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ON THE INTERESTING IDENTITY $\sum_{j=0}^{2016} \binom{2016}{j}(-1)^j(2015^{2015} - j)^{2016} = 2016!$

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Abstract: We shall apply combinatorial arguments to provide the following identity relating factorial, binomial numbers and polynomial values at complex points: let $P(x) \in \mathbb{C}[x]$ be a polynomial with degree $k \geq 0$ and leading coefficient b_k. Then

$$\sum_{j=0}^{k} \binom{k}{j}(-1)^j P(z-j) = b_k k!, \text{ for all } z \in \mathbb{C}.$$

As an immediate consequence, we obtain the identity in the title.

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1. Introduction

We begin by saying that actually we desire to prove a more general identity than the proposed one in the title, but we thought that this title would excite curiosity of this equation.

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There are many interesting identities involving binomial numbers, for instance

$$T_n = \binom{n-1}{0} + 2\binom{n-1}{1} + \binom{n-1}{2},$$

where T_n is the nth triangular number (sequence A000217 in OEIS [2]) and

$$\begin{aligned} n^5 &= \binom{n-1}{0} + 31\binom{n-1}{1} + 180\binom{n-1}{2} \\ &+ 390\binom{n-1}{3} + 360\binom{n-1}{4} + 120\binom{n-1}{5}, \end{aligned}$$

Another recent interesting application of binomial coefficients can be found for example in [3, 4, 5, 6, 7].

We shall provide an identity relating factorial, binomial numbers and polynomial values at complex points. We state it as a theorem.

Theorem 1. *Let $P(x) \in \mathbb{C}[x]$ be a polynomial with degree $k \geq 0$ and leading coefficient b_k. Then*

$$\sum_{j=0}^{k} \binom{k}{j} (-1)^j P(z-j) = b_k k!, \text{ for all } z \in \mathbb{C}.$$

As can be seen by the reader, many interesting identities may be derivated from the Theorem, for instance the one in the title is obtained by choosing $P(x) = x^{2016}$ and $z = 2015^{2015}$.

2. The proof

Before we go on, we recall that the *Stirling number of second kind* $S(m, n)$ is defined as the number of partitions of $\{1, \ldots, m\}$ into exactly n subsets. For instance, $S(1, 2) = 0$, $S(2, 2) = 1$ and $S(3, 2) = 3$.

Evidently, $S(m, n) = 0$ if $n < 1$ or $n > m$. Since there is just one way to partition $\{1, \ldots, m\}$ into a single block and $\{1\} \cup \cdots \cup \{m\}$ is the unique (unordered) way to express it as the disjoint union of m nonempty subsets, it follows that $S(m, 1) = 1 = S(m, m)$.

Let $Sur(m, n)$ be the number of surjections (or onto functions) from a set of m elements to a set of n elements. The principle of inclusion and exclusion tells us that

$$Sur(m, n) = \sum_{i=0}^{n} \binom{n}{i} (-1)^i (n-i)^m, \tag{1}$$

here $(n-i)^m$ is the number of functions from a set of m elements to a set of n elements where at least i specified range elements are not hit by the function. Now, re-index by replacing i with $n-j$ and use the well known fact that $Sur(m,n) = n!S(m,n)$, then equation (1) becomes

$$\sum_{j=0}^{n}\binom{n}{j}(-1)^j j^m = (-1)^n n! S(m,n) \tag{2}$$

This is Identity 210 in the nice book by Benjamin and Quinn [1].

Set $P(x) = \sum_{m=0}^{k} b_m x^m$. We then have

$$\begin{aligned}\sum_{j=0}^{k}\binom{k}{j}(-1)^j P(z-j) &= \sum_{j=0}^{k}\binom{k}{j}(-1)^j\sum_{m=0}^{k} b_m (z-j)^m\\ &= \sum_{j=0}^{k}\binom{k}{j}(-1)^j\sum_{m=0}^{k} b_m \sum_{i=0}^{m}\binom{m}{i} z^i(-j)^{m-i}\\ &= \sum_{m=0}^{k} b_m \sum_{i=0}^{m}\binom{m}{i} z^i(-1)^{m-i}\sum_{j=0}^{k}\binom{k}{j}(-1)^j j^{m-i}\\ &= \sum_{m=0}^{k} b_m \sum_{i=0}^{m}\binom{m}{i} z^i(-1)^{m-i+k} k! S(m-i,k),\end{aligned}$$

using the binomial theorem and the equation (2). Now the Stirling number $S(m-i,k)$ is equal to 1 when $m=k$ and $i=0$, and 0 for all other allowable values of m and i. Thus the sum becomes $b_k\binom{m}{0}z^0(-1)^{2k}k! = b_k k!$ as claimed.

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