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SUFFICIENT CONDITIONS FOR OSCILLATION OF SECOND ORDER NEUTRAL ADVANCED DIFFERENCE EQUATIONS

A. Murugesan[1 §], K. Ammamuthu[2]

[1]Department of Mathematics
Government Arts College (Autonomous)
Salem - 636 007, Tamil Nadu, INDIA
[2]Department of Mathematics
Arignar Anna Government Arts College
Vadachennaimalai, Attur-636 121
Tamil Nadu, INDIA

Abstract: The aim of this paper is to study the oscillation of the second order neutral advanced difference equations

$$\Delta(r(n)\Delta[x(n)+p(n)x(\tau(n))])+q(n)x(\sigma(n))=0, \quad n=0,1,2,....$$

Obtained results are based on the new comparison theorems that enable us to reduce problem of the oscillation of the second order equation the oscillation of the first order equations. Obtained comparison principles essentially simplify the examination of the studied equations.

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1. Introduction

In this paper, we are concerned with the oscillation of the solutions of second

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[§]Correspondence author

order neutral advanced equations of the form

$$\Delta(r(n)\Delta[x(n)+p(n)x(\tau(n))])+q(n)x(\sigma(n))=0, \quad n=0,1,2,..., \tag{1}$$

where Δ is the forward difference operator given by $\Delta x(n)=x(n+1)-x(n)$, $\{p(n)\}_{n\geq 0}$, $\{q(n)\}_{n\geq 0}$ and $\{r(n)\}_{n\geq 0}$ assumed to be infinite sequences of real numbers with $q(n)>0$, $r(n)>0$. Also $\{\tau(n)\}_{n\geq 0}$ and $\{\sigma(n)\}_{n\geq 0}$ are sequences of positive integers. Further the following conditions are assumed for its use in the sequel.

(H_1) $\sigma(n)>n+1$ and $\{\sigma(n)\}_{n\geq 0}$ is nondecreasing;

(H_2) $\lim_{n\to\infty}\tau(n)=\infty$;

(H_3) $\Delta\tau(n)\geq\tau_0>0$;

(H_4) $\tau\circ\sigma=\sigma\circ\tau$;

(H_5) $R(n)=\sum_{s=0}^{n}\frac{1}{r(s)}\to\infty$ as $n\to\infty$.

Let $\{x(n)\}$ be a real sequence. We will also define a companion or associated sequence $\{z(n)\}$ of it by

$$z(n)=x(n)+p(n)x(\tau(n)), \quad n\geq 0, \tag{2}$$

where $\{p(n)\}$ and $\{\tau(n)\}$ have been defined above.

Let n_0 be a fixed nonnegative integer. By a solution of (1), we mean a nontrivial real sequence $\{x(n)\}$ which is defined for $n\geq min\{n_0,\tau(n_0)\}$ and satisfies the equation (1) for $n\geq n_0$. A solution $\{x(n)\}$ of (1) is said to be oscillatory if for every positive integer $N>0$, there exists an $n\geq N$ such that $x(n)x(n+1)\leq 0$, otherwise $\{x(n)\}$ is said to be nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

Recently, there has been a lot of interest in studying the oscillatory behavior of difference equations. See, for example [1-6, 10] and references cited there in.They have mainly concerned with the oscillation and nonoscillatory of solutions of (1). The advanced equations have wide use. For example, the population of the future limit to population growth can be described through (1) with $p(n)\equiv 0$. Sternal et al. [9] showed that $-1\leq p_1\leq p_n\leq 0$ together with $\sum_{n=0}^{\infty}\frac{1}{r_n}=\infty$ and $\sum_{n=0}^{\infty}q_n=\infty$ guarantee the oscillation of unbounded solutions of the neutral equation

$$\Delta\left(r_n\Delta\left[u_n+p_nu_{n-k}\right]\right)+q_nf\left(u_{n-l}\right)=0.$$

For the same equation R.N. Rath et al. [8] established oscillation criteria . This results has been improved and generalized by other authors. We mention Tripathy [11] who studied oscillation of

$$\Delta\left(r_n \Delta\left[y_n + p_n y_{n-m}\right]\right) + f_n H_1\left(y_{n-k_1}\right) - g_n H_2\left(y_{n-k_2}\right) = q_n$$

under the conditions

$$\sum_{n=0}^{\infty} \frac{1}{r_n} < \infty \quad \textit{and} \quad \sum_{n=0}^{\infty} \frac{1}{r_n} = \infty.$$

Zhang et al. [12] established oscillation criteria for the equation (1) with

$$p(n) \equiv 0 \quad \textit{and} \quad \sum_{n=n_0}^{\infty} q(n) = \infty.$$

In this paper, we discuss the oscillation of solutions of (1). We obtain some better sufficient conditions for (1) to be oscillatory. They are delicate criteria. Our technique permits to relax restrictions usually imposed on the coefficients of (1). So that our results are of high generality and can be easily extended also to the nonlinear neutral difference equations. Obtained results are easily applicable and are illustrated on a suitable example.

In the sequel for convenience when we write a functional inequality without specifying its domain of validity, we assume that it holds for all sufficiently large positive integer n.

2. Some Lemmas

First we state a lemma which is due to [5].

Lemma 1. *Assume that* $\tau(n) = n - \tau$, τ *is a positive integer and* $-1 < p \leq p(n) \leq 0$. *Assume further that* $\{x(n)\}$ *is an eventually positive solution of (1) and* $\{z(n)\}$ *is its associated sequence defined by (2). If* $\{\Delta z(n)\}$ *is eventually negative or if* $\lim_{n\to\infty} \sup x(n) > 0$, *then* $\{z(n)\}$ *is eventually positive.*

Lemma 2. *If* $\{x(n)\}$ *is an eventually positive solution of (1), then its associated sequence* $\{z(n)\}$ *defined by (2) satisfies* $z(n) > 0$, $r(n)\Delta z(n) > 0$ *and* $\Delta\left(r(n)\Delta z(n)\right) < 0$ *eventually.*

Proof. Assume that $\{x(n)\}$ is an eventually positive solution of (1). Then it follows from (1) that

$$\Delta\left(r(n)\Delta z(n)\right) = -q(n)x\left(\sigma(n)\right) < 0.$$

Consequently, $\{r(n)\Delta z(n)\}$ is decreasing and thus either $\Delta z(n) > 0$ or $\Delta z(n) < 0$, eventually. If we let $\Delta z(n) < 0$, then also $r(n)\Delta z(n) < -c < 0$ and summing this from n_1 to $n-1$, we have

$$z(n) \leq z(n_1) - c\sum_{s=n_1}^{n-1}\frac{1}{r(s)} \to -\infty \quad as \quad n \to \infty.$$

This contradicts the positivity of $\{z(n)\}$ and the proof is complete. □

Lemma 3. *If $\tau(n) = n - \tau$, τ is a positive integer, $-1 \leq p \leq p(n) \leq 0$ and $\{x(n)\}$ is an eventually positive solution of (1) such that $\limsup_{n\to\infty} x(n) > 0$, then its associated sequence $\{z(n)\}$ defined by (2) satisfies $z(n) > 0$, $r(n)\Delta z(n) > 0$ and $\Delta(r(n)\Delta z(n)) < 0$, eventually.*

Proof. Assume that $\{x(n)\}$ is an eventually positive solution of (1) such that

$$\limsup_{n\to\infty} x(n) > 0.$$

Then it follows from (1) that

$$\Delta(r(n)\Delta z(n)) = -q(n)x(\sigma(n)) < 0.$$

Consequently $\{r(n)\Delta z(n)\}$ is decreasing and thus either $\Delta z(n) > 0$ or $\Delta z(n) < 0$, eventually. If we let $\Delta z(n) < 0$, then by Lemma 1, $z(n) > 0$ eventually. Then also $r(n)\Delta z(n) < -c < 0$ and summing this from n_1 to $n-1$, we have

$$z(n) \leq z(n_1) - c\sum_{s=n_1}^{n-1}\frac{1}{r(s)} \to -\infty \quad as \quad n \to \infty.$$

This contradicts the positivity of $\{z(n)\}$ and hence $\Delta z(n) > 0$. Since $\limsup_{n\to\infty} x(n) > 0$, by Lemma 1 we have $z(n) > 0$ eventually and the proof is complete. □

A slight modification in the proof of the Theorem 3 in [7] leads to the following lemma about the advanced difference inequality.

Lemma 4. *Consider the advanced difference inequality*

$$\Delta x(n) - q(n)x(\sigma(n)) \geq 0. \tag{3}$$

where $\{q(n)\}$ *and* $\{\sigma(n)\}$ *are defined in (1). If*

$$\liminf_{n\to\infty} \frac{1}{\sigma(n)-n-1} \sum_{s=n+1}^{\sigma(n)-1} q(s) > \limsup_{n\to\infty} \frac{(\sigma(n)-n-1)^{\sigma(n)-n-1}}{(\sigma(n)-n)^{\sigma(n)-n}},$$

then (3) has no eventually positive solution.

For our further references, let us denote

$$Q(n) = min\,\{q(n), q(\tau(n))\}\,; \tag{4}$$

$$Q_1(n) = \frac{1}{r(n)} \sum_{s=n}^{\infty} Q(s); \tag{5}$$

and

$$Q_2(n) = Q(n)(R(n) - R(n_1 - 1)), \tag{6}$$

where $n \geq n_1$, n_1 is a large enough.

3. Main Results

Theorem 5. *Let* $\tau(n) \geq n$ *and* $0 \leq p(n) \leq p_0 < \infty$. *Assume that at least one of the first order advanced difference inequalities*

$$\Delta w(n) - \frac{\tau_0}{\tau_0 + p_0} Q_1(n) w(\sigma(n)) \geq 0; \tag{7}$$

$$\Delta w(n) - \frac{\tau_0}{\tau_0 + p_0} Q_2(n) w(\sigma(n)) \geq 0; \tag{8}$$

has no positive solution. Then (1) is oscillatory.

Proof. Assume that $\{x(n)\}$ is an eventually positive solution of (1). Then from (2), we have

$$z(\sigma(n)) = x(\sigma(n)) + p(\sigma(n))x(\tau(\sigma(n))) \leq x(\sigma(n)) + p_0 x(\sigma(\tau(n))) \tag{9}$$

where we used to the hypothesis (H_4).

On the other hand, it follows from (1) that

$$\Delta(r(n)\Delta z(n)) + q(n)x(\sigma(n)) = 0; \tag{10}$$

and more over, taking (H_3) into account, we have

$$\begin{aligned} 0 &= \frac{1}{\Delta z(n)}\Delta\left(r(\tau(n))\frac{\Delta z(\tau(n))}{\Delta z(n)}\right) + q(\tau(n))x(\sigma(\tau(n))) \\ &\geq \frac{p_0}{\tau_0}\Delta\left(r(\tau(n))\frac{\Delta z(\tau(n))}{\Delta z(n)}\right) + p_0 q(\tau(n))x(\sigma(\tau(n))). \end{aligned} \tag{11}$$

Combining (10) and (11), we are lead to

$$\Delta(r(n)\Delta z(n)) + \frac{p_0}{\tau_0}\Delta\left(r(\tau(n))\frac{\Delta z(\tau(n))}{\Delta \tau(n)}\right) \\ + q(n)x(\sigma(n)) + p_0 q(\tau(n))x(\sigma(\tau(n))) \leq 0;$$

which in view of (9) and (4) provides

$$\Delta(r(n)\Delta z(n)) + \frac{p_0}{\tau_0}\Delta\left(r(\tau(n))\frac{\Delta z(\tau(n))}{\Delta \tau(n)}\right) + Q(n)z(\sigma(n)) \leq 0. \tag{12}$$

Summing the previous inequality from n to ∞, we get

$$r(n)\Delta z(n) + \frac{p_0}{\tau_0}\left(r(\tau(n))\frac{\Delta z(\tau(n))}{\Delta \tau(n)}\right) \geq \sum_{s=n}^{\infty} Q(s)z(\sigma(s)). \tag{13}$$

On the other hand, since $\{r(n)\Delta z(n)\}$ is decreasing and $\tau(n) \geq n$, it follows from (13) that

$$r(n)\Delta z(n)\left(1 + \frac{p_0}{\tau_0}\right) \geq \sum_{s=n}^{\infty} Q(s)z(\sigma(s)). \tag{14}$$

Using that $\{z(\sigma(n))\}$ is increasing, an summation from n_1 to $n-1$, yields,

$$\begin{aligned} z(n) &\geq \frac{\tau_0}{\tau_0 + p_0}\sum_{k=n_1}^{n-1}\frac{1}{r(k)}\sum_{s=k}^{\infty} Q(s)z(\sigma(s)) \\ &\geq \frac{\tau_0}{\tau_0 + p_0}\sum_{k=n_1}^{n-1}\frac{z(\sigma(k))}{r(k)}\sum_{s=k}^{\infty} Q(s). \end{aligned}$$

That is,

$$z(n) \geq \frac{\tau_0}{\tau_0 + p_0}\sum_{k=n_1}^{n-1} Q_1(k)z(\sigma(k)). \tag{15}$$

Let us denote the right hand side of (15) by $w(n)$. Then $w(n) > 0$ and using $z(n) \geq w(n)$, one can see that

$$\Delta w(n) = \frac{\tau_0}{\tau_0 + p_0} Q_1(n) z(\sigma(n)) \geq \frac{\tau_0}{\tau_0 + p_0} Q_1(n) w(\sigma(n)).$$

Thus $\{w(n)\}$ is a positive solution of (7). This contradicts our assumptions and thus the absence of the eventually positive solutions of (7) implies the oscillatory of (1).

Now, we shall show that the absence of the eventually positive solutions of (8) also yields the oscillation of (1). An summation of (14) from n_1 to $n-1$, provides

$$\begin{aligned} z(n) &\geq \frac{\tau_0}{\tau_0 + p_0} \sum_{k=n_1}^{n-1} \frac{1}{r(k)} \sum_{s=k}^{\infty} Q(s) z(\sigma(s)) \\ &\geq \frac{\tau_0}{\tau_0 + p_0} \sum_{k=n_1}^{n-1} \frac{1}{r(k)} \sum_{s=k}^{\infty} Q(s) z(\sigma(s)) \\ &= \frac{\tau_0}{\tau_0 + p_0} \sum_{s=n_1}^{n-1} Q(s) z(\sigma(s)) \sum_{k=n_1}^{\infty} \frac{1}{r(k)}. \end{aligned}$$

That is,

$$z(n) \geq \frac{\tau_0}{\tau_0 + p_0} \sum_{s=n_1}^{n-1} Q_2(s) z(\sigma(s)). \tag{16}$$

Let us denote the right hand side of (16) by $w(n)$. Then $w(n) > 0$ and using that$z(n) \geq w(n)$; one can see that $\{w(n)\}$ is an eventually positive solution of (8). This is a contradiction and the proof is complete now. □

Theorem 6. *Let $\tau(n) \geq n$ and $0 \leq p(n) \leq p_0 < \infty$. Assume that at least one of the following conditions*

$$\liminf_{n\to\infty} \frac{1}{\sigma(n) - n - 1} \sum_{k=n+1}^{\sigma(n)-1} Q_1(k) > \left(\frac{\tau_0 + p_0}{\tau_0}\right)$$

$$\limsup_{n\to\infty} \frac{(\sigma(n) - n - 1)^{\sigma(n)-n-1}}{(\sigma(n) - n)^{\sigma(n)-n}}; \tag{17}$$

$$\liminf_{n\to\infty} \frac{1}{\sigma(n)-n-1} \sum_{k=n+1}^{\sigma(n)-1} Q_2(k) > \left(\frac{\tau_0+p_0}{\tau_0}\right)$$
$$\limsup_{n\to\infty} \frac{(\sigma(n)-n-1)^{\sigma(n)-n-1}}{(\sigma(n)-n)^{\sigma(n)-n}} \tag{18}$$

holds. Then (1) is oscillatory.

Proof. Lemma 4 guarantees that (7) and (8) have no eventually positive solutions provided that (17) and (18) hold respectively. The assertion now follows from Theorem 5. □

For our incoming references, let us denote

$$Q_3(n) = \frac{\Delta\tau(n)}{r(\tau(n))} \sum_{s=n}^{\infty} Q(s); \quad Q_4(n) = Q(n)(R(\tau(n)) - R(\tau(n_1-1))). \tag{19}$$

Theorem 7. *Assume that $\tau(n) = n - \tau$ and $\sigma(n) = n + \sigma$ where τ and σ are positive integers and $0 \le p(n) \le p_0 < \infty$. Assume further that at least one of the first order advanced difference inequalities*

$$\Delta w(n) - \frac{1}{1+p_0} Q_3(n) w(n+\tau+\sigma) \ge 0; \tag{20}$$

$$\Delta w(n) - \frac{1}{1+p_0} Q_4(n) w(n+\tau+\sigma) \ge 0; \tag{21}$$

has no eventually positive solution. Then (1) is oscillatory.

Proof. Assume that (20) has no eventually positive solution. Without loss of generality, we suppose that $\{x(n)\}$ is an eventually positive solution of (1). Then its associated sequence $\{z(n)\}$ defined by (2) satisfies (13). That is,

$$r(n)\Delta z(n) + p_0 r(n-\tau)\Delta z(n-\tau) \ge \sum_{s=n}^{\infty} Q(s) z(s+\sigma), \tag{22}$$

where $\tau_0 = 1$ and $\Delta\tau(n) = 1$.

On the other hand, since $\{r(n)\Delta z(n)\}$ is decreasing, then it follows from (22) that

$$r(n-z)\Delta z(n-\tau)(1+p_0) \ge \sum_{s=n}^{\infty} Q(s) z(s+\sigma). \tag{23}$$

Multiplying by $\frac{1}{r(n-\tau)}$ and then summing from n_1 to $n-1$, we get

$$z(n-\tau) \geq \frac{1}{1+p_0} \sum_{k=n_1}^{n-1} \frac{1}{r(k-z)} \sum_{s=k}^{\infty} Q(s) z(s+\sigma)$$

$$\geq \frac{1}{1+p_0} \sum_{k=n_1}^{n-1} z(k+\sigma) \frac{1}{r(k-z)} \sum_{s=k}^{\infty} Q(s).$$

That is,

$$z(n-\tau) \geq \frac{1}{1+p_0} \sum_{k=n_1}^{n-1} Q_3(k) z(k+\sigma). \tag{24}$$

Let us denote the right hand side of (24) by $w(n)$. Noting that $z(n-\tau) \geq w(n)$. One can see that $\{w(n)\}$ is an eventually positive solution of (20). This contradicts our assumptions and thus the absence of the positive solutions of (20) implies the oscillation of (1).

Now we shall show that the absence of the positive solutions of (21) implies the oscillation of (1). An summation of (23) from n_1 to $n-1$, gives

$$z(n-\tau) \geq \frac{1}{1+p_0} \sum_{k=n_1}^{n-1} \frac{1}{r(k-\tau)} \sum_{s=k}^{\infty} Q(s) z(s+\sigma)$$

$$\geq \frac{1}{1+p_0} \sum_{k=n_1}^{n-1} \frac{1}{r(k-\tau)} \sum_{s=k}^{n-1} Q(s) z(s+\sigma)$$

$$= \frac{1}{1+p_0} \sum_{s=n_1}^{n-1} Q(s) z(s+\sigma) \sum_{k=n_1-\tau}^{s-\tau} \frac{1}{r(k)}.$$

That is,

$$z(n-\tau) \geq \frac{1}{1+p_0} \sum_{s=n_1}^{n-1} Q_4(s) z(s+\sigma). \tag{25}$$

Let us denote the right hand side of (25) by $w(n)$. Then $w(n) > 0$ and using that $z(n-\tau) \geq w(n)$, one can see that $\{w(n)\}$ is an eventually positive solution of (21). This is a contradiction and the proof is complete now. □

Theorem 8. *Assume that $\tau(n) = n - \tau$, $\sigma(n) = n + \sigma$, where τ and σ are positive integers and $0 \le p(n) \le p_0 < \infty$. Assume further that at least one of the following conditions*

$$\liminf_{n\to\infty} \sum_{s=n+1}^{n+\tau+\sigma-1} Q_3(s) > (1+p_0)\left(\frac{\tau+\sigma-1}{\tau+\sigma}\right)^{\tau+\sigma}; \tag{26}$$

$$\liminf_{n\to\infty} \sum_{s=n+1}^{n+\tau+\sigma-1} Q_4(s) > (1+p_0)\left(\frac{\tau+\sigma-1}{\tau+\sigma}\right)^{\tau+\sigma}; \tag{27}$$

holds. Then (1) is oscillatory.

Proof. Lemma 4 guarantees that (20) and (21) have no eventually positive solutions provided that (26) and (27) hold respectively. The assertion now follows from Theorem 7. □

Theorem 9. *Assume that $\tau(n) = n - \tau$, $\sigma(n) = n + \sigma$, where τ and σ are positive integers and $-1 < p \le p(n) \le 0$. Assume further that at least one of the first order advanced difference inequalities (20) and (21) has no eventually positive solution. Then every solution of (1) is either oscillatory or tends to zero as $n \to \infty$.*

Theorem 10. *Assume that $\tau(n) = n - \tau$, $\sigma(n) = n + \sigma$, where τ and σ are positive integers and $-1 < p \le p(n) \le 0$. Assume further that at least one of the conditions (26) and (27) holds. Then every solution of (1) is either oscillatory or tends to zero as $n \to \infty$.*

Proofs of Theorem 9 and 10.

Without loss of generality, we may assume that $\{x(n)\}$ is an eventually positive solution of (1) such that $\limsup_{n\to\infty} x(n) > 0$. Then, by Lemma 3, we have $z(n) > 0$, $r(n)\Delta z(n) > 0$ and $\Delta(r(n)\Delta z(n)) > 0$, eventually.

Theorem 9 and Theorem 10 can be proved by applying the procedures that are used in the Theorem 7 and 8 respectively.

Example 11. We consider the second order neutral advanced difference equation

$$\Delta\left(\frac{1}{2^n}\Delta\left[x(n) + \frac{1}{n+1}x(n+1)\right]\right) + \frac{1}{2^n}x(n+2) = 0; \quad n = 0, 1, 2..., \tag{28}$$

where $p(n) = \frac{1}{n+1}$, $r(n)\frac{1}{2^n}$, $q(n) = \frac{1}{2^n}$, $\sigma(n) = n+2$ and $\tau(n) = n+1$. Clearly, $p_0 = 1$, $\tau_0 = 1$, $Q(n) = \frac{1}{2^{n+1}}$ and $Q_1(n) = 1$.

Also,

$$\liminf_{n\to\infty} \frac{1}{\sigma(n)-n-1} \sum_{s=n+1}^{\sigma(n)-1} Q_1(s) = 1$$
$$> \left(\frac{\tau_0 + p_0}{\tau_0}\right) \lim_{n\to\infty} sup \frac{(\sigma(n)-n-1)^{\sigma(n)-n-1}}{(\sigma(n)-n)^{\sigma(n)-1}}$$
$$= \frac{1}{2}.$$

Then by Theorem 6, every solution of (28) is oscillatory.

Remark 12. All our conclusions can be very easily extend to nonlinear neutral difference equations of the form

$$\Delta(r(n)\Delta\left[x(n) + p(n)x(\tau(n))\right] + q(n)f(x(\sigma(n)) = 0. \tag{29}$$

Adding the additional condition

$$\frac{f(x)}{x} \geq \lambda$$

the reader can verify that our results here hold also for (29), provided that we replace in the assumption of our achievements the sequences $\{q(n)\}$ by $\{\lambda q(n)\}$.

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