International Journal of Pure and Applied Mathematics
Volume 105 No. 2 2015, 223-234
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v105i2.9
ijpam.eu

ON THE HYPERSURFACE OF A FINSLER SPACE WITH RANDERS CHANGE OF GENERALIZED (α, β) METRIC

Gauree Shanker[1], Vijeta Singh[2 §]
[1,2]Department of Mathematics and Statistics
Banasthali University
Banasthali, Rajasthan, 304022, INDIA

Abstract: The study of special Finsler spaces has been introduced by M. Matsumoto [6]. The purpose of the present paper is to study hypersurfaces of a Finsler space with (α, β)-metric $L^{\square} = \alpha + \epsilon\beta + \kappa\dfrac{\beta^2}{\alpha} + \beta$. We have obtained the conditions for this hypersurface to be hyperplane of 1st, 2nd kind but not of 3rd kind.

AMS Subject Classification: 53B40, 53C60
Key Words: finsler space, hypersurface, induced cartan connection, hyperplane of 1st, 2nd, 3rd kind

1. Introduction

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space i.e. a pair consisting of an n-dimensional differential manifold M^n equipped with a fundamental function $L(x, y)$. The concept of the (α, β)-metric was introduced by M. Matsumoto [6] and has studied by many authors such as Shibata and others ([3], [5], [6], [8]), where $\alpha^2 = a_{ij}(x)y^iy^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M^n. It has plenty of applications in various fields such as physics, mechanics, seismology, biology and ecology ([1], [2], [7], [10]).

Received: September 9, 2015

§Correspondence author

A change of Finsler metric $L(x,y) \to L^{\square}(x,y) = L(x,y) + b_i(x)y^i$ is called Randers change of metric. The notion of a Randers change ws proposed by Matsumoto, named by Hashiguchi and Ichijyo [4] and studied in detail by Shibata [9]. In the present paper we are introducing Randers change of general (α,β) metric, $L^{\square} = \alpha + \epsilon\beta + \kappa\frac{\beta^2}{\alpha} + \beta$.

A hypersurface M^{n-1} of the M^n may be represented parametrically by the equation $x^i = x^i(u)$, $= 1, ..., n-1$, where u are Gaussian coordinates on M^{n-1}. Since the function $L^{\square} = (L(x(u), y(u,v))$ gives rise to a Finsler metric of M^{n-1},we get an $(n-1)$-dimensional Finsler space $F^{\square(n-1)} = (M^{n-1}, L^{\square}(u,v))$. The hypersurface of Finler Space with some given special metrics has been studied by authors.

In the present paper, we consider an n-dimensional Finsler space $F^{\square n} = (M^n, L^{\square})$ with (α,β)-metric $L^{\square} = \alpha + \epsilon\beta + \kappa\frac{\beta^2}{\alpha} + \beta$ and the hypersurface of F^n with $b_i = \partial_i b$ being the gradient of a scalar function $b(x)$. We prove the conditions for this hypersurface to be a hyperplane of 1st kind, 2nd kind and we also prove that this hypersurface is not a hyperplane of 3rd kind.

2. Preliminaries

Let M^n be a real smooth manifold of dimension n and let $F^{\square n} = (M^n, L^{\square})$ be a Finsler space on a differentiable manifold M^n endowed with a fundamental function $L^{\square}(x,y)$, where

$$L^{\square} = \alpha + \epsilon\beta + \kappa\frac{\beta^2}{\alpha} + \beta. \tag{2.1}$$

The derivatives of the (2.1) with respect to α and β is given by

$$\begin{aligned} L^{\square}_{\alpha} &= \frac{\alpha^2 - \kappa\beta^2}{\alpha^2}, & L^{\square}_{\beta} &= \frac{(\epsilon+1)\alpha + 2\kappa\beta}{\alpha}, \\ L^{\square}_{\alpha\alpha} &= \frac{2\kappa\beta^2}{\alpha^3}, & L^{\square}_{\beta\beta} &= \frac{2\kappa}{\alpha}, \\ L^{\square}_{\alpha\beta} &= -\frac{2\kappa\beta}{\alpha^2}, \end{aligned} \tag{2.2}$$

where

$$L^{\square}_{\alpha} = \frac{\partial L^{\square}}{\partial \alpha}, \qquad L^{\square}_{\beta} = \frac{\partial L^{\square}}{\partial \beta}, \qquad L^{\square}_{\alpha\alpha} = \frac{\partial L^{\square}_{\alpha}}{\partial \alpha},$$

$$L^{*}_{\beta\beta} = \frac{\partial L^{*}_{\beta}}{\partial \beta}, \qquad L^{*}_{\alpha\beta} = \frac{\partial L^{*}_{\alpha}}{\partial \beta}$$

In the special Finsler space $F^{*n} = (M^n, L^{*})$ the normalized element of support $l_i \dot{\partial}_i L^{*}$ and the angular metric tensor $h^{*}_{ij} = L^{*}\dot{\partial}_i\dot{\partial}_j L^{*}$:

$$l^{*}_i = \alpha^{-1} L^{*}_{\alpha} y_i + L^{*}_{\beta} b_i, \tag{2.3}$$

$$h^{*}_{ij} = p a_{ij} + q_0 b_i b_j + q_1(b_i Y_j + b_j Y_i) + q_2 Y_i Y_j, \tag{2.4}$$

where

$$\begin{aligned}
Y_i &= a_{ij} y_j,\\
p &= L^{*} L^{*}_{\alpha}\alpha^{-1} = \frac{\alpha^4 + \alpha^3\beta(\epsilon+1) - \alpha\beta^3\kappa(\epsilon+1) - \kappa^2\beta^4}{\alpha^4},\\
q_0 &= L^{*} L^{*}_{\beta\beta} = \frac{2\kappa\{\alpha^2 + \alpha\beta(\epsilon+1) + \kappa\beta^2\}}{\alpha^2},\\
q_1 &= L^{*} L^{*}_{\alpha\beta}\alpha^{-1} = -\frac{2\kappa\beta\{\alpha^2 + \alpha\beta(\epsilon+1) + \kappa\beta^2\}}{\alpha^4},\\
q_2 &= L^{*}\alpha^{-2}(L^{*}_{\alpha\alpha} - L^{*}_{\alpha}\alpha^{-1})\\
&= -\frac{\alpha^4 + \alpha^3\beta(\epsilon+1) - 2\kappa\alpha^2\beta^2 - 3\kappa(\epsilon+1)\alpha\beta^3 - 3\kappa^2\beta^4}{\alpha^6}.
\end{aligned} \tag{2.5}$$

The fundamental tensor $g_{ij} = (1/2)\dot{\partial}_i\dot{\partial}_j L^{*2}$ is given by

$$g^{*}_{ij} = p a_{ij} + p_0 b_i b_j + p_1(b_i Y_j + b_j Y_i) + p_2 Y_i Y_j, \tag{2.6}$$

where

$$\begin{aligned}
p_0 &= q_0 + L^{*2}_{\beta} = \frac{\alpha^2\{2\kappa + (\epsilon+1)^2\} + \alpha\{6\kappa(\epsilon+1)\beta\} + 6\kappa^2\beta^2}{\alpha^2},\\
p_1 &= q_1 + L^{*-1} p L^{*}_{\beta} = \frac{(\epsilon+1)\alpha^3 - 3\kappa(\epsilon+1)\alpha\beta^2 - 4\kappa^2\beta^3}{\alpha^4},\\
p_2 &= q_2 + p^2 L^{*-2} = -\frac{(\epsilon+1)\alpha^3 - 3\kappa(\epsilon+1)\alpha\beta^3 - 4\kappa^2\beta^4}{\alpha^6}.
\end{aligned} \tag{2.7}$$

Now, the reciprocal tensor g^{*ij} of g^{*}_{ij} is given by

$$g^{*ij} = p^{-1} a^{ij} - s_0 b^i b^j - s_1(b^i y^j + b^j y^i) - s_2 y^i y^j, \tag{2.8}$$

where

$$b^i = a^{ij} b_j;$$

$$s_0 = \frac{\{pp_0 + (p_0p_2 - p_1^2)\alpha^2\}}{\zeta p},$$
$$s_1 = \frac{\{pp_1 - (p_0p_2 - p_1^2)\beta\}}{\zeta p},$$
$$s_2 = \frac{\{pp_2 + (p_0p_2 - p_1^2)b^2\}}{\zeta p}, \qquad b^2 = a_{ij}b^ib^j,$$
$$\zeta = p(p + p_0b^2 + p_1\beta) + (p_0p_2 - p_1^2)(\alpha^2b^2 - \beta^2). \tag{2.9}$$

For general Finsler space F^n the hv-torsion tensor is given by $C_{ijk} = p_1(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_im_jm_k$, where

$$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_1q_0, \qquad m_i = b_i - \alpha^{-2}\beta Y_i. \tag{2.10}$$

Here m_i is a non-vanishing covariant vector orthogonal to the element of support y^i.

Let $\{^{\,i}_{jk}\}$ be the component of Christoffel symbols of the associated Riemannian space R^n and let ∇_k be covariant differentiation with respect to x^k relative to this Christoffel symbols. We will use the following tensors:

$$2E_{ij} = b_{ij} + b_{ji}, \qquad 2F_{ij} = b_{ij} - b_{ji}, \tag{2.11}$$

where $b_{ij} = \nabla_j b_i$.

Let $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$, be the Cartan connection of F^n. The difference tensor $D^i_{jk} = F^i_{jk} - \{^{\,i}_{jk}\}$ of the special Finsler space F^n is given by

$$\begin{aligned} D^i_{jk} =& B^iE_{jk} + F^i_kB_j + F^i_jB_k + B^i_jb_{0k} + B^i_kb_{0j} - b_{0m}g^{im}B_{jk} \\ &- C^i_{jm}A^m_k - C^i_{km}A^m_j + C_{jkm}A^m_sg^{is} \\ &+ \lambda^s(C^i_{jm}C^m_{sk} + C^i_{km}C^m_{sj} - C^m_{jk}C^i_{ms}), \end{aligned} \tag{2.12}$$

where:

$$\begin{aligned} &B_k = p_0b_k + p_1y_k, && B^i = g^{ij}B_j, \\ &F^k = g^{kj}F_{ji}, && B_{ij} = \frac{p_1(a_{ij} - \alpha^{-2}Y_iY_j) + (\partial p_0/\partial\beta)m_im_j}{2}, \\ &A^m_k = B^m_kE_{00} + B^mE_{k0} && B^k_i = g^{kj}B_{ji}, \\ &\quad + B_kF^m_0 + B_0F^m_k, && \\ &\lambda^m = B^mE_{00} + 2B_0F^m_0, && B_0 = B_iy^i. \end{aligned} \tag{2.13}$$

where 0 denote contraction with y^i except for the quantities p_0, q_0 and s_0 .

3. Induced Cartan Connection

Let F^{n-1} be a hypersurface of F^n given by the equations $x^i = x^i(u)$. The element of support y^i of F^n is to be taken tangential to F^{n-1}, that is

$$y^i = B^i_\alpha(u)v^\alpha, \tag{3.1}$$

The metric tensor $g_{\alpha\beta}$ and the v-torsion tensor $C_{\alpha\beta\gamma}$ are given by

$$g_{\alpha\beta} = g_{ij}B^i_\alpha B^j_\beta, \qquad C_{\alpha\beta\gamma} = C_{ijk}B^i_\alpha B^j_\beta B^k_\gamma \tag{3.2}$$

At each point u^α of F^{n-1}, a unit normal vector $N^i(u, v)$ is defined by

$$g_{ij}B^i_\alpha N^j = 0, \qquad g_{ij}N^iN^j = 1 \tag{3.3}$$

For the angular metric tensor h_{ij}, we have

$$h_{\alpha\beta} = h_{ij}B^i_\alpha B^j_\beta, \qquad h_{ij}B^i_\alpha N^j = 0, \qquad h_{ij}N^iN^j = 1 \tag{3.4}$$

The inverse projection factors $B^i_\alpha(u, v)$ of B^i_α are defined as

$$B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta, \tag{3.5}$$

where $g^{\alpha\beta}$ is the inverse of the metric tensor $g_{\alpha\beta}$ of F^{n-1}.

From (3.3) and (3.5), it follows that

$$B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \qquad B^i_\alpha N_i = 0, \qquad N^iB^\alpha_i = 0, \qquad N^iN_i = 1, \tag{3.6}$$

and further

$$B^i_\alpha B^\alpha_j + N^iN_j = \delta^i_j. \tag{3.7}$$

For induced Cartan connection $IC\Gamma = (F^\alpha_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ on F^{n-1}, the second fundamental h-tensor $H_{\alpha\beta}$ and the normal curvature vector H_α are given by

$$H_{\alpha\beta} = N_i(B^i_{\alpha\beta} + F^i_{jk}B^j_\alpha B^k_\beta) + M_\alpha H_\beta, \qquad H_\alpha = N_i(B^i_{0\alpha} + G^i_jB^j_\alpha), \tag{3.8}$$

where $M_\alpha = C_{ijk}B^i_\alpha N^jN^k$, $B^i_{\alpha\beta} = \partial^2x^i/\partial u^\alpha\partial u^\beta$, and $B^i_{0\alpha} = B^i_{\beta\alpha}v^\beta$. It is clear that $H_{\alpha\beta}$ is not symmetric and

$$H_{\alpha\beta} - H_{\beta\alpha} = M_\alpha H_\beta - M_\beta H_\alpha. \tag{3.9}$$

Equation (3.8) yields

$$H_{0\alpha} = H_{\beta\alpha} v^{\beta} = H_{\alpha}, \qquad H_{\alpha 0} = H_{\alpha\beta} v^{\beta} = H_{\alpha} + M_{\alpha} H_0 \tag{3.10}$$

The second fundamental v-tensor $M_{\alpha\beta}$ is defined as:

$$M_{\alpha\beta} = C_{ijk} B^i_{\alpha} B^j_{\beta} N^k. \tag{3.11}$$

The relative h- and v- covariant derivatives of B^i_{α} and N^i are given by

$$B^i_{\alpha|\beta} = H_{\alpha\beta} N^i, \quad B^i_{\alpha}|_{\beta} = M_{\alpha\beta} N^i, \quad N^i_{|\beta} = -H_{\alpha\beta} B^{\alpha}_j g^{ij}, \quad N^i|_{\beta = -M_{\alpha\beta}} B^{\alpha}_j g^{ij}. \tag{3.12}$$

Let $X_i(x, y)$ be a vector field of F^n, the relative h- and v- covriant derivatives of X_i are given by

$$X_{i|\beta} = X_{i|j} B^j_{\beta} + X_{i|j} N^j H_{\beta}, \qquad X_{i|\beta} = X_i|_j B^j_{\beta}. \tag{3.13}$$

Matsumoto [3] defined different kind of hyperplanes and obtained their characterstics conditions, which are given in the following lemmas.

Lemma 3.1. *For Finsler space F^n a hypersurface F^{n-1} is a hyperplane of the first kind if and only if $H_{\alpha} = 0$ or equivalently $H_0 = 0$.*

Lemma 3.2. *For Finsler space F^n a hypersurface F^{n-1} is a hyperplane of the second kind if and only if $H_{\alpha\beta} = 0$.*

Lemma 3.3. *For Finsler space F^n a hypersurface F^{n-1} is a hyperplane of the third kind if and only if $H_{\alpha\beta} = 0 = M_{\alpha\beta}$.*

4. Hypersurface F^{n-1} of the Finsler Space with Randers Change of General Metric $L = \alpha + \epsilon\beta + \kappa\dfrac{\beta^2}{\alpha}$

Let us consider a special Finsler metric $L^{*} = \alpha + \epsilon\beta + \kappa\dfrac{\beta^2}{\alpha} + \beta$ with a gradient $b_i(x) = \partial_i b$.

From parametric equation $x^i = x^i(u^{\alpha})$ of $F^{n-1}(c)$, we get $\partial_{\alpha} b(x(u)) = 0 = b_i B^i_{\alpha}$ so that $b_i(x)$ are regarded as covariant components of a normal vector field of $F^{n-1}(c)$. Therefore, along the $F^{*n-1}(c)$ we have

$$b_i B^{*i}_{\alpha} = 0, \qquad b_i y^i = 0. \tag{4.1}$$

Therefore induced metric $L^*(u,v)$ of $F^{*n-1}(c)$ is given by

$$L^*(u,v) = \sqrt{a_{\alpha\beta}v^\alpha v^\beta}, \qquad a_{\alpha\beta} = a_{ij}B^{*i}_\alpha B^{*j}_\beta \tag{4.2}$$

which is a Riemannian metric.

At a point of $F^{*(n-1)}(c)$, from (2.5), (2.7) and (2.9), we have

$$\begin{aligned} &p = 1, && q_0 = 2\kappa, && q_1 = 0, && q_2 = -\frac{1}{\alpha}^2, \\ &p_0 = 2\kappa + (\epsilon+1)^2, && p_1 = \frac{\epsilon+1}{\alpha}, && p_2 = 0, && \zeta = 1 + 2\kappa b^2, \\ &s_0 = \frac{2\kappa}{1+2\kappa b^2}, && s_1 = \frac{\epsilon+1}{\alpha(1+2\kappa b^2)}, && s_2 = -\frac{(\epsilon+1)^2 b^2}{\alpha^2(1+2\kappa b^2)} && \end{aligned} \tag{4.3}$$

Therefore (2.8) gives

$$g^{*ij} = a^{ij} - \frac{2\kappa}{1+2\kappa b^2}b^i b^j - \frac{(\epsilon+1)}{\alpha(1+2\kappa b^2)}(b^i y^j + b^j y^i) + \frac{(\epsilon+1)^2 b^2}{\alpha^2(1+2\kappa b^2)}, \tag{4.4}$$

Using (4.1), we get

$$g^{*ij}b_i b_j = \frac{b^2}{1+2\kappa b^2} \tag{4.5}$$

which gives

$$b_i(x(u)) = \sqrt{\frac{b^2}{1+2\kappa b^2}}N^*_i, \tag{4.6}$$

where b is the length of the vector b^i. Using (4.1) and (4.6) we get

$$b^i = a^{ij}b_j = \sqrt{b^2(1+2\kappa b^2)}N^{*i} + (\epsilon+1)b^2\alpha^{-1}y^i. \tag{4.7}$$

Theorem 4.1. *In the special Finsler hypersurface $F^{*(n-1)}(c)$,the induced metric is a Riemannian metric given by (4.2) and the scalar function $b(x)$ is given by (4.5)and (4.6).*

The angular metric tensor and metric tensor of F^{*n} are given by

$$h^*_{ij} = a_{ij} + 2\kappa b_i b_j - \frac{y_i y_j}{\alpha^2}, \tag{4.8}$$

$$g^*_{ij} = a_{ij} + [2\kappa + (\epsilon+1)^2]b_i b_j + \frac{(\epsilon+1)}{\alpha}(b_i y_j + b_j y_i). \tag{4.9}$$

If $h^{(a)}_{\alpha\beta}$ denote angular metric tensor of the Riemmanian metric $a_{ij}(x)$ then, using (4.1),(4.8) and (3.4) for induced metric $L^{\square}$, we get

$$h^{\square}_{\alpha\beta} = h^{\square(a)}_{\alpha\beta}. \tag{4.10}$$

From (2.7), we get

$$\frac{\partial p_0}{\partial \beta} = \frac{6\kappa(\epsilon+1)}{\alpha} + \frac{12\kappa^2\beta^2}{\alpha^2}. \tag{4.11}$$

Thus, along $F^{\square(n-1)}(c)$, $\frac{\partial p_0}{\partial \beta} = \frac{6\kappa(\epsilon+1)}{\alpha}$, and therefore (2.10) gives $\gamma_1 = 0, m_i = b_i$.

Then the hv-tensor becomes

$$C^{\square}_{ijk} = \frac{(\epsilon+1)}{2\alpha}(h_{ij}b_k + h_j k b_i + h_{ki}b_j), \tag{4.12}$$

and therefore using (3.4), (3.11) and (4.1), we get

$$M^{\square}_{\alpha\beta} = \left(\frac{\epsilon+1}{2\alpha}\right)\sqrt{\frac{b^2}{1+2\kappa b^2}}\, h^{\square}_{\alpha\beta}, \qquad M^{\square}_{\alpha} = 0 \tag{4.13}$$

and hence from (3.9) it follows that $H^{\square}_{\alpha\beta}$ is symmetric. Thus we have the following.

Theorem 4.2. *The second fundamental v tensor of special Finsler hypersurface $F^{\square(n-1)}(c)$ vanishes and the second fundamental h- tensor $H^{\square}_{\alpha\beta}$ is symmetric.*

Next taking h-covariant derivative of (4.1) with respect to the induced connection, we get

$$b_{i|\beta}B^{\square i}_{\alpha} + b_i B^{\square i}_{\alpha|\beta} = 0 \tag{4.14}$$

Applying (3.13) for the vector b_i, we get

$$b_{i|\beta}B^{\square i}_{\alpha} = b_{i|j}B^{\square j}_{\beta} + b_{i|j}N^{\square j}H^{\square}_{\beta}. \tag{4.15}$$

Using this and $B^{\square i}_{\alpha|\beta} = H^{\square}_{\alpha\beta}N^{\square i}$,(4.14) becomes

$$b_{i|j}B^{\square i}_{\alpha}B^{\square j}_{\beta} + b_i|_j B^{\square i}_{\alpha}N^{\square j}H^{\square}_{\beta} + b_i H^{\square}_{\alpha\beta}N^{\square i} = 0 \tag{4.16}$$

Since $b_i|_j = -b_h C^h_{ij}$ for Finsler space F^n , using (4.6) and (4.13), we get

$$b_i|_j B^{\square i}_\alpha N^{\square j} = -\sqrt{\frac{b^2}{1+2\kappa b^2}} M^{\square}_\alpha = 0 \tag{4.17}$$

Thus (4.16) gives

$$\sqrt{\frac{b^2}{1+2\kappa b^2}} H^{\square}_{\alpha\beta} + b_{i|j} B^{\square i}_\alpha B^{\square j}_\beta = 0 \tag{4.18}$$

Since $H^{\square}_{\alpha\beta}$ is symmetric, it is clear that $b_{i|j}$ is symmetric.

Further contracting (4.18) with v^β, we get

$$\sqrt{\frac{b^2}{1+2\kappa b^2}} H^{\square}_\alpha + b_{i|j} B^{\square i}_\alpha y^j = 0, \tag{4.19}$$

and then with v^α, we get,

$$\sqrt{\frac{b^2}{1+2\kappa b^2}} H^{\square}_0 + b_{i|j} y^i y^j = 0. \tag{4.20}$$

In the view of Lemma 1, the hypersurface $F^{\square(n-1)}(c)$ is a hyperplane of the first kind if and only if $b_{i|j} y^i y^j = 0$. Here $b_{i|j}$ being the covariant derivative with respect to the Cartan connection of $F^{\square n}$ may depend on y^i.

Since b_i is a gradient vector, from (2.11) for induced metric $L^{\square}$, we have $E^{\square}_{ij} = b_{ij}, F^{\square}_{ij} = 0$. Thus (2.12) reduces to

$$\begin{aligned} D^{\square i}_{jk} =& B^{\square i} b_{jk} + B^{\square i}_j b_{0k} + B^{\square i}_k b_{0j} - b_{0m} g^{im} B^{\square}_{jk} - C^{\square i}_{jm} A^{\square m}_k - C^{\square i}_{km} A^{\square m}_j \\ &+ C^{\square}_{jkm} A^{\square m}_s g^{is} + \lambda^s (C^{\square i}_{jm} C^{\square m}_{sk} + C^{\square i}_{km} C^{\square m}_{sj} - C^{\square m}_{jk} C^{\square i}_{ms}) \end{aligned} \tag{4.21}$$

In view of (4.3) and (4.4), the relations in (2.13) become to

$$\begin{aligned} & B^{\square}_i = \{2\kappa + (\epsilon+1)^2\} b_i + \frac{(\epsilon+1)}{\alpha} y^i, \quad && B^{\square i} = \frac{2\kappa}{1+2\kappa b^2} b^i + \frac{(\epsilon+1)}{\alpha(1+2\kappa b^2)} y^i, \\ & B^{\square}_{ij} = \frac{1}{\alpha}\frac{(\epsilon+1)}{\alpha} a_{ij} - \frac{1}{\alpha^3}(\epsilon+1) y_i y_j && \\ & \quad + \frac{6\kappa(\epsilon+1)}{\alpha} b_i b_j, && B^{\square i}_j = g^{ki} B_{kj}, \\ & A^{\square m}_k = B^{\square m}_k b_{00} + B^{\square m} b_{k0}, && \lambda^{\square m} = B^{\square m} b_{00}. \end{aligned} \tag{4.22}$$

By virtue of (4.1) we have $B^{\square}_{i0}$, $B^{\square i}_{0} = 0$ which leads $A^{\square m}_{0} = B^{\square m} b_{00}$. Therefore we have

$$D^{\square i}_{j0} = B^{\square i}_{j0} + B^{\square i}_{j} b_{00} - B^{\square m} C^{\square i}_{jm} b_{00},$$

$$D^{\square i}_{00} = B^{\square i}_{00} = \left[\frac{2\kappa}{1+2\kappa b^2} b^i + \frac{(\epsilon+1)}{\alpha(1+2\kappa b^2)} y^i \right] b_{00} \tag{4.23}$$

Using the relation (4.1), we get

$$b_i D^{\square i}_{j0} = \frac{2\kappa b^2}{1+2\kappa b^2} b_{j0} + \frac{(\epsilon+1)(1+6\kappa b^2)}{2\alpha(1+2\kappa b^2)} b_j b_{00} - \frac{2\kappa}{1+2\kappa b^2} b^m b_i C^{\square i}_{jm} b_{00}, \tag{4.24}$$

$$b_i D^{\square i}_{00} = \frac{2\kappa b^2}{1+2\kappa b^2} b_{00} \tag{4.25}$$

From (4.12), it follows that $b^m b_i C^{\square i}_{jm} B^{\square i}_{\alpha} = b^2 M^{\square}_{\alpha} = 0$.

Therefore the relation $b_{i|j} = b_{ij} - D^r_{ij} b_r$, and equations (4.25) and (4.26) give

$$b_{i|j} y^i y^j = b_{00} - D^{\square r}_{00} b_r = \frac{1}{1+2\kappa b^2} b_{00} \tag{4.26}$$

Consequently (4.19) and (4.20) can be written as

$$\sqrt{\frac{b^2}{1+2\kappa b^2}} H^{\square}_{\alpha} + b_{i|0} B^{\square i}_{\alpha\alpha} = 0, \tag{4.27}$$

$$\sqrt{\frac{b^2}{1+2\kappa b^2}} H^{\square}_{0} + \frac{1}{1+2\kappa b^2} b_{00} = 0 \tag{4.28}$$

Thus the condition $H^{\square}_{00}$ for induced metric is equivalent to b_{00}, where b_{ij} does not depend on y^i. Since y^i is to satisfy (4.1), the condition is written as $b_{ij} y^i y^j = (b_i y^i)(c_j y^j)$ for some $c_j(x)$, so that we have

$$2b_{ij} = b_i c_j + b_j c_i. \tag{4.29}$$

Using(4.1), it follows that

$$b_{00} = 0, \qquad b_{ij} B^{\square i}_{\alpha} B^{\square j}_{\beta} = 0, \qquad b_{ij} B^{\square i}_{\alpha} y^j = 0. \tag{4.30}$$

Again (4.22) and (4.29) gives

$$b_{i0} b^i = \frac{c_0 b^2}{2}, \qquad \lambda^{\square m} = 0, \qquad A^{\square i}_{j} B^{\square j}_{\beta} = 0,$$

$$B^{*}_{ij} B^{*i}_{\alpha} B^{*j}_{\beta} = \frac{(\epsilon + 1)}{2\alpha} h_{\alpha\beta}. \tag{4.31}$$

Using (3.11), (4.4), (4.7), (4.13) and (4.21) we get

$$b_r D^{*r}_{ij} B^{*i}_{\alpha} B^{*j}_{\beta} = -\frac{(\epsilon + 1) c_0 b^2}{2\alpha(1 + 2\kappa b^2)^2} h^{*}_{\alpha\beta}. \tag{4.32}$$

Therefore the relation (4.18) reduces to

$$\sqrt{\frac{b^2}{1 + 2\kappa b^2}} H^{*}_{\alpha\beta} + \frac{(\epsilon + 1) c_0 b^2}{2\alpha(1 + 2\kappa b^2)^2} h^{*}_{\alpha\beta} = 0. \tag{4.33}$$

Theorem 4.3. *The necessary and sufficient condition for $F^{*(n-1)}(c)$ to be hyperplane of 1st kind is (4.29) and in this case the second fundamental tensor of $F^{*(n-1)}(c)$ is proportional to its angular metric tensor.*

In view of lemma (3.3), $F^{*(n-1)}(c)$ is a hyperplane of second kind if and only if $H^{*}_{\alpha\beta} = 0$. Thus from (4.33), we get $c_0 = c_i(x)y^i = 0$. Therefore there exist a function $e(x)$ such that $c_i(x) = e(x)b_i(x)$. Thus (4.29) gives

$$b_{ij} = e b_i b_j. \tag{4.34}$$

Theorem 4.4. *The necessary and sufficient condition for $F^{*(n-1)}(c)$ to be a hyperplane of 2nd kind is (4.34) .*

Finally from (4.13) and lemma (3.3) we see that $F^{*(n-1)}(c)$ is not a hyperplane of the 3rd kind.

Theorem 4.5. *The hypersurface $F^{*(n-1)}(c)$ of Finsler space with Randers change of General (α, β) metric is not a hyperplane of 3rd kind.*

References

[1] P. L. Antonelli, A. Bona and M. A. Slawinski, *Seismic rays as Finsler geodesics*, Nonlinear Analysis RWA, **5**(2003), 711-722.

[2] P. L. Antonelli, R. S. Ingarden and M. Matsumoto, The theory of Sprays and Finsler spaces with Applications in Physics and Biology, *Kluwer Academic , Dordrecht, the Netherlands*, 1993.

[3] M. Hashiguchi, S. Hojo and M. Matsumoto, *On Landsberg spaces of two dimensions with (α, β)-metric*, Journal of the Korean Mathematical Society, **10**(1973), 17-26.

[4] M. Hashiguchi and Y. Ichijiyo, *Randers spaces with rectilinear geodesics*, Reports of the Faculty of science of Kagoshima University, **13**(1980), 33-40.

[5] M. Matsumoto, *Projectively flat Finsler spaces with* (α, β)*-metric*, Reports on Mathematical Physics, **30**(1991), **1**, 15-20.

[6] M. Matsumoto, *Theory of Finsler spaces with* (α, β)*-metric*, Rep. Math. Phys., **30**(1991).

[7] M. Rafie and B. Razaei, *On Einstein Matsumoto metrics*, Nonlinear Analysis RWA, 63(2012), 882-886.

[8] C. Shibata, *On Finsler spaces with an* (α, β)*-metric*, Journal of Hokkaido University of Education. Section II A, **35**(1984), **1**, 1-16.

[9] C. Shibata, *On Invariant tensors of* β*-changes of Finsler metrics*, Journal of Mathematics of Koyo University, **24**(1984), **1**, 163-188.

[10] H. Shimada and S. V. Sabau, *Introduction to Matsumoto metrics*, Nonlinear Analysis, 63(2005), 165-168.

[11] U. P. Singh and Bindu Kumari, *On a hypersurface of a Matsumoto space*, Indian J. pure appl. Math., **32**(2001), **4**, 521-531.

[12] I. L. Yong Lee, Ha-Yong Park and Yong-Duk Lee, *On a hypersuface of a special Finsler space with a metric* $\alpha + \dfrac{\beta^2}{\alpha}$, Korian J. Math. Sciences, **8**(2001), **1**, 93-101.