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A CONTRIBUTION TO THE STUDY OF THE PROBABILISTIC HAUSDORFF METRIC

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Abstract: The purpose of the present paper is to study several important properties of the probabilistic Hausdorff metric. Indeed, we prove that the collection of nonempty compact subsets of a probabilistic metric space is complete with respect to the probabilistic Hausdorff metric if and only if the probabilistic metric space is complete . As a consequence, the relationship between completion of the probabilistic Hausdorff metric for a given PM space and probabilistic Hausdorff metric of its completion is examined. Finally, Deterministic counterpart of these results is deduced. Our work extend some results of [2] and [12].

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1. Introduction and Preliminaries

The concept of probabilistic metric space was first introduced and studied in 1942 by K.Menger [7]. It is a probabilistic generalization of metric space in which the distance $d(x, y)$ between two point x, y was replaced by a real function F_{xy} whose value $F_{xy}(t)$, for any positive real t, is interpreted as the probability

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that the distance between x and y is less than or equal to t. The study of these spaces was performed extensively by B.Schweizer and A.Sklar [9].

We briefly recall some definitions and known results about probabilistic metric spaces, for more detail we refer the reader to [9], Recall that a non-negative real function f defined on $\overline{\mathbb{R}^+} = \mathbb{R}^+ \cup \{+\infty\}$ is known as a distance distribution function (briefly, a d.d.f) if it is non-decreasing, left continuous on $]0, +\infty[$, with $f(0) = 0$ and $f(+\infty) = 1$. The set of all d.d.f will be denoted by Δ^+.

A simple example of distribution function is Heavyside function defined by:

$$H(t) = \begin{cases} 0 & \text{if} \quad t \leq 0, \\ 1 & \text{if} \quad t > 0. \end{cases}$$

In the sequel we will define some functions, on $\mathbb{R}^+$ and consider them automatically extended to $\overline{\mathbb{R}^+}$.

According to [9] a commutative, associative and non-decreasing mapping $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ is called a triangle function if and only if $\tau(f, H) = f$ for all $f \in \Delta^+$.

A triangle function τ is called sup-continuous, if for any family $\{F_i : i \in I\}$ of Δ^+ and any $G \in \Delta^+$ we have $\tau(\sup_{i\in I} F_i, G) = \sup_{i\in I} \tau(F_i, G)$.

Definition 1.1. Let f and g be in Δ^+, let h be in $(0, 1]$, and let $(f,g;h)$ denote the condition

$$0 \leq g(x) \leq f(x + h) + h \quad \text{for all} \quad x \in (0, \frac{1}{h})$$

The modified Lévy distance is the function d_L defined on $\Delta^+ \times \Delta^+$ by

$$d_L(f, g) = \inf\{h : \text{ both } (f, g; h) \text{ and } (g, f; h) \text{ hold}\}.$$

Lemma 1.1. *([9])*

(1) *The pair (Δ^+, d_L) is a compact metric space.*

(2) *For any $t > 0$*

$$f(t) > 1 - t \quad \textit{if and only if} \quad d_L(f, H) < t$$

(3) *If f and g are in Δ^+ and $f \leq g$. Then $d_L(g, H) \leq d_L(f, H)$*

Definition 1.2. By a probabilistic metric space we mean a triplet (M, F, τ) such that M is nonempty set,τ is a triangle function and F is a mapping from $M \times M$ into Δ^+ satisfying the following conditions for all p, q, r in M :

(i) $F_{pp} = H$,

(ii) $F_{pq} \neq H$ if $p \neq q$,

(iii) $F_{pq} = F_{qp}$;

(iv) $F_{pq} \geq \tau(F_{pr}, F_{rq})$.

For a given PM space (M, F, τ), B. Schweizer and A.Sklar ([9]) point out that If τ is continuous, then F generates a first countable Hausdorff topology $\mathfrak{T}_F$ on M which has as a base the family of sets of the form $\{N_x(t) : x \in M, t > 0\}$, where $N_x(t) = \{y \in M : F_{xy}(t) > 1 - t\}$. On the other hand $\{U_t : t > 0\}$ is a base for a uniformity $\mathcal{U}(F)$ on M compatible with $\mathfrak{T}_F$, where $U_t = \{(x, y) \in M \times M : F_{xy}(t) > 1 - t\}$ for all $t > 0$. $\mathcal{U}(F)$ is called the strong uniformity induced by F. In the sequel, when we speak about a PM space (M, F, τ), we always assume that τ is continuous.

In virtue of $\mathfrak{T}_F$, a sequence $(x_n)_{n\in\mathbb{N}}$ in M, is said to be convergent to x (we write $x_n \to x$ or $\lim\limits_{n\to\infty} x_n = x$) if

$$\forall t > 0 \; \exists n_0 \in \mathbb{N} \quad \text{such that } n \geq n_0 \Rightarrow F_{x_n x}(t) > 1 - t,$$

A sequence $(x_n)_{n\in\mathbb{N}}$ in M is said to be Cauchy if

$$\forall t > 0 \; \exists n_0 \in \mathbb{N} \quad \text{such that } n, m \geq n_0 \Rightarrow F_{x_n x}(t) > 1 - t,$$

The PM space (M, F, τ) is said to be complete, if every Cauchy sequence in M converges to some point in M.

Following [10] (see also, [9]) a completion of a PM space (M, F, τ) is a pair $((N, G, \sigma), \Phi)$, where (N, G, σ) is a complete PM space and Φ is an isometry from M into N (i.e. $F_{pq} = G_{\Phi(p)\Phi(q)}$ for all $p, q \in M$) such that $\Phi(M)$ is dense in N. It is well known that every metric space has a completion which is unique up to isometry see for instance [4]. Sherwood in [10] established the following result (see also [9])

Lemma 1.2. *([9],[10]) Every PM space (M, F, τ) with a continuous triangle function has a completion that is unique up to isometry.*

Indeed, The construction of the completion of a PM space (M, F, τ) with continuous triangle function follows the line as in the classical metric case.

In 1968 R.J. Egbert [3] extended the classical Hausdorff distance to the probabilistic setting by introducing the notion of probabilistic Hausdorff metric in the case of Menger space. Thereafter, Tardiff [11] enlarged this notion to the context of general probabilistic metric spaces (see also [9]). Let (M, F, τ)

be a PM space and denote by $\mathcal{P}_0(M)$ the family of all nonempty subsets of M. Given $p \in M$, $B \in \mathcal{P}_0(M)$. "The probabilistic distance" form p to B is defined as

$$F_{pB}(t) = F_{Bp}(t) = \sup_{q \in B} F_{pq}(t) \quad \text{for all } t \in [0, +\infty)$$

Given $A, B \in \mathcal{P}_0(M)$ and defined

$$\mathcal{H}_F^-(A,B)(t) = \begin{cases} 0 & \text{if } t = 0 \\ \sup\limits_{0<s<t} \inf\limits_{p \in A} F_{pB}(s) & \text{if } t \in (0, +\infty) \end{cases}$$

and

$$\mathcal{H}_F^+(A,B)(t) = \begin{cases} 0 & \text{if } t = 0 \\ \sup\limits_{0<s<t} \inf\limits_{q \in B} F_{Aq}(s) & \text{if } t \in (0, +\infty) \end{cases}$$

The probabilistic Hausdorff distance between A and B is the d.d.f:

$$\mathcal{H}_F(A,B) = \min\{\mathcal{H}_F^-(A,B), \mathcal{H}_F^+(A,B)\}$$

In [9] it was proved that If the triangle function τ is sup-continuous. Then the triplet $(\mathcal{C}_0(M), \mathcal{H}_F, \tau)$ is a PM space, where $\mathcal{C}_0(M)$ denote the collection of nonempty closed subset of M. Moreover $\mathcal{H}_F$ satisfies the following properties, where $cl_{\mathfrak{T}_F}$ denotes the closure of with respect to the strong topology.

Lemma 1.3. *([9],[2],[6]) Let (M, F, τ) a PM space. Let $p, q \in M$ and $A, B \in \mathcal{P}_0(M)$. Then we have the following*

(1) *$x \in cl_{\mathfrak{T}_F}(A)$ if and only if $F_{xA} = H$.*

(2) *$\mathcal{H}_F(\{p\}, \{q\}) = F_{pq}$ for all $p, q \in M$.*

(3) *$F_{pB} \geq \mathcal{H}_F^-(A,B) \geq \mathcal{H}_F(A,B)$.*

(4) *$\mathcal{H}_F^+(A,B) = \mathcal{H}_F^-(B,A)$, $\mathcal{H}_F(A,B) = \mathcal{H}_F(cl_{\mathfrak{T}_F}(A), cl_{\mathfrak{T}_F}(B))$, and $\mathcal{H}_F(A,B) = H$ if and only if $cl_{\mathfrak{T}_F}(A) = cl_{\mathfrak{T}_F}(B)$.*

(5) *If $\mathcal{H}_F(A,B)(t) > 1 - t$ for some $t > 0$ then*

$$\forall p \in A \quad \exists q \in B \quad \text{such that} \quad F_{pq}(t) > 1 - t$$

$$\forall q \in B \quad \exists p \in A \quad \text{such that} \quad F_{pq}(t) > 1 - t$$

(6) *If τ is sup-continuous then $F_{pB} \geq \tau(F_{pA}, \mathcal{H}_F(A,B))$*

2. Main Results

We begin this section by studying completeness property of the probabilistic Hausdorff metric. The first result is a direct consequences of [[6],Th 12].

Theorem 2.1. *A PM space (M, F, τ) with sup-continuous triangle function is complete if and only if the PM space $(\mathcal{C}_0(M), \mathcal{H}_F, \tau)$ is complete.*

To state our next result we recall that a subset Y of an uniform space $(X,\mathcal{U})$ is said to be totally bounded if for every $U \in \mathcal{U}$ there exists a finite subset Z of X such that $Y \subset U(Z) = \bigcup_{z \in Z} U(z)$. where $U(z) = \{x \in X : (x, z) \in U\}$. A subset Y of a uniform space $(X,\mathcal{U})$ is compact if and only if Y is complete and totally bounded ([[5], Ch. 6, Th. 32]). Let $\mathcal{K}_0(M)$ denote the collection of nonempty compact subset of PM space (M, F, τ).

Theorem 2.2. *A PM space (M, F, τ) with sup-continuous triangle function is complete if and only if PM space $(\mathcal{K}_0(M), \mathcal{H}_F, \tau)$.*

Proof. At first suppose that $(\mathcal{K}_0(M), \mathcal{H}_F, \tau)$ is a complete PM space. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in M and consider the sequence $(A_n)_{n\in\mathbb{N}}$ where $A_n = \{x_n\}$ for all $n \in \mathbb{N}$. Then, from Lemma 1.3 (2) $(A_n)_{n\in\mathbb{N}}$ is Cauchy sequence in $(\mathcal{K}_0(M), \mathcal{H}_F, \tau)$, so there exists $A \in \mathcal{K}_0(M)$ such that $A_n \to A$ with respect to $\mathcal{H}_F$. it is easy to show that $(x_n)_{n\in\mathbb{N}}$ converge to every $x \in A$. hence (M, F, τ) is complete.

Conversely, by Theorem 2.1 $(\mathcal{C}_0(M), \mathcal{H}_F, \tau)$ is a complete PM space. Since $\mathcal{K}_0(M) \subset \mathcal{C}_0(M)$, it is enough to prove that $\mathcal{K}_0(M)$ is closed in $\mathcal{C}_0(M)$ with respect to $\mathcal{T}_{\mathcal{H}_F}$. Indeed, Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{K}_0(M)$ such that A_n converge to $A \in \mathcal{C}_0(M)$ with respect to $\mathcal{T}_{\mathcal{H}_F}$. Let $t > 0$, since τ is uniformity continuous, then there is μ_t such that for all $G, K \in \Delta^+$

$$d_L(G, \varepsilon_0) < \mu_t \quad \Rightarrow \quad d_L(\tau(K, G), K) < \frac{t}{2}. \tag{2.1}$$

The convergence of $(A_n)_{n\in\mathbb{N}}$ and (2) of Lemma 1.1 implies that there exists $N \in \mathbb{N}$ such that

$$d_L(\mathcal{H}_F(A_N, A), H) < \frac{t}{2}. \tag{2.2}$$

Now, let $p \in A$, by (5) of Lemma 1.3 and (2.2) of Lemma 1.1, there is $q \in A_N$ such that

$$d_L(F_{pq}, \varepsilon_0) < \frac{t}{2}, \tag{2.3}$$

since A_N is compact, so A_N is totally bounded, Then there exists a finite subset $Z \subset M$ such that $A_N \subset U_{\mu_t}(Z)$, which implies the existence of $z \in Z$ such that

$$d_L(F_{qz}, \varepsilon_0) < \mu_t. \tag{2.4}$$

Hence form Lemma 1.1 (1), (2) and the inequalities (2.1), (2.3) and (2.4) we have

$$\begin{aligned} d_L(F_{pz}, \varepsilon_0) &\leq d_L(\tau(F_{pq}, F_{qz}), \varepsilon_0) \\ &\leq d_L(\tau(F_{pq}, F_{qz}), F_{pq}) + d_L(F_{pq}, \varepsilon_0) \\ &< \frac{t}{2} + \frac{t}{2} = t \end{aligned}$$

This shows that $A \subseteq U_t(Z)$. Thus, A is totally bounded and since A is closed in the complete PM space (M, F, τ), it follows that A is complete. Hence, $A \in \mathcal{K}_0(M)$. □

In the sequel we will refer to $(\widetilde{M}, \widetilde{F}, \tau)$ as the completion of the PM space (M, F, τ). Our next object is to find the relationship between $(\widetilde{\mathcal{C}_0(M)}, \widetilde{\mathcal{H}_F}, \tau)$ and $(\mathcal{C}_0(\widetilde{M}), \mathcal{H}_{\widetilde{F}}, \tau)$ (resp, $(\widetilde{\mathcal{K}_0(M)}, \widetilde{\mathcal{H}_F}, \tau)$ and $(\mathcal{K}_0(\widetilde{M}), \mathcal{H}_{\widetilde{F}}, \tau)$).

Theorem 2.3. *Let (M, F, τ) be a PM space with sup-continuous triangle function. Then, $(\mathcal{C}_0(\widetilde{M}), \mathcal{H}_{\widetilde{F}}, \tau)$ and $(\widetilde{\mathcal{C}_0(M)}, \widetilde{\mathcal{H}_F}, \tau)$ are isometric.*

Proof. Let (M, F, τ) be a PM space and consider the mapping Φ from $\mathcal{C}_0(M)$ into $\mathcal{C}_0(\widetilde{M})$ defined by $\Phi(A) = cl_{\mathfrak{T}_{\widetilde{F}}}(A)$ for all $A \in \mathcal{C}_0(M)$. We will claim that Φ is an isometry from $(\mathcal{C}_0(M), \mathcal{H}_F, \tau)$ to $(\mathcal{C}_0(\widetilde{M}), \mathcal{H}_{\widetilde{F}}, \tau)$. Let $A, B \in \mathcal{C}_0(M)$, from Lemma 1.3 (4) we have for all $t > 0$

$$\begin{aligned} \mathcal{H}_{\widetilde{F}}(\phi(A), \phi(B))(t) &= \mathcal{H}_{\widetilde{F}}(A, B)(t) \\ &= \min\{\sup_{s<t} \inf_{a\in A} \sup_{b\in B} \widetilde{F}_{ab}(s), \sup_{s<t} \inf_{b\in B} \sup_{a\in A} \widetilde{F}_{ba}(s)\} \\ &= \min\{\sup_{s<t} \inf_{a\in A} \sup_{b\in B} F_{ab}(s), \sup_{s<t} \inf_{b\in B} \sup_{a\in A} F_{ba}(s)\} \\ &= \mathcal{H}_F(A, B)(t) \end{aligned}$$

In the next step we show that $\phi(\mathcal{C}_0(M))$ is dense in $\mathcal{C}_0(\widetilde{M})$. Equivalently we need to show that

$$\forall A \in \mathcal{C}_0(\widetilde{M}),\ \forall t > 0 \quad \exists B \in \phi(\mathcal{C}_0(M))\ ;\ \mathcal{H}_{\widetilde{F}}(A, B)(t) > 1 - t$$

Let $A \in \mathcal{C}_0(\widetilde{M})$, $t > 0$ and $0 < s < t$. Since τ is uniformly continuous there is $\mu_s > 0$ such that for all $G, K \in \Delta^+$

$$G(\mu_s) > 1 - \mu_s \text{ and } K(\mu_s) > 1 - \mu_s \quad \Rightarrow \quad \tau(G, K)(s) > 1 - s. \tag{2.5}$$

Putting $t_1 = \min\{s, \mu_s\}$, since M is dense in $\widetilde{M}$, then for each $a \in A$ there exists $x_a \in M$ such that $\widetilde{F}_{ax_a}(t_1) > 1 - t_1$. Putting $B = \{x_a : a \in A\}$ and $C = cl_{\mathfrak{T}_F}(B)$, we will show that $\mathcal{H}_{\widetilde{F}}(A, \phi(C))(t) > 1 - t$, i.e. $\mathcal{H}_{\widetilde{F}}(A, C)(t) > 1 - t$ by Lemma 1.3 (5). To this end take $a \in A$ then there exists x_a such that $\widetilde{F}_{ax_a}(t_1) > 1 - t_1$, so $\widetilde{F}_{aC}(t_1) \geq \widetilde{F}_{ax_a}(t_1) > 1 - t_1 > 1 - t$, then $\inf\limits_{a \in A} \widetilde{F}_{aC}(t_1) > 1 - t$ .Hence we conclude that

$$\mathcal{H}^-_{\widetilde{F}}(A, C)(t) > 1 - t. \tag{2.6}$$

Next, let $c \in C$, then there exists $x_a \in B$ such that $F_{cx_a}(t_1) > 1 - t_1$ and $a \in A$ with $\widetilde{F}_{x_a a}(t_1) > 1 - t_1$. Then we have

$$\widetilde{F}_{cx_a}(\mu_s) \geq \widetilde{F}_{cx_a}(t_1) > 1 - t_1 \geq 1 - \mu_s \tag{2.7}$$

and

$$\widetilde{F}_{x_a a}(\mu_s) \geq \widetilde{F}_{x_a a}(t_1) > 1 - t_1 \geq 1 - \mu_s \tag{2.8}$$

Combining (2.5) with (2.7) and (2.8), we obtain

$$\widetilde{F}_{cA}(s) \geq \widetilde{F}_{ca}(s) \geq \tau(\widetilde{F}_{cx_a}, \widetilde{F}_{x_a a})(s) > 1 - s > 1 - t.$$

Consequently

$$\mathcal{H}^-_{\widetilde{F}}(C, A)(t) > 1 - t \tag{2.9}$$

Hence,from (2.6) and (2.9) we conclude that $\mathcal{H}_{\widetilde{F}}(A, C)(t) > 1 - t$. This shows that $\phi(\mathcal{C}_0(M))$ is dense in $\mathcal{C}_0(\widetilde{M})$ and since by Theorem 2.1 $(\mathcal{C}_0(\widetilde{M}), \mathcal{H}_{\widetilde{F}}, \tau)$ is a complete PM space, we deduce that $(\mathcal{C}_0(\widetilde{M}), \mathcal{H}_{\widetilde{F}}, \tau)$ is a completion of $(\mathcal{C}_0(M), \mathcal{H}_F, \tau)$. Thus, by Theorem 1.2 we conclude that $(\mathcal{C}_0(\widetilde{M}), \mathcal{H}_{\widetilde{F}}, \tau)$ and $(\widetilde{\mathcal{C}_0(M)}, \widetilde{\mathcal{H}_F}, \tau)$ are isometric. □

Now we will show that Theorem 2.3 remain valid if we replace $\mathcal{C}_0(M)$ by $\mathcal{K}_0(M)$.

Lemma 2.1. *Let (M, F, τ) be a PM space with sup-continuous triangle function. If K is a dense subset of M, then $\mathcal{K}_0(K)$ is dense in the PM space $(\mathcal{K}_0(M), \mathcal{H}_F, \tau)$.*

Proof. Let $A \in \mathcal{K}_0(M)$, $t > 0$ and $0 < s < t$. Since τ is uniformly continuous, then there is $\mu_s > 0$ such that for all $G, R \in \Delta^+$

$$G(\mu_s) > 1 - \mu_s \quad \text{and} \quad R(\mu_s) > 1 - \mu_s \Rightarrow \tau(G,R)(s) > 1 - s.$$

Putting $t_1 = \min\{s, \mu_s\}$. Since A is compact, then A is totally bounded, so there exists a finite subset $Y = \{y_1, \dots, y_n\}$ such that $A \subset \bigcup_{1 \le i \le n} U_t(y_i) = \bigcup_{1 \le i \le n} N_{y_i}(t)$.By hypothesis K is dense in M, then for each $i \in \{1, \dots, n\}$ there exists $z_i \in K$ such that $F_{z_i y_i}(t_1) > 1 - t_1$. Putting $Z = \{z_1, \dots, z_n\}$. Applying the same arguments as in Theorem 2.3 we show that $\mathcal{H}_F^-(Z, A)(t) > 1 - t$ and $\mathcal{H}_F^-(A, Z)(t) > 1 - t$ hence we conclude that $\mathcal{H}_F(A, Z)(t) > 1 - t$. □

Theorem 2.4. *Let (M, F, τ) be a PM space with sup-continuous triangle function. Then, $(\mathcal{K}_0(\widetilde{M}), \mathcal{H}_{\widetilde{F}}, \tau)$ and $(\widetilde{\mathcal{K}_0(M)}, \widetilde{\mathcal{H}_F}, \tau)$ are isometric.*

Proof. Using the fact the restriction to $\mathcal{K}_0(M)$ of the map Φ defined in Theorem 2.3 is the identity, Lemma 2.1 and Theorem 2.2 we deduce that $(\mathcal{K}_0(\widetilde{M}), \mathcal{H}_{\widetilde{F}}, \tau)$ is a completion of $(\mathcal{K}_0(M), \mathcal{H}_F, \tau)$. The conclusion follows immediately from Theorem 1.2. □

3. Relative Results

In this section we will show a counterpart of Theorem 2.3 and Theorem 2.4 in the case of metric spaces. In the sequel $(\widetilde{M}, \widetilde{d})$ denote the completion of given metric space (M, d). It is well known (see, [1]) that for a given metric space (M, d) the Hausdorff metric on the collection af all closed subsets $\mathcal{C}_0(M)$ is defined as the infinite valued metric $\mathcal{H}_d : \mathcal{C}_0(M) \times \mathcal{C}_0(M) \to [0, \infty]$ by $\mathcal{H}_d(A, B) = \max\{\mathcal{H}_d^-(A, B), \mathcal{H}_d^+(A, B)\}$, where

$$\mathcal{H}_d^-(A, B) = \sup_{a \in A} d(a, B) \text{ and } \mathcal{H}_d^+(A, B) = \sup_{b \in B} d(A, b)$$

In [8] Michael proved that a metric space (M, d) is complete if and only if $(\mathcal{C}_0(M), \mathcal{H}_d)$ is complete.

Let (M, d) be a metric space and consider the mapping $F^d : M \times M \to \Delta^+$ defined by

$$F^d_{pq}(t) = \frac{t}{t + d(p,q)} \quad \text{for all } p, q \in M,\ t > 0.$$

It is easy to check that (M, F^d, π) is a PM space where $\pi(f,) = f.g$ for all $f, g \in \Delta^+$ and that (M, F^d, π) is complete if and only if (M, d) is complete. In addition we have

Lemma 3.1. *Let (M, d) be a metric space then*

(i) *$\mathcal{H}_{F^d}$ and $F^{\mathcal{H}_d}$ coincide in $\mathcal{C}_0(M)$.*

(ii) *$\widetilde{F}^d$ and $F^{\widetilde{d}}$ coincide in $\widetilde{M}$.*

Proof. (i) Let $A, B \subset \mathcal{C}_0(M)$ and $s > 0$, we can easily show that $F^d_{pB}(s) = \frac{s}{s+d(p,B)}$, then $\inf\limits_{a\in A} F^d_{aB}(s) = \frac{s}{s+\sup\limits_{a\in A} d(a,B)}$. Thus

$$\mathcal{H}^-_{F^d}(A, B)(t) = \sup_{s<t} \frac{s}{s + \sup\limits_{a\in A} d(a, B)} = \frac{t}{t + \mathcal{H}^-_d(A, B)}(t) = F^{\mathcal{H}^-_d}_{AB}(t)$$

similarly we show that $\mathcal{H}^+_{F^d}(A, B) = F^{\mathcal{H}^+_d}_{AB}$. Hence, $\mathcal{H}_{F^d} = F^{\mathcal{H}_d}$.

(ii) Let $\widetilde{p}, \widetilde{q} \in \widetilde{M}$ and $(p_n)_{n\in\mathbb{N}}$, $(q_n)_{n\in\mathbb{N}}$ two Cauchy sequences in (M, d) such that $(p_n)_{n\in\mathbb{N}} \in \widetilde{p}$ and $(q_n)_{n\in\mathbb{N}} \in \widetilde{q}$ Then for each $t > 0$ we have $\widetilde{F}^d_{\widetilde{p}\widetilde{q}}(t) = \lim\limits_n Fp_nq_n(t) = \frac{t}{t+\lim\limits_n d(p_n,q_n)} = \frac{t}{t+\widetilde{d}(\widetilde{p},\widetilde{q})} = F^{\widetilde{d}}_{\widetilde{p}\widetilde{q}}(t)$. Hence, $\widetilde{F}^d$ and $F^{\widetilde{d}}$ coincide in $\widetilde{M}$. □

Theorem 3.1. *Let (M, d) be a metric space. Then*

(i) *$(\mathcal{C}_0(\widetilde{M}), \mathcal{H}_{\widetilde{d}})$ and $(\widetilde{\mathcal{C}_0(M)}, \widetilde{\mathcal{H}}_d)$ are isometric.*

(ii) *$(\mathcal{K}_0(\widetilde{M}), \mathcal{H}_{\widetilde{d}})$ and $(\widetilde{\mathcal{K}_0(M)}, \widetilde{\mathcal{H}}_d)$ are isometric.*

Proof. From Lemma 3.1 we have $\mathcal{H}_{\widetilde{F}^d} = \mathcal{H}_{F^{\widetilde{d}}} = F^{\mathcal{H}_{\widetilde{d}}}$ on $\mathcal{C}_0(\widetilde{M})$ and $\widetilde{\mathcal{H}_{F^d}} = \widetilde{F^{\mathcal{H}_d}} = F^{\widetilde{\mathcal{H}_d}}$ on $\widetilde{\mathcal{C}_0(M)}$. from Theorem 2.3 there is and isometry Φ from $(\mathcal{C}_0(\widetilde{M}), F^{\mathcal{H}_{\widetilde{d}}})$ into $(\widetilde{\mathcal{C}_0(M)}, F^{\widetilde{\mathcal{H}_d}})$. hence Φ is an isometry from $(\mathcal{C}_0(\widetilde{M}), \mathcal{H}_{\widetilde{d}})$ into $(\widetilde{\mathcal{C}_0(M)}, \widetilde{\mathcal{H}}_d)$. (ii) follows similarly by replacing Theorem 2.3 by Theorem 2.4. □

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