$\begin{aligned} H_1 &= X\ ,\quad X \text{ is well ordered} \\ H_2 &= \{(xy)\ /\ x,y \in X\ ,\ x > y\} \\ &\vdots \\ H_n &= \{(ab)c)\ a,b,c,bc \in \cup_{i=1}^{n-1} H_i\ ,\ a > b,\ b \leq c\ ,\ ab > c\} \end{aligned}$$

Now put $H = \cup_{n=1}^{\infty} H_n$. Then the set H forms a basis for the free Lie algebra F. [2]

The lower central series of F is defined inductively by:

$$\gamma_1(F) = F\ ,\ \gamma_{k+1}(F) = [\gamma_k(F), F]\ ,\ k \geq 1.$$

The second term $\gamma_2(F)$ is called the derived subalgebra and it is denoted by $F^{'}$. $F^{''}$ will be the the derived subalgebra of $F^{'}$.

As an easy consequence of the Jacobi identity we have $[\gamma_k(F), \gamma_n] \subseteq \gamma_{k+n}(F)$. Therefore any element in F of length$\geq n$ is contained in $\gamma_n(F)$. Hence $H_n \subseteq \gamma_n(F)$. The terms of the lower central series of F are not finitely generated as subalgebras for $n \geq 2$. Free generating sets for $\gamma_n(F)$ are given by A.L.Smelkin [3]. He has proved that the set C_n defined as

$$C_n = \{x = [a,b]\ /\ \ a,b \in H\ ,\ \text{length}(x) \geq n\ ,\ x \in H\ ,\ \text{length}(b) < m\}$$

is a set of free generators for $\gamma_n(F)$.

In this work we study the structure of the Lie algebra $\gamma_n(F)/[\gamma_n(F), F^{'}]$. Since $(\gamma_n(F))^{'}$ is contained in $\gamma_n(F)$ then $\gamma_n(F)/[\gamma_n(F), F^{'}]$ is abelian. Our main results are:

1) $\gamma_n(F)/[\gamma_n(F), F^{'}]$ is infinitely generated.

2) If $n = 2, 3, 4$ and $m \geq 2$ or $n = 5$ and $m = 2$ then $\gamma_n(F)/[\gamma_n(F), F^{'}]$ is free abelian of infinite rank.

2. The Lie Algebra $\gamma_n(F/F^{''})$

Let F be the free Lie algebra generated by the set $X = \{x_1, x_2, ..., x_n\}$.

The proof of the following theorem can be found in [1].

Theorem 1. (see [1]) *Let M be the free metabelian Lie algebra $F/F^{''}$. Then the derived subalgebra $M^{'}$ is a free abelian Lie algebra freely generated by the set C_2 modulo $F^{''}$.*

Lemma 2. *The algebra $\gamma_n(F/F^{''})$ is free abelian of infinite rank.*

Proof. $\gamma_n(F/F'') \subseteq \gamma_2(F/F'') = F'/F''$. Hence $\gamma_n(F/F'')$ is free abelian

$$\gamma_n(F/F'') = \gamma_n(F) + F''/F'' \simeq \gamma_n(F)/\gamma_n(F) \cap F''.$$

The algebra $\gamma_n(F)/\gamma_n(F) \cap F''$ freely generated by the set C_n modulo $\gamma_n(F) \cap F''$. So $\gamma_n(F/F'')$ freely generated by C_n modulo F''. Therefore rank of the algebra $\gamma_n(F/F'')$ is infinite. □

3. The Algebra $(\gamma_n(F) \cap F'')/[\gamma_n(F), F']$

In this section we use the method introduced in [4] for group case. The following theorem may be found in [5].

Theorem 3. *The algebra $\gamma_m(F)/\gamma_{m+1}(F)$ is free abelian with basis H_m.*

Theorem 4. *1. $(\gamma_n(F) \cap F'')/[\gamma_n(F), F']$ is finitely generated.*

2. If $n = 2, 3, 4$ and $m \geq 2$ then $(\gamma_n(F) \cap F'')/[\gamma_n(F), F']$ is free abelian.

3. If $n = 5$ and $m = 2$ then $(\gamma_n(F) \cap F'')/[\gamma_n(F), F']$ is free abelian.

Proof. Consider the algebra $F'/\gamma_n(F)$. We have

$$\gamma_{n+1}(F') = [\gamma_n(F'), F'] \subseteq [\gamma_n(F), F].$$

Hence there is an epimorphism

$$\gamma_n(F) \cap F''/\gamma_{n+1}(F') \longrightarrow \gamma_n(F) \cap F''/[\gamma_n(F), F'].$$

But $\gamma_n(F) \cap F''/\gamma_{n+1}(F') \subseteq F'/\gamma_{n+1}(F)$. It is well known that the algebra $F'/\gamma_{n+1}(F')$ is finitely generated. Hence the Lie algebra $(\gamma_n(F) \cap F'')/\gamma_{n+1}(F')$ is also finitely generated and so its homomorphic image

$$\gamma_n(F) \cap F''/[\gamma_n(F), F']$$

is finitely generated.

2. Let $n = 2, 3, 4$ and $m \geq 2$. Then

$$(\gamma_n(F) \cap F'')/[\gamma_n(F), F'] = F''/[\gamma_n(F), F'].$$

Computing

$$\gamma_2(F''/[\gamma_n(F), F']) \simeq \gamma_2(F'') + [\gamma_n(F), F']/[\gamma_n(F), F']$$

$$
\begin{aligned}
&\simeq \gamma_2(F^{''})/\gamma_2(F^{''}) \cap [\gamma_n(F), F^{'}] \\
&= \gamma_2(F^{''})/\gamma_2(F^{''})
\end{aligned}
$$

Hence the result follows.

3. $n = 5$ and $m = 2$. Then $\gamma_n(F) \cap F^{''} \subseteq F^{''}$. Hence the result is immediate from part 2 of the theorem. □

Corollary 5. *1) The algebra* $\gamma_n(F)/[\gamma_n(F), F^{'}]$ *is infinitely generated.*

2) If $n = 2, 3, 4$ *and* $m \geq 2$ *or* $n = 5$ *and* $m = 2$ *then* $\gamma_n(F)/[\gamma_n(F), F^{'}]$ *is free abelian of infinite rank.*

Proof. 1) $\gamma_n(F/F^{''})$ is free abelian of infinite rank by lemma(2). Thus

$$
\begin{aligned}
(\gamma_n(F)/[\gamma_n(F), F^{'}]) \diagup (\gamma_n(F) \cap F^{''})/[\gamma_n(F), F^{'}]) &\simeq \gamma_n(F) \diagup \gamma_n(F) \cap F^{''} \\
&\simeq \gamma_n(F) + F^{''} \diagup F^{''} \\
&\simeq \gamma_n(F/F^{''})
\end{aligned}
$$

and

$$
\gamma_n(F) \diagup [\gamma_n(F), F^{'}] \simeq \gamma_n(F) \cap F^{''} \diagup [\gamma_n(F), F^{'}] \oplus \gamma_n(F/F^{''})
$$

Hence the result follows.

2) If $n = 2, 3, 4$ and $m \geq 2$ or $n = 5$ and $m = 2$ then the algebra $(\gamma_n(F) \cap F^{''} \diagup [\gamma_n(F), F^{'}]$ is free abelian by Theorem(4). Rest of the proof is immediate from part (1) of the corollary. □

Corollary 6. $\gamma_n(F) \diagup [\gamma_n(F), F^{'}]$ *is free abelian if and only if* $\gamma_n(F) \cap F^{''} \diagup [\gamma_n(F), F^{'}]$ *is free abelian.*

References

[1] M. Boral, *Subalgebras of Free Nilpotent and Polynilpotent Lie Algebras*, Ph.D. Thesis, Univ. of st. Andrews, 1977.

[2] N. Bourbaki, *Groupes et Algebres de Lie*, Chapitre II., Hermann, Paris, 1972.

[3] A.L. Smelkin, Free polynilpotent groups (three parts): *Soviet Math. Dokl.* (Eng. Trans.), **4** (1963), 950-953, 1963; *Izvest. Akad. Nauk. S.S.S.R. Ser. Mat.* (Russian) **28** (1964), 91-122; *Dokl. Akad. Nauk. S.S.S.R.*, **169** (1966), 1024-1025.

[4] D. Spellmann, The structure of the abelian groups Fn/(Fn,F'), *J. Alg.* (1976), 301-308.

[5] M. Ward, Basic commutators, *Philos Trans. Roy. Soc. London, Ser A*, **264** (1964),343-412.