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ADJACENCY IN THE LATTICE OF ČECH CLOSURE OPERATORS

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Abstract: In this paper we investigate the adjacency relations in the lattice of Čech closure operators on a fixed set with special reference to T_1 Čech closure operators. The existence of upper and lower neighbours of some Čech closure operators are demonstrated. The concept of simple expansions of a Čech closure operator is also introduced.

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1. Introduction

Edward Čech in [3] introduced the concept of Čech closure operators on a set X, as a generalisation of Kuratowski closure operators. He showed that the set of all Čech closure operators on a set X is a complete lattice. This lattice is complemented if and only if X is finite[12]. A Čech closure operator on a set X is left fixed by every automorphism of $L(X)$ if and only if it is

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quasi discrete[13]. Here we consider some problems related to adjacency in the lattice of Čech closure operators on a set X. Hewitt introduced the concept of topological expansions for strengthening a given topology [4]. If τ is topology on a given set X and A is a subset of X not belonging to τ, then the simple expansion of τ by A is the smallest topology containing τ and A. We also investigate simple expansions in L(X).

2. Preleminaries

Let X be a set. A Čech closure operator on a set X is a function

$$V : P(X) \rightarrow P(X)$$

satisfying

$$V(\phi) = \phi, \quad A \subseteq V(A),$$

and $V(A \cup B) = V(A) \cup V(B)$ for every $A,\ B \in P(X)$. Here $P(X)$ denotes the power set of X. For brevity we call V a closure operator on X and the pair (X, V) a closure space.

A subset S of a closure space (X, V) is said to be closed if $V(S) = S$, and is said to be open if its complement is closed. The collection of all open sets in a closure space (X, V) is a topology on X, called the topology associated with V. A closure operator V is said to be topological if and only if V is idempotent.

Let $I : P(X) \rightarrow P(X)$ be given by

$$I(A) = \begin{cases} \phi & ; \text{ if } A = \phi \\ X & ; \text{ otherwise.} \end{cases}$$

Then I is a closure operator on X. This closure operator is the topological closure operator associated with the indiscrete topology on X and is called the indiscrete closure operator. The closure operator D on X given by $D(A) = A$ for all $A \in P(X)$, is the topological closure operator associated with the discrete topology on X, called the discrete closure operator.
Let V_1, V_2 be two closure operators on set X. Then V_1 is said to be coarser than V_2 if $V_2(A) \subseteq V_1(A)$ for every $A \in P(X)$ and is denoted by $V_1 \leq V_2$. This relation in the set of all closure operators on X is a partial order. The set of all closure operators on X forms a lattice under this partial order and is denoted by L(X). The smallest element of this lattice is the indiscrete closure operator I and the largest is the discrete closure operator D.

Let $\{V_a; a \in \mathscr{A}\}$ where $\mathscr{A}$ is some indexing set, be a non empty collection of closure operators on $L(X)$. A closure operator V which is the infimum of $\{V_a\}$

in $L(X)$ is given by $V(A) = \bigcup_{a\in\mathscr{A}} V_a(A)$ for $A \in P(X)$. If U is the supremum of a non empty collection $\{V_a\}$ in $L(X)$ and $A \in P(X)$, then $x \in U(A)$ if and only if for each finite cover $\{A_1, A_2, \ldots, A_n\}$ of A, there exists an A_i such that $x \in V_a(A_i)$ for each $a \in \mathscr{A}$.
For $x, y \in X$, $x \neq y$, define $V_{x,y}$ on $P(X)$ as follows.

$$V_{x,y}(A) = \begin{cases} \phi & ; \text{ if } A = \phi \\ X - \{y\} & ; \text{ if } A = \{x\} \\ X & ; \text{ otherwise.} \end{cases}$$

Then $V_{x,y}$ is a closure operator on X. Such closure operators are called infra closure operators on X. They are precisely the atoms of $L(X)$.

Dual atoms in the lattice of topologies are called ultra topologies. Frölich proved that the ultra topologies on X are precisely the topologies of the form $P(X - \{a\}) \cup \mathscr{U}$ where $a \in X$ and $\mathscr{U}$ is an ultra filter on X, not containing $\{a\}$. The closure operator associated with the ultra topology $P(X - \{a\}) \cup \mathscr{U}$ is given by

$$V(A) = \begin{cases} A & ; \text{ if } a \in A \text{ or } X - A \in \mathscr{U} \\ A \cup \{a\} & ; \text{ otherwise} \end{cases}$$

and is called the ultra closure operator. In $L(X)$, it is proved that infra closure operators are smaller than or equal to any non principal ultra closure operator[12].

The co-finite closure operator on X is given by the function $C_0 : P(X) \to P(X)$ defined by

$$C_0(A) = \begin{cases} A & ; \text{ if } A \text{ is finite} \\ X & ; \text{ otherwise} \end{cases}$$

More generally we define C_α on $P(X)$, where α is any infinite cardinal number such that $\alpha \leq |X|$, by,

$$C_\alpha(A) = \begin{cases} A & ; \text{ if } |A| < \alpha \\ X & ; \text{ otherwise} \end{cases}$$

When $\alpha = \aleph_0$, we get that $C_\alpha = C_{\aleph_0} = C_0$.

3. On Upper Neighbours of Čech Closure Operators

In this section, we discuss some properties related to the adjacency in the lattice $L(X)$ of closure operators. Here various types of upper neighbours in $L(X)$ like topological upper neighbours, strictly topological upper neighbours are defined and investigated.

Definition 1. Let U and V be two closure operators on X such that $U < V$. Then V is called an upper neighbour of U if for a closure operator W on X such that $U \leq W \leq V$, either $U = W$ or $W = V$.

Remark 2. If V is an upper neighbour of U, then U is called a lower neighbour of V. In this case, U and V are said to be adjacent.

Example 3.

(a) The closure operator $V_{x,y}$ and the indiscrete closure operator I are adjacent. $V_{x,y}$ is an upper neighbour of I.

(b) The discrete closure operator D and the ultra closure operator are adjacent.

Definition 4. Let U and V be two topological closure operators on X such that $U < V$. Then V is called a topological upper neighbour of U, if W is any closure operator such that $U \leq W \leq V$, then W is not topological. If V is a topological upper neighbour of the topological closure operator U, then U is a topological lower neighbour of V. In this case U and V are said to be topologically adjacent.

Definition 5. Let U and V be two topological closure operators on X. Then V is called a strict topological upper neighbour of the topological closure operator U if there is no closure operator W on X such that $U < W < V$. If V is a strict topological upper neighbour of the topological closure operator U, then U is a strict topological lower neighbour of V.Then U and V are said to be strictly topologically adjacent.

Example 6. The discrete closure operator D and an ultra closure operator are strictly topologically adjacent.

Remark 7. (i) Let V be a closure operator on X. Then for $x, y \in X$, $x \neq y$, $V_{x,y} \nleq V$ if and only if $y \in V(\{x\})$.

(ii) Let V be a closure operator on X such that $V \neq I$, and $V_{x,y} \nleq V$,then the closure operator $U = V \vee V_{x,y}$ is given by

$$U(S) = \begin{cases} V(S) - \{y\}, & \text{if } x \in S \text{ and } y \notin V(S - \{x\}) \\ V(S) & \textit{otherwise.} \end{cases}$$

Then clearly, U is finer than V, but U need not be an upper neighbour of V.

Lemma 8. *Let V be a closure operator on X such that $V \neq I$ on $L(X)$. For $x, y \in X$, $x \neq y$, let $V_{x,y} \nleq V$. Then the closure operator $U = V \vee V_{x,y}$ is an upper neighbour of V if and only if y is in the closure of every element of X.*

Proof. If U is an upper neighbour of V and W is a closure operator on X such that $V \leq W \leq U$, then either $V = W$ or $W = U$. Suppose $W \neq V$. Then

$$W(S) = \begin{cases} V(S) - \{y\} & ; \; if S = \{x\} \\ V(S) & ; \; otherwise. \end{cases}$$

is the least closure operator finer than V and coarser than U. Therefore $W = U$ implies y is in the closure of every element of X.
On the other hand

$$U(S) = \begin{cases} V(S) - \{y\} & ; \; if S = \{x\} \\ V(S) & ; \; otherwise. \end{cases}$$

Obviously U is an upper neighbour of V. □

Example 9. Let $X = \{x, y, z, p\}$, V be a closure operator on X defined by $V(\phi) = \phi$, $V(\{x\}) = \{x, y\} = V(\{y\})$, $V(\{z\}) = \{x, y, z\}$, $V(\{p\}) = \{x, y, p\}$. Then $U = V \vee V_{x,y}$ and V are strictly topologically adjacent.

In the following results X is an infinite set and α denotes an infinite cardinal number such that $\alpha \leq |X|$.

Definition 10. Let $a, b \in X$ with $a \neq b$. Define a function $C^{\alpha}_{a,b} : P(X) \to P(X)$ by,

$$C^{\alpha}_{a,b}(S) = \begin{cases} S & ; \quad \text{if } |S| < \alpha \; a \notin S \\ S \cup \{b\} & ; \quad \text{if } |S| < \alpha, a \in S \\ X & ; \quad \text{if } |S| \geq \alpha \end{cases}$$

Then $C^{\alpha}_{a,b}$ is a closure operator on X and $C^{\alpha}_{a,b} < C_{\alpha}$.

Theorem 11. *The closure operator $C^{\alpha}_{a,b}$ is a lower neighbour C_{α} and any lower neighbour of C_{α} is of the form $C^{\alpha}_{a,b}$.*

Proof. Let V be any closure operator satisfying $C^{\alpha}_{a,b} \leq V \leq C_{\alpha}$. Then for any S with $|S| \geq \alpha$, $V(S) = X$. If $x \in X$, $x \neq a$, then $\{x\} \subseteq V(\{x\}) \subseteq C^{\alpha}_{a,b}(\{x\})$. Thus $V(\{x\}) = \{x\}$ for every $x \in X$, $x \neq a$. Also since $C^{\alpha}_{a,b} < V$, $\{a\} \subseteq V(\{a\}) \subset C^{\alpha}_{a,b}(\{a\}) = \{a, b\}$. Hence $V(\{a\}) = \{a\}$. This shows that $V = C_{\alpha}$ and thus $C^{\alpha}_{a,b}$ is a lower neighbour of C_{α}. If $V \in L(X)$ and $V < C_{\alpha}$, then there is some $a \in X$ with $V(\{a\}) \neq \{a\}$. If $|S| \geq \alpha$, since '$V < C_{\alpha}$, we have $C^{\alpha}_{a,b}(S) = C_{\alpha}(S) = X \subseteq V(S)$. Thus $V(S) = X$. If $S \subset X$, $|S| < \alpha$ and $a \notin S$, then $S = C^{\alpha}_{a,b}(S) = C_{\alpha}(S) \subseteq V(S)$. If $|S| < \alpha$ and $a \in S$, then $b \in V(S)$ and $C^{\alpha}_{a,b}(S) = S \cup \{b\}$. Hence $C^{\alpha}_{a,b}(S) = S \cup \{b\} \subseteq V(S)$. Thus $V \leq C^{\alpha}_{a,b}$. □

Remark 12. If $\alpha = \aleph_0$, then $C^{\alpha}_{a,b}$ is given by

$$W_{a,b}(S) = \begin{cases} S & ; \text{ if S is finite and } a \notin S \\ S \cup \{b\} & ; \text{ if S is finite and } a \in S \\ X & ; \text{ if S is infinite.} \end{cases}$$

and is a lower neighbour of C_0, the co-finite closure operator and any lower neighbour of C_0 is of the form $W_{a,b}$.

Definition 13. [13] Let $x \in X$. Suppose A is an infinite subset of X not containing x. Define $V_{A,x} : P(X) \to P(X)$ as

$$V_{A,x}(S) = \begin{cases} S & ; \ if\, S\, is\, finite \\ X - \{x\} & ; \ if\, S \text{ is infinite, } S - A \text{ finite and } x \notin S \\ X & ; \ otherwise. \end{cases}$$

Then $V_{A,x}$ is a closure operator on X and $C_0 < V_{A,x}$.

Remark 14. If $B \subseteq A \subseteq X$, then $V_{B,x} \leq V_{A,x}$.

We can generalise this definition as

Definition 15. Let X be any set. Let $\alpha \leq |A|$ and x an element of $(X - A)$. Then define $C^{\alpha}_{A,x} : P(X) \longrightarrow P(X)$ by,

$$C^{\alpha}_{A,x}(S) = \begin{cases} S & \text{if } |S| < \alpha \\ X - \{x\} & \text{if } |S| \geq \alpha, |S - A| < \alpha \text{ and } x \notin S \\ X & \text{otherwise} \end{cases}$$

Then $C^{\alpha}_{A,x}$ is an element of $L(X)$ and $C_{\alpha} < C^{\alpha}_{A,x}$ and thus $C^{\alpha}_{A,x} \in (C_{\alpha}, D)$.

Remark 16. If $B \subseteq A \subseteq X$ such that $\alpha \leq |B|$, then $y \in C^{\alpha}_{A,x}(S), S \subseteq X$ implies that $y \in C^{\alpha}_{B,x}(S)$. Thus $C^{\alpha}_{B,x} \leq C^{\alpha}_{A,x}$.

Theorem 17. *If A and B are two distinct subsets of X such that $\alpha \leq |A|$ and $\alpha \leq |B|$ and $x \in X - (A \cup B)$, then $C^{\alpha}_{A,x}$ and $C^{\alpha}_{B,x}$ are equal if and only if $|A \Delta B| < \alpha$.*

Proof. Suppose $|A \Delta B| < \alpha$. Then $|A - B| < \alpha$ and $|B - A| < \alpha$. Thus we have $|S - A| < \alpha$ if and only if $|S - B| < \alpha$. Thus $C^{\alpha}_{A,x}$ and $C^{\alpha}_{B,x}$ are equal, by definition.
Conversely assume that $C^{\alpha}_{A,x} = C^{\alpha}_{B,x}$. Then $C^{\alpha}_{A,x}(A) = C^{\alpha}_{B,x}(A) = X - \{x\}$. Hence $|A - B| < \alpha$. Also $C^{\alpha}_{A,x}(B) = C^{\alpha}_{B,x}(B) = X - \{x\}$. Then $|B - A| < \alpha$. Thus $|A \Delta B| < \alpha$. □

Lemma 18. *The closure operator $C^{\alpha}_{A,x}$ is topological if and only if $|X - A| < \alpha$.*

Proof. If $|X - A| < \alpha$, $C^{\alpha}_{A,x} = C^{\alpha}_{X-\{x\},x}$ by Theorem 17 . Hence if $|S| \geq \alpha$, $|S - A| < \alpha$ and $x \notin S$, then $C^{\alpha}_{A,x}(C^{\alpha}_{A,x}(S)) = C^{\alpha}_{A,x}(X - \{x\}) = X - \{x\} = C^{\alpha}_{A,x}(S)$. If $|X - A| \geq \alpha$, then $C^{\alpha}_{A,x}(A) = X - \{x\}$ and $C^{\alpha}_{A,x}(C^{\alpha}_{A,x}(A)) = X$. Thus $C^{\alpha}_{A,x}$ is not topological. □

Lemma 19. *If $A \subset X$, $|A| \geq \alpha$, and $x \in X - A$, then there is a subset D of A such that $C^{\alpha}_{D,x} < C^{\alpha}_{A,x}$.*

Proof. Since $|A| \geq \alpha$, there exists a subset B of A with $|B| = \alpha$. Since $\alpha + \alpha = \alpha$, B has a subset D with $|D| = \alpha$ and $|B \Delta D| = \alpha$. Hence by Theorem 17, $C^{\alpha}_{D,x} < C^{\alpha}_{B,x} \leq C^{\alpha}_{A,x}$.

□

Lemma 20. *Let V be a closure operator on X such that $C_{\alpha} < V$. Then $C^{\alpha}_{A,x} \leq V$ if and only if $x \notin V(A)$.*

Proof. Suppose $C^{\alpha}_{A,x} \leq V$. Then $V(A) \subseteq C^{\alpha}_{A,x}(A) = X - \{x\}$. Now if $x \notin V(A)$, $V(A) \subseteq X - \{x\} = C^{\alpha}_{A,x}(A)$. Hence the result. □

Remark 21. When $\alpha = \aleph_0$, we get that $C^{\alpha}_{A,x} = V_{A,x}$. Then $V_{A,x} \leq V$ if and only if $x \notin V(A)$.

Theorem 22. *The closure operator C_{α} has no upper neighbour in the lattice of closure operators.*

Proof. Let V be any closure operator in $L(X)$ satisfying $C_{\alpha} < V$. When $|S| < \alpha$, $C_{\alpha}(S) = S$. Choose $A \subset X$ with $|A| \geq \alpha$ and $V(A) \subset C_{\alpha}(A) = X$. Thus there is $x \in X - V(A)$. Hence $C^{\alpha}_{A,x} \leq V$ by Lemma 20. Thus there are closure operators $C^{\alpha}_{D,x}$ and $C_{B,x}$ such that $C^{\alpha}_{D,x} < C^{\alpha}_{B,x} \leq C^{\alpha}_{A,x}$. Hence $C_{\alpha} < C^{\alpha}_{D,x} < C^{\alpha}_{B,x} \leq C^{\alpha}_{A,x} \leq V$. □

Remark 23. If $\alpha = \aleph_0$, C_{α} is the co-finite closure operator and thus above theorem says co-finite closure operator has no upper neighbour in the lattice of closure operators.

Theorem 24. *Let V be any closure operator on X such that $C_0 < V$. Then V is the join of all closure operators of the form $V_{A,x}$ where A is an infinite subset of X and $x \in X - A$.*

Proof. Let U be the join of all closure operators of the form $V_{A,x} \leq V$. Clearly, $U \leq V$. If $U \neq V$, then there is a $B \subseteq X$ and $x \in U(B)$, but $x \notin V(B)$. If B is finite, $V(B) = U(B) = B$. Thus if B is infinite and $x \notin B$. Then $V_{B,x} \leq V$. Consequently, we have that $V_{B,x} \leq U$. Then $x \notin U(B)$, which is a contradiction. Hence the result.

□

We have by Lemma 18, $V_{A,x}$ is topological if and only if $X - A$ is finite. We denote $V_{A,x}$ by V_x when $X - A$ is finite.

Theorem 25. *Let $x \in X$. Then V_x has no upper neighbour in $[C_0, D]$.*

Proof. Let $V \in [C_0, D]$ such that $V_x < V$. Then there is $A \subset X$ such that $V(A) \subset V_x(A)$. Then there is a $y \in V_x(A)$ such that $y \notin V(A)$. Then $y \neq x$. For, if $y = x$, then $x \notin A$. Then $x = y \in V_x(A) = X - \{x\}$, a contradiction. Further we claim that A is infinite. For, if A is finite, $y \notin V_x(A) = A$, a contradiction. Then $V_{A,y} \leq V$, since $y \notin V(A)$. There is an infinite subset C of A such that $V_{C,y} < V_{A,y}$. Let $V^{'} = V_x \bigvee V_{C,y}$. Then $V_x < V$ and $V_{C,y} < V_{A,y} \leq V$ implies that $V^{'} = V_x \bigvee V_{C,y} < V$. Also $V_x < V_x \bigvee V_{C,y} = V^{'}$. Thus V_x does not have an upper neighbour. □

By modifying this proof we can also prove that $V_{A,x}$ cannot have an upper neighbour in $L(X)$.

Theorem 26. *Let A be an infinite subset of X and $x \in X - A$. Then $V_{A,x}$ has no upper neighbour in $L(X)$.*

Proof. Let V be a closure operator such that $V_{A,x} < V$. Then there exists $B \subset X$ such that $V(B) \subset V_{A,x}(B)$. Then there exists $y \in V_{A,x}(B)$ such that $y \notin V(B)$. Then B is infinite and $y \notin B$.
Case (i) : $y \neq x$
Since $y \notin V(B)$ we get that $V_{B,y} \leq V$. By Lemma 19, there exists a subset C

of B such that $V_{C,y} < V_{B,y} \leq V$. Let $V' = V_{A,x} \bigvee V_{C,y}$. Then $V_{A,x} < V'$ for otherwise $V_{C,y} \leq V_{A,x}$. Also we have that
$V' < V$ since $V' = V_{A,x} \bigvee V_{C,y} < V_{A,x} \bigvee V_{B,y} \leq V$.
Thus $V_{A,x} < V' < V$.
Case (ii): $y = x$
Then $x \in V_{A,x}(B)$. Then $B-A$ is infinite. Let $\{b_1, b_2, \ldots\}$ be a countable infinite subset of $B - A$. Let $C = \{b_2, b_4, \ldots\}$. Then $V_{C,x} < V_{B,x}$. But $x = y \notin V(B)$. Then $V_{B,x} \leq V$. Thus $V_{C,x} < V_{B,x} < V$. Let $V' = V_{A\cup C,x}$.
Then $V_{A,x} < V'$. Also $V' = V_{A\cup C,x} < V_{A\cup B,x}$, $x \notin V(B)$. Also $V(A) \subset V_{A,x}(A) = X - \{x\}$. Thus $x \notin V(A)$ Then $x \notin V(A \cup B) = V(A) \cup V(B)$. Thus $V_{A\cup B,x} \leq V$. Consequently, we obtain that $V_{A,x}$ has no upper neighbour in both the cases. □

Remark 27. In the lattice of T_1 topologies, the smallest element co-finite topology and the atoms have upper neighbours. But in $L(X)$, co-finite closure operator C_0 and its strictly topological upper neighbour V_x do not have an upper neighbour.

Theorem 28. *Suppose A and B are two infinite subsets of X, $x \notin A \cup B$, then $V_{A,x} \bigvee V_{B,x} = V_{A\cup B,x}$.*

Proof. By Remark 14, $V_{A,x} \leq V_{A\cup B,x}(S)$ and $V_{B,x} \leq V_{A\cup B,x}(S)$. Hence $V_{A,x} \bigvee V_{B,x} \leq V_{A\cup B,x}$.

To prove $V_{A\cup B,x} \leq V_{A,x} \bigvee V_{B,x}$, it is enough to prove that $x \notin (V_{A,x} \bigvee V_{B,x})(A \cup B)$, by Remark 21. We have $(V_{A,x} \bigvee V_{B,x})(A) \subseteq V_{A,x}(A)$ and $(V_{A,x} \bigvee V_{B,x})(B) \subseteq V_{B,x}(B)$. Since $x \notin V_{A,x}(A)$, $x \notin (V_{A,x} \bigvee V_{B,x})(A)$. Similarly we can prove that $x \notin (V_{A,x} \bigvee V_{B,x})(B)$. Therefore $x \notin (V_{A,x} \bigvee V_{B,x})(A\cup B) = V$. □

Theorem 29. *If $A_1, A_2, \ldots, A_n$ is a finite collection of infinite subsets of X such that $x \notin A_i$ for any $i \in \{1, 2, \ldots, n\}$, then $V_{A_1,x} \bigvee V_{A_2,x} \bigvee \ldots \bigvee V_{A_n,x} = V_{A_1\cup A_2\cup\ldots\cup A_n,x}$.*

Proof. Similar to the proof of Theorem 28. □

4. Simple Expansions in $L(X)$

The concept of simple expansions is very important in the study of upper neighbours in the lattice of topologies. The simple expansion $\tau(A)$ of the topology τ by a subset A of X, $A \notin \tau$, is the join of τ and the atom $\{\phi, A, X\}$. In this section we introduce an analogous concept in the lattice of Čech closure operators.

Definition 30. Let A be a non empty proper subset of X and $x \in A$. Define, $V_{(A,x)} : P(X) \to P(X)$ by,

$$V_{(A,x)}(S) = \begin{cases} \phi & ; \text{ if } S = \phi \\ X - \{x\} & ; \text{ if } S \subseteq X - A \text{ and } S \neq \phi \\ X & ; \text{ otherwise} \end{cases}$$

Then $V_{(A,x)}$ is a closure operator on X.

Remark 31. 1. If $A = X - \{y\}$, x, $y \in X$ and $x \neq y$, then $V_{(A,x)}$ is

$$V_{(X-\{y\},x)}(S) = \begin{cases} \phi & ; \text{ if } S = \phi \\ X - \{x\} & ; \text{ if } S = \{y\} \\ X & ; \text{ otherwise} \end{cases}$$

This shows that $V_{(X-\{y\},x)} = V_{y,x}$.
2. If $A = \{x\}$

$$V_{(\{x\},x)}(S) = \begin{cases} \phi & ; \text{ if } S = \phi \\ X - \{x\} & ; \text{ if } x \notin S \neq \phi \\ X & ; \text{ otherwise} \end{cases}$$

Theorem 32. *The closure operator $V_{(A,x)}$ is topological if and only if $A = \{x\}$.*

Proof. If $S = \phi$ or $S = X$, $V_{(\{x\},x)}\ (V_{(\{x\},x)}(S)) = V_{(\{x\},x)}(S)$. If $x \notin S$, $V_{(\{x\},x)}(V_{(\{x\},x)}(S)) = V_{(\{x\},x)}(X - \{x\}) = X - \{x\} = V_{(\{x\},x)}(S)$. Conversely assume that $V_{(A,x)}$ is topological. Let $S \subseteq X - A$, $S \neq \phi$. Then $V_{(A,x)}(X - \{x\}) = X - \{x\}$. Thus $X - \{x\} \subseteq X - A$. Since A is non empty $A = \{x\}$. □

Remark 33. The closed subsets of X with respect to the closure operator $V_{(\{x\},x)}$ are ϕ, $X - \{x\}$ and X. Hence $V_{(\{x\},x)}$ is the topological closure operator of the infra topology $\{\phi, \{x\}, X\}$. The closure operator $V_{(X-\{y\},x)}$ is not topological when $|X| \geq 3$.

Lemma 34. *If A is a proper subset of X, then for every closure operator of the form $V_{(A,x)}$ on X, there is an element $y \in X - A$ such that, $V_{(X-\{y\},x)} \leq V_{(A,x)} \leq V_{(\{x\},x)}$.*

Proof. If $|X| = 2$, then $A \neq X$, otherwise $V_{(A,x)} = I$, the indiscrete closure operator. In this case, $V_{(X-\{y\},x)} = V_{(A,x)} = V_{(\{x\},x)}$. Let $|X| \geq 3$. Then $A \neq X$. Let $y \in X - A$, then clearly, $V_{(X-\{y\},x)} \leq V_{(A,x)} \leq V_{(\{x\},x)}$. □

If $|X| \geq 3$, then the closure operator $V_{(A,x)}$ is not topological for any proper subsets A of X such that A is different from $\{x\}$. In this case the topology associated with $V_{(A,x)}$ is the indiscrete topology.

Using the above concept we introduce simple expansion of a closure operator in the following way.

Definition 35. Let V be any closure operator on X and A be a subset of X such that $x \in A$. The closure operator $V_A^x = V \bigvee V_{(A,x)}$ is called a simple expansion of V by A at x. The closure operator V_A^x is given by,

$$V_A^x(S) = \begin{cases} V(S) - \{x\} & ; \quad \text{if } S \cap (X - A) \neq \phi \text{ and } x \notin V(S \cap A) \\ V(S) & ; \quad \text{otherwise} \end{cases}$$

Lemma 36. *$V_A^x = V$ if and only if $x \notin V(X - A)$.*

Proof. Suppose $V_A^x(S) = V$ then we have $V_A^x(S) = V(S)$ for every subset S of X. Hence if $S \cap (X - A) \neq \phi$ and $x \notin V(S \cap A)$ then $x \notin V(S)$. This implies $x \notin V(X - A)$. If $x \notin V(X - A)$, then from the definition $V_A^x = V$. □

Theorem 37. *The simple expansion ${C_0}_A^x$ of C_0 by A at x is topological if and only if A is finite.*

Proof. We have,

$${C_0}_A^x(S) = \begin{cases} S & ; \quad \text{if } S \text{ is finite,} \\ X - \{x\} & ; \quad \text{if } S \cap (X - A) \neq \phi \text{ and } x \notin C_0(S \cap A) \\ X & ; \quad \text{otherwise} \end{cases}$$

If ${C_0}_A^x$ is topological, then ${C_0}_A^x({C_0}_A^x(S)) = {C_0}_A^x(S)$ for every subset S of X. Thus if $S \cap (X - A) \neq \phi$ and $x \notin C_0(S \cap A)$, then ${C_0}_A^x(S) = X - \{x\}$ and ${C_0}_A^x(X - \{x\}) = X - \{x\}$. This is when $x \notin C_0(S \cap A)$ implies $S \cap A$ is finite. Since ${C_0}_A^x(X - \{x\}) = X - \{x\}$, A must be finite. If A is finite, then for every infinite set S not containing x, we have $x \notin C_0(S \cap A)$. Therefore for every infinite subset S of X such that $x \notin S$, ${C_0}_A^x(S) = X - \{x\}$ and ${C_0}_A^x(X - \{x\}) = X - \{x\}$. □

Lemma 38. *The closure operator $V_{A,x}$ is a simple expansion of C_0 by $X - A$ at x. In other words $V_{A,x} = V_{(X-A,x)} \bigvee C_0$.*

Proof. As we have noted before, $C_0 \leq V_{A,x}$ and $V_{(X-A,x)} \leq V_{A,x}$. Therefore $V_{(X-A,x)} \bigvee C_0 \leq V_{A,x}$. Let $S \subseteq X$. Now suppose $x \in (V_{(X-A,x)} \bigvee C_0)(S)$. Then for every finite cover $\{S_i\}$ of S, there exists a S_j such that $x \in C_0(S_j)$ and $x \in V_{(X-A,x)}(S_j)$. Thus S_j is infinite and S_j not contained in A. Since $S_j \subseteq S$, we have $x \in V_{A,x}(S)$. Hence $V_{A,x} \leq V_{(X-A,x)} \bigvee C_0$. □

Remark 39. 1. The simple expansion ${C_0}_A^x$ of C_0 by A at x when A is finite is V_x.
2. The simple expansion ${C_0}_A^x$ of C_0 by A at x and the simple expansion ${C_0}_B^x$ of C_0 by B at x are equal if and only if $A \Delta B$ is finite.
3.Let V be a closure operator on X. If $X - A$ is a singleton set, say $\{y\}$, then the simple expansion V_A^x of V by A at x is an upper neighbour of V if and only if x is in the closure of every element of X. (Refer Lemma 8)

Proposition 40. *Let V be a closure operator on X, $B \subseteq A \subseteq X$ and $x \in B$. Then $V_A^x \leq V_B^x$.*

Proof. If $B \subseteq A$, then $V_{(A,x)} \leq V_{(B,x)}$, Thus $V \bigvee V_{(A,x)} \leq V \bigvee V_{(B,x)}$. Hence $V_A^x \leq V_B^x$. □

Example 41. The converse of this result is not true. Let X be the set of all positive integers. Take $A = \{x \in X \,|\, x = 1 \text{ or } x \text{ is even } \}$ and $B = \{x \in X \,|\, x = 3 \text{ or } x \text{ is even } \}$. Then $X - A$, $X - B$ are not comparable and $A \Delta B$ is finite. But $C_{0(X-A)}^x$ and $C_{0(X-B)}^x$ are equal.

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References

[1] Agashe P. and Levine N., Adjacent Topologies, *J. Math. Tokushima Univ.*, **7** (1973), 21-35.

[2] Birkhoff G., *Lattice Theory*, Amer. Math. Soc. Colloq. Pub. **25**, Third Edn. (1967), doi: http//dx.doi.org/10.2307/2268183.

[3] Čech E., *Topological Spaces*, Rev. Ed. Wiley, New York(1966), Library of Congress Catalogue Card Number 66-13066.

[4] Hewitt E., A Problem of Set Theoretic Topology, *Duke Math. J.*, **10** (1943), 309-333, doi: 10.1215/S0012-7094-43-01029-4.

[5] Kunheenkutty M., *On Simple Expansions of Topologies*, Proceedings, K. M. A. National Seminar on New Vistas in Topology, KKTM Govt College, Pullut, Kodungallur, Kerala (2005), 11-15.

[6] Kunheenkutty M., *Comparison of Simple Expansions of Topologies*, Proceedings, International Seminar on Recent Trends in Topology and its Applications, St. Joseph's College, Irinjalakuda, Kerala (2009), 99-105.

[7] Kunheenkutty M., *Some Problems on Local Properties of the Lattice of Topologies*, Thesis for Ph. D. Degree, University of Calicut(2013).

[8] Kunheenkutty M., On Simple Expansions of Topologies by Singletons, *Bulletin of Kerala Mathematics Association*, **10**, No. 1 (2013), 1-6.

[9] Larson R. E. and Andima S.J., The Lattice of Topologies: A Survey, *Rocky Mountain J. Math.*, **5** (1975), 177-198, doi: 10.1216/RMJ-1975-5-2-177.

[10] Levine N., Simple Extensions of Topologies, *Amer. Math. Mon.*, **71** (1964) 22-25, doi: 10.2307/2311297.

[11] Ramachandran P.T., *Some Problems in Set Topology Relating Group of Homeomorphisms and Order*, Thesis for Ph. D. Degree, CUSAT (1985).

[12] Ramachandran P.T., Complementation in the Lattice of Čech Closure Operators, *Indian J. Pure. App. Math.*, **18** No. 2 (1987), 152-158.

[13] Ramachandran P.T., *Fixed Points of the Lattice of Čech Closure Operators*, Proceedings, U.G.C. Sponsored National Seminar on Discrete Analysis at Sri Ramakrishna Mission Vidyalaya College of Arts and Science (Autonomous), Coimbatore 641020 (1999), 29-31.

[14] Ramachandran P.T., Complemented Elements in the Lattice of Čech Closure Operators, *J. Comp. & Math. Sci.* **3** No. 5 (2012), 498-501.

[15] Robert R. Stoll, *Set Theory and Logic*, Dover Publication, Inc. New York(1961), Library of Congress Catalog Card Number 79-87812.

[16] Steiner A. K., The Lattice of Topologies: Stucture and Complementation, *Trans. Amer. Math. Soc.* **122** (1966) 379-398, doi: 10.2307/1994555.