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ON LOCALLY PROJECTIVELY FLAT SPECIAL EXPONENTIAL FINSLER METRIC

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Abstract: In the present paper, we introduced a special exponential Finsler metric and studied projectively flat Finsler space with this metric and obtained the necessary and Sufficient conditions for special exponential Finsler metric to be locally projectively flat.

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Key Words: locally projectively flat, exponential Finsler metric, Douglas space, Weyl tensor, Minkowskian space, Riemannian space

1. Introduction

In projective Finsler geometry , we have a remarkable theorem called Rapcsak theorem, which plays an important role in the projective geometry of Finsler space. In fact this theorem gives the necessary and sufficient conditions that a Finsler space is projective to another Finsler space. Many interesting results on projectively flat Finsler space are given by many Finslerists (for example [3], [4], [6], [8], [12], . . .). First of all we give the following

Definition 1. A projective change is a mapping from $F^n = (M, L)$ to $\bar{F}^n = (M, \bar{L})$, which is a diffeomorphism and maps geodesic of $F^n$is a geodesic

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of $\bar{F}^n$. If any geodesic of $F^n$is a geodesic of $\bar{F}^n$ and the converse is also true, then the change $L \to \bar{L}$of the metric is called a projective change and F^n is said to be projective to$\bar{F}^n$.

A Finsler space $F^n$is projective to another Finsler space $\bar{F}^n$, if and only if there exists a positively homogeneous scalar field of degree one in y, $p(x, y)$ such that (see [15])

$$\bar{G}^i = G^i(x, y) + p(x, y)y^i, \tag{1}$$

where $G^i$is geodesic coefficient and $p(x, y)$ is called projective factor of the projective change. Moreover, Rapcsak in [14] proved the theorem stated as:

Let $F^n = (M, L)$, $\bar{F}^n = (M, \bar{L})$ be two Finsler spaces on a common underlying manifold $M$of dimension n. The change $L \to \bar{L}$ of the metric is a projective change, if and only if $\bar{L}$ satisfies

$$\bar{L}|i - \frac{\partial \bar{L}|k}{\partial y^i} y^k = 0, \tag{2}$$

where "|" denotes the $h-$ covariant derivative of $\bar{L}$ on F^n. We say that F^n is projective to $\bar{F}^n$ with the projective factor given by

$$P = \frac{\bar{L}|ky^k}{2\bar{L}}. \tag{3}$$

Definition 2. If there exists a projective change $L \to \bar{L}$ of a Finsler space $F^n = (M, L)$ such that the Finsler space $\bar{F}^n = (M, \bar{L})$ is a locally Minkowskian space, then $F^n$is called locally projectively flat.

If $\bar{F}^n$ is a locally Minkowskian, then $\bar{G}^i = 0$, and equation (1) implies that

$$G^i = -py^i. \tag{4}$$

According to [16], a Finsler metric Lon a differentiable manifold $M$is locally projectively flat if any point $p \in M$, there is a local coordinate system (x^i) in M, such that every geodesic $c = c(t)$ is straight, i.e. $x^i(t) = f(t)a^i + b^i$, where $f(t)$ is a C^∞ function, $a^i$and $b^i$are constants.

The following lemma is very important in Finsler geometry which gives the requirement for any Finsler metric to be locally projectively flat and will be used to established our main results in continuation.

Lemma. (see [5]) *A Finsler space $F^n = (M, L)$ is locally projectively flat if and only if*

$$\frac{\partial L}{\partial x^i} - \frac{\partial^2 L}{\partial x^k \partial y^i} y^k = 0. \tag{5}$$

In projective Finsler geometry, there are two important projectively invariant tensors namely the Weyl tensor and the Douglas tensor. These tensors provides us with some important information about the projective properties of Finsler metrics. For example, it is shown that these tensors vanish simultaneously in a locally projectively flat Finsler space. The Douglas tensor is defined in terms of G^i as follows

$$D^h_{ijk} = G^h_{ijk} - \frac{y^h G_{ijk} + \sigma_{(ijk)}\{\delta^h_i G_{jk}\}}{n+1},$$

where

$$\begin{aligned} G^i =&\frac{1}{2}\Gamma^i_{jk} y^i y^j, \\ G^h_{ijk} =&\frac{\partial G^h_{jk}}{\partial y^i}, \\ G_{ij} =&G^r_{ijr}, \\ G_{ijk} =&\frac{\partial G_{ij}}{\partial y^k}, \end{aligned} \tag{6}$$

and $\sigma_{(ijk)}$ means interchange of indices $i, j$and $k$and addition. It is easily proved that the Douglas tensor is invariant under projective change (1).

If the Douglas tensor is equal to zero, a Finsler space is called a Douglas space and its metric is called a Douglas metric.

The Weyl tensor is defined as follows

$$W^i_{jk} = R^i_{jk} + \frac{(H_{jk} - H_{kj})y^i + \delta^i_j H_k - \delta^i_k H_j}{n+1},$$

where

$$\begin{aligned} R^i_{jk} =&\frac{\partial G^i_j}{\partial x^k} - \frac{\partial G^i_k}{\partial x^j} - G^i_{jm} G^m_k + G^i_{km} G^m_j, \\ H^h_{ijk} =&\frac{\partial R^h_{jk}}{\partial y^i}, \\ H_{ij} =&H^k_{ijk}, \end{aligned}$$

$$H_i = \frac{(nH_{ki} + H_{ik})y^k}{n-1}.$$

It can be easily proved that the Weyl tensor is invariant under projective change (1).

A Finsler space F^n, $n \succ 2$ is locally projectively flat if and only if $W^i_{jk} = 0$ and $D^h_{ijk} = 0$, see [10].

The projective change between two Finsler spaces with $(\alpha, \beta)-$metric have been studied by (see [2], [13]). Here we find necessary and sufficient conditions for each Finsler space with some special (α, β)-metrics to be locally projectively flat and calculate scalar curvature of each locally projectively flat Finsler space with the given metric.

The author introduced and studied three kinds of Finsler metric in the papers [17], [18], [19]. In the present article a special exponential metric with respect to that of studied in [17] has been considered.

2. Geodesic Coefficient OF Finsler Space of a Special Exponential (α, β)-Metric

Here we obtain the relationship between the geodesic coefficient of $L$and α by considering each Finsler space with the given metric. Let $\bar{G}^i$and G^ibe the geodesic coefficients of $L$and α respectively. A. Rapcsak [14] proved that

$$\bar{G}^i = G^i + \frac{L|k y^k}{2\bar{L}} y^i + \frac{L}{2}\bar{g}^{il}\left\{\frac{\partial L|k}{\partial y^l} y^k - L|l\right\}, \tag{7}$$

where $L|k$denotes the $h-$covariant derivative of Lon (M, α).

We now find the relationship between $\bar{G}^i$and G^i for the Finsler space with the given (α, β) metric, where β is closed i.e. $b_{i|j} - b_{j|i} = 0$, in other word, β is closed if

$$\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i} = 0.$$

Taking $h-$covariant derivative of Lon (M, α) gives

$$\begin{aligned} L|k =& (\alpha e^{\beta/\alpha})_{|k} \\ =& \alpha_{|k} e^{\beta/\alpha} + \alpha(\frac{\beta}{\alpha})e^{\beta/\alpha} \\ =& (\frac{\alpha\alpha_{|k} + \alpha\beta_{|k} - \alpha_{|k}\beta}{\alpha})e^{\beta/\alpha}. \end{aligned}$$

Since $\beta_{|k} = b_{i|k} y^i$and $\alpha_{|k} = 0$ on (M, α), therefore

$$L_{|k} = b_{i|j} y^i e^{\beta/\alpha},$$

and similarly

$$L|j = b_{i|j} y^i e^{\beta/\alpha}.$$

Differentiating $L|k$ with respect to $y^j$and contracting with y^k gives

$$\frac{\partial L|k}{\partial y^j} y^k = b_{j|k} y^k e^{\beta/\alpha} + b_{i|k} y^i y^k \frac{\partial}{\partial y^j}(e^{\beta/\alpha}).$$

Subtracting $L|j$ from the above expression yields

$$\frac{\partial L|k}{\partial y^j} y^k - L|j = (b_{j|k} - b_{k|j}) y^k e^{\beta/\alpha} + b_{i|k} y^i y^k \frac{\partial}{\partial y^j}(e^{\beta/\alpha}), \tag{8}$$

since β is closed therefore the equation (8) implies

$$\frac{\partial L|k}{\partial y^j} y^k L|j = bi|ky^i y^k \frac{\partial}{\partial y^j}(e^{\beta/\alpha}), \tag{9}$$

and hence (7) shows that

$$\bar{G}^i = G^i + \frac{bj|ky^j y^k}{2\alpha} y^i + \frac{L}{2} g^{ij} br|ky^k y^r \frac{\partial}{\partial y^j}(e^{\beta/\alpha}). \tag{10}$$

Moreover from theoem [14] which asserts, the change $L \to \bar{L}$ of the metric is a Projective change if and only if , $\bar{L}$ satisfies in relation (2), and also β is parallel with respect to α if

$$bi|j = \frac{\partial b_i}{\partial x^j} - b_k \Gamma^k_{ij} = 0,$$

where $\Gamma^k_{ij} = \frac{1}{2} g^{rk} \{ \frac{\partial g_{ri}}{\partial x^j} + \frac{\partial g_{rj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^r} \}$, denote the Christoffel symbols of α.

Now if β is parallel with respect to α, then from equations (8) and (5), (M, α) is projective to (M, L), and equations (10) and (6) imply that

$$\bar{G}^i = G^i = \frac{1}{2} \Gamma^i_{jk}(x) y^j y^k.$$

Definition 3. A Finsler metric L is called Berwald metric, if the geodesic coefficients $G^i$are quadratic in y^i, that is if there are local functions $G^i_{jk}(x)$ on M,such that

$$G^i = \frac{1}{2} G^i_{jk}(x) y^j y^k.$$

If L is a Berwald metric, The space $F^n=(M,L)$ is called a Berwald space. Hence definition 3 and above relation shows that $L$is a Berwald metric.

3. Locally Projectively Flat Exponential Finsler Metric

In Finsler geometry, the flag curvature $K(\rho,y)$ is an analogue of the sectional curvature in Riemannian geometry, where

$$K(\rho,y)=\frac{g_y(R_y(u),u)}{g_y(y,y)g_y(u,u)-g_y(y,u)g_y(u,u)}.$$

In Riemannian geometry, the Beltrami theorem says that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. We will now find the conditions that a Finsler space with the so-called exponential $(\alpha,\beta)-$metric,where α is of constant sectional curvature and the one-form β is closed, is to be locally projectively flat. In this section we state and prove the following theorems.

Theorem 1. *Let $L=\alpha e^{\beta/\alpha}$ be an $(\alpha,\beta)-$metric on an $n-$dimensional differentiable manifold M, where α is a Riemanniam metric and β is 1-form defined on M, then Lis locally projectively flat, if and only if*

$$\frac{\partial}{\partial y^i}(\frac{\beta}{\alpha})\{\frac{\partial\beta}{\partial x^k}-(\frac{\beta}{\alpha})\frac{\partial\alpha}{\partial x^k}\}y^k-(1-\frac{\beta}{\alpha})\{\frac{\partial}{\partial y^i}(\frac{\partial\alpha}{\partial x^k})y^k-\frac{\partial\alpha}{\partial x^i}\} +(\frac{\partial b_i}{\partial x^k}-\frac{\partial b_k}{\partial x^i})y^k=0 \quad (11)$$

Proof. Suppose that $L=\alpha e^{\beta/\alpha}$ be locally projectively flat. Equation (5) implies that

$$\frac{\partial}{\partial y^i}(\frac{\partial L}{\partial x^k})y^k-\frac{\partial L}{\partial x^i}=0. \quad (5')$$

Now

$$\frac{\partial L}{\partial x^k}=\frac{\partial}{\partial x^k}(\alpha e^{\beta/\alpha})=\{\frac{\partial\alpha}{\partial x^k}+\frac{\partial\beta}{\partial x^k}-(\frac{\beta}{\alpha})\frac{\partial\alpha}{\partial x^k}\}e^{\beta/\alpha}. \quad (12)$$

Differentiating (12) with respect to $y^i$and then contracting with y^k, we receive

$$\frac{\partial}{\partial y^i}(\frac{\partial L}{\partial x^k})y^k = \{\frac{\partial}{\partial y^i}(\frac{\partial \alpha}{\partial x^k})y^k + \frac{\partial b_i}{\partial x^k}y^k - (\frac{\beta}{\alpha})\frac{\partial}{\partial y^i}(\frac{\partial \alpha}{\partial x^k})y^k + \frac{\partial}{\partial y^i}(\frac{\beta}{\alpha})\frac{\partial \beta}{\partial x^k}y^k - (\frac{\beta}{\alpha})\frac{\partial \alpha}{\partial x^k}\frac{\partial}{\partial y^i}(\frac{\beta}{\alpha})y^k\}e^{\beta/\alpha}. \quad (13)$$

We may re-write relation (12) as

$$\frac{\partial L}{\partial x^i} = [\frac{\partial \alpha}{\partial x^i} + \frac{\partial b_k}{\partial x^i}y^k - (\frac{\beta}{\alpha})\frac{\partial \alpha}{\partial x^i}]e^{\beta/\alpha}. \quad (14)$$

From the equations (13), (14), (5)' and the fact that $e^{\beta/\alpha} \neq 0$ we obtain the relation (11). The converse also follows easily. □

Theorem 2. *An exponential metric $L = \alpha e^{\beta/\alpha}$ is locally projectively flat if and only if α is locally projectively flat and β is closed , provided that*

$$\frac{\partial \alpha}{\partial y^i} = (\frac{\beta}{\alpha})b_i.$$

Proof. Suppose $L = \alpha e^{\beta/\alpha}$, we have

$$\frac{\partial L}{\partial x^i} - \frac{\partial^2 L}{\partial y^i \partial x^j}y^j = \{(1 - \frac{\beta}{\alpha})(\frac{\partial \alpha}{\partial x^i} - \frac{\partial^2 \alpha}{\partial y^i \partial x^j}y^j) + (\frac{\partial b_j}{\partial x^i} - \frac{\partial b_i}{\partial x^j}) - \frac{1}{\alpha^2}(1 - \frac{\partial \alpha}{\partial x^j})(\alpha b_i - \beta\frac{\partial \alpha}{\partial y^i})y^j\}e^{\beta/\alpha}.$$

Since $e^{\beta/\alpha} \neq 0$, we received from the relation (5), necessary and sufficient conditions for L to be locally projectively flat are obtained. □

Theorem 3. *Let Lbe a locally projectively flat special exponential metric. Suppose that α is locally projectively flat, then*

$$\frac{\partial}{\partial y^i}(\frac{\beta}{\alpha})(P - Q) = \frac{1}{2\beta}(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i})y^j, \quad (15)$$

where $P = \frac{1}{2\alpha}\frac{\partial \alpha}{\partial x^i}y^i$, $Q = \frac{1}{2\beta}\frac{\partial \beta}{\partial x^i}y^i$, and the scalar curvature of L is given by

$$KL^2 = \mu\alpha^2 + (\frac{\beta}{\alpha})^2(P - Q)^2 - 2(\frac{\beta}{\alpha})(P - Q)P + \{\frac{\partial}{\partial x^i}(\frac{\beta}{\alpha})(P - Q) + (\frac{\beta}{\alpha})(\frac{\partial P}{\partial x^i} - \frac{\partial Q}{\partial x^i})\}y^i, \quad (16)$$

where K and μ are the constant sectional curvatures of $L$and α respectively.

Proof. Since α is locally projectively flat, equation (5) gives

$$\frac{\partial}{\partial y^i}(\frac{\partial \alpha}{\partial x^j})y^j - \frac{\partial \alpha}{\partial x^i} = 0. \tag{17}$$

Again by (3), we may write the projection factor as

$$P = \frac{1}{2\alpha}(\frac{\partial \alpha}{\partial x^j})y^j.$$

Now since L is locally projectively flat, therefore theorem 1 holds. Hence form relations (11) and (17), we have

$$\frac{\partial}{\partial y^i}(\frac{\beta}{\alpha})\{\frac{\partial \beta}{\partial x^j} - (\frac{\beta}{\alpha})\frac{\partial \alpha}{\partial x^j}\}y^j = (\frac{\partial b_j}{\partial x^i} - \frac{\partial b_i}{\partial x^j})y^j,$$

which by definitions of $P$and Qgives

$$2\beta\frac{\partial}{\partial y^i}(\frac{\beta}{\alpha})(P - Q) = (\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i})y^j.$$

That proves the first part of the theorem.

To prove the second part, from (3) we get

$$\overline{P} = \frac{1}{2L}\frac{\partial L}{\partial x^i}y^i, \tag{18}$$

which by using relation (12), we obtain

$$\overline{P} = (1 - \frac{\beta}{\alpha})P + \frac{\beta}{\alpha}Q. \tag{19}$$

The scalar curvature of $L$is given by

$$R^i_h g_{ij} = KL^2(g_{jh} - l_j l_h),$$

where $l_i = \frac{\partial L}{\partial y^i}$, and R^i_h are the coefficients of Riemann curvature of L.

Contracting above by g^{kj}, we get

$$R^k_h = K(L^2\delta^k_h - l^k l_h) = K(L^2\delta^k_h - L\frac{\partial L}{\partial y^h}y^k),$$

where $l^k = \frac{y^k}{L}$. For $h = k = i$, the above relation gives

$$R^i_i = K(n-1)L^2. \tag{20}$$

Again we have $G^i = \overline{G}^i + \overline{P}y^i$, where $\overline{P}$ is the projection factor given by (18), since $\overline{G}^i = 0$, then

$$G^i = \overline{P}y^i. \tag{21}$$

Substituting in the Riemann curvature coefficients

$$R^i_h = 2\frac{\partial G^i}{\partial x^h} - y^j\frac{\partial^2 G^i}{\partial x^j \partial x^h} + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k},$$

and putting $i = h$, we obtain

$$R^i_i = (n-1)\overline{P}^2 - (n-1)\frac{\partial \overline{P}}{\partial x^i}y^i. \tag{22}$$

From relations (20) and (22), we get

$$K = \frac{1}{L^2}(\overline{P}^2 - \frac{\partial \overline{P}}{\partial x^i}y^i), \tag{23}$$

or equivalently $KL^2 = \overline{P}^2 - \frac{\partial \overline{P}}{\partial x^i}y^i$.

Replacing (19) into (23), we have

$$\begin{aligned} KL^2 = (1-\frac{\beta}{\alpha})^2P^2 + (\frac{\beta}{\alpha})^2Q^2 + 2\frac{\beta}{\alpha}(1-\frac{\beta}{\alpha})PQ \\ - \{\frac{\partial}{\partial x^i}(\frac{\beta}{\alpha})(Q-P) + (1-\frac{\beta}{\alpha})\frac{\partial P}{\partial x^i} + (\frac{\beta}{\alpha})\frac{\partial Q}{\partial x^i}\}y^i \end{aligned} \tag{24}$$

Sinceα is locally projectively flat, therefore equation (23) can give the constant sectional curvature of α as follows

$$\mu\alpha^2 = P^2 - \frac{\partial P}{\partial x^i}y^i. \tag{25}$$

Relations (24) and (25) implies the second part of the theorem. □

Theorem 4. *Let $F^n = (M, L)$ be a locally projectively flat Finsler space with $L = \alpha e^{\beta/\alpha}$. If α is locally projectively flat and β is closed, then $F^n$is either Minkowskian space or a Riemannian space of non-zero constant sectional curvature given by $KL^2 = \mu\alpha^2$, provided $\frac{\partial}{\partial y^i}(\frac{\beta}{\alpha}) \neq 0$.*

Proof. If β is closed then equation, then (15) gives

$$\frac{\partial}{\partial y^i}(\frac{\beta}{\alpha})(P-Q)=0.$$

Since according to the given condition, $\frac{\partial}{\partial y^i}(\frac{\beta}{\alpha})\neq 0$ therefore $P-Q=0$ i.e. $P=Q$ and hence from (19), we get

$$\overline{P}=P \tag{26}$$

Since $L$and α are locally projectively flat, therefore equation (21) gives

$$\overline{G}=\overline{P}y^i \quad \text{and} \quad G^i=Py^i,$$

where $\overline{G}^i$and $G^i$are the geodesic coefficients of $L$and αrespectively. With the help of relation (26), the above relations give

$$\overline{G}^i=G^i.$$

This implies that $L$is a Berwald metric and hence $L$is a locally projectively flat Berwald Metric. Therefore F^n is either a locally Minkowskian space or a Riemannian space of constant sectional curvature [1].

If F^n is a locally Minkowskian space then $\overline{G}^i=0$.This shows that $\overline{P}=P=0$.Therefore equations (23) and (25) imply that $K=\mu=0$. If $F^n$is a Riemannian space then from equation (16) and $P=Q$, we have

$$KL^2=\mu\alpha^2,$$

which is the required result by which the proof of the theorem is completed. □

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