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ON P-SEMISIMPLE IN KK-ALGEBRAS

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Abstract: In the present paper, we introduce the notion of p-semisimple in KK-algebras and investigate some related properties. We also give some necessary and sufficient conditions under which a branchwise commutative KK-algebra is also commutative.

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1. Introduction and Preliminaries

In 1992, E.H. Roh et al. [3] studied the p-semisimple in BCI-algebras and obtained some related properties. In 2012, S. Asawasamrit and A. Sudprasert [2] introduced a new algebraic structure called a KK-algebra and described the relationships between ideals and congruences in this algebra. Furthermore, they defined a quotient KK-algebra and studied its properties. In this paper, we extend the notion of p-semisimple in BCI-algebras to KK-algebras, and investigate its properties.

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We will now summarize some of the results from [2], that we require to prove the results in this paper.

A *KK-algebra* was defined as an algebra $(X, *, 0)$ with a binary operation $*$ and a nullary element 0 such that for all $x, y, z \in X$, the following properties are satisfied:

(KK-1) $(x * y) * ((y * z) * (x * z)) = 0$;

(KK-2) $0 * x = x$;

(KK-3) $x * y = 0$ and $y * x = 0$ if and only if $x = y$.

As an example, let $*$ be defined on an abelian group G by letting $x * y = x^{-1} \cdot y$, where $x, y \in G$, $\cdot$ is the multiplication operator in G, and e is the unit element of G. Then $(G, \cdot, e)$ is a KK-algebra.

For a KK-algebra $(X, *, 0)$, we introduced in [2] a binary relation $\leq$ on X by defining $x \leq y$ if and only if $y * x = 0$ and proved that $(X, \leq)$ is a partially ordered set. In the remainder of this paper, we will usually abbreviate $(X, *, 0)$ to X. It was then proved in [2] that the following properties are true for a KK-algebra. For any $x, y, z \in X$:

(P-1) $x * ((x * y) * y) = 0$;

(P-2) $x * x = 0$;

(P-3) $x * (y * z) = y * (x * z)$;

(P-4) $((x * y) * y) * y = x * y$;

(P-5) $(x * y) * 0 = (x * 0) * (y * 0)$;

(P-6) $(x * y) * ((z * x) * (z * y)) = 0$;

(P-7) If $x \leq y$ then $y * z \leq x * z$;

(P-8) If $x \leq y$ then $z * x \leq z * y$.

In [2], a *closed subset* A of a KK-algebra X was defined as a set A such that if $x, y \in A$ then $x * y \in A$. Further, a non-empty subset A of a KK-algebra X was defined to be an *ideal* of X if it satisfies the following conditions:

(I-1) $0 \in A$.

(I-2) For any $x, y \in X$, if $x * y \in A$ and $x \in A$, then $y \in A$.

In [2], we also defined x to be a *positive element* of X if $x * 0 = 0$, that is if $0 \leq x$. From this definition, the element 0 of X is positive. Note that if x is any element in a KK-algebra X, then $((x * 0) * 0) * x$ is a positive element of X for every $x \in X$.

2. p-Semisimple KK-Algebras

In this section, we define a p-semisimple in a KK-algebra and prove some of its basic properties.

Definition 2.1 *A KK-algebra X is called p-semisimple if $(x * 0) * 0 = x$ for all $x \in X$.*

In [2], we defined an element a in a KK-algebra X to be a *minimal* element if $a * x = 0$ (i.e. $x \leq a$) implied $x = a$ for all $x \in X$.

Proposition 2.2 The following conditions are equivalent for a KK-algebra X:

(1) a is a minimal element of X;

(2) $(a * 0) * 0 = a$;

(3) there is $x \in X$ such that $a = x * 0$;

(4) for all $x \in X$, $x * a = (a * 0) * (x * 0)$;

(5) for all $x \in X$, $x * a = (a * x) * 0$.

Let's first give equivalent conditions of p-semisimplicity. From proposition 2.2, we have the following theorem.

Theorem 2.3 Given a KK-algebra X, then the following conditions are equivalent:

(1) X is a p-semisimple;

(2) every element x in X is minimal;

(3) $X = \{x * 0 : x \in X\}$.

Theorem 2.4 Given a KK-algebra X, then the following conditions are equivalent:

(1) X is a p-semisimple;

(2) for any $x, y \in X$, $(x * 0) * y = (y * 0) * x$;

(3) for any $x \in X$, $x * 0 = 0$ implies $x = 0$;

(4) for any $a, x \in X$, $(x * a) * a = x$;

(5) $X = \{x * a : x \in X\}$ for any $a \in X$.

Proof. Let X be a KK-algebra and $a, x, y \in X$.

(1) $\Rightarrow$ (2). Let X be a p-semisimple, by theorem 2.2, it follows that for any element in X is minimal. Since $(y * 0) * x = (y * 0) * ((x * 0) * 0) = (x * 0) * ((y * 0) * 0) = (x * 0) * y$.

(2) $\Rightarrow$ (3). Assume that $x * 0 = 0$, by (2) it follows that $x = 0 * x = (0 * 0) * x = (x * 0) * 0 = 0$, proving that $x = 0$.

(3) $\Rightarrow$ (1). Suppose that (3) holds, by proposition 2.12.[2], $(((x * 0) * 0) * x) * 0 = 0$, and by hypothesis, we get that $((x * 0) * 0) * x = 0$. Since $(x * 0) * 0$ is a minimal element of X, it follows that $x = (x * 0) * 0$. Therefore X is a p-semisimple.

(1) $\Rightarrow$ (4). Suppose that X be a p-semisimple. It follows that a and x are minimal elements of X. Since $(x * a) * a \leq x$, then $(x * a) * a = x$.

(4) $\Rightarrow$ (5). Suppose that (4) holds in X, then $x = (x*a)*a = y*a$, whenever $y = x * a$.

(5) $\Rightarrow$ (1). Obvious. □

Now, we give some examples and show some properties for a p-semisimple in KK-algebras.

Example 2.5 Suppose that $(G; \cdot, e)$ is an abelian group with e is a unity element in G. Define a binary operation $*$ on X by putting $x*y = x^{-1} \cdot y$, Then we get that $(G, *, 0)$ is a KK-algebras. And since $(x * e) * e = (x^{-1} \cdot e)^{-1} \cdot e = (x^{-1})^{-1} = x$, for any $x \in X$. Hence $(G; \cdot, e)$ is a p-semisimple.

Example 2.6 Let $(X; *, 0)$ be a p-semisimple algebra. Define another binary operation $\cdot$ on X as follow: $x \cdot y = (x * 0) * y$. Then $(X; \cdot, 0)$ is an abelian group with 0 as the unity element. In fact, by theorem 2.4, $x \cdot y = (x * 0) * y = (y * 0) * x = y \cdot x$, then the operation $\cdot$ satisfies the commutative law. Also since $x \cdot (y \cdot z) = (x * 0) * ((y * 0) * z) = (y * 0) * ((x * 0) * z) = y \cdot (x \cdot z)$, then the commutative law gives $(x \cdot y) \cdot z = z \cdot (x \cdot y) = x \cdot (z \cdot y) = x \cdot (y \cdot z)$. So, the operation

$\cdot$ meets the associative law. Moreover, since $x \cdot 0 = (x * 0) * 0 = x$, as X is a p-semisimple, so 0 is the unit element of X. Finally, the inverse element of any element x in X is $x * 0$. That is because $(x * 0) \cdot x = ((x * 0) * 0) * x = x * x = 0$ and $x \cdot (x * 0) = (x * 0) * (x * 0) = 0$. We call the group $(X; \cdot, 0)$ the *adjoint abelian group* of $(X; *, 0)$.

Theorem 2.7 Given a KK-algebra X, then the following conditions are equivalent:

(1) $(x * 0) * 0 = x$ for any $x \in X$;

(2) $(x * 0) * y = (y * 0) * x$ for any $x, y \in X$;

(3) $x * 0 = 0$ implies $x = 0$.

Proof. Let X be a KK-algebra and $x, y \in X$.

(1) $\Rightarrow$ (2). Suppose that X has the property (1). It follows that $(x*0)*y = (x * 0) * ((y * 0) * 0) = (y * 0) * ((x * 0) * 0) = (y * 0) * x$.

(2) $\Rightarrow$ (3). Suppose that $(x * 0) * y = (y * 0) * x$ such that $x * 0 = 0$. Then $x = (0 * 0) * x = (x * 0) * 0 = 0$.

(3) $\Rightarrow$ (1). Suppose that (3) holds in X, then $(x * 0) * (x * 0) = 0$. From (P-3), $x * ((x * 0) * 0) = 0$, which means $(x * 0) * 0 \leq x$. And since $0 = x * x \leq ((x * 0) * 0) * x$. Therefore $[((x * 0) * 0) * x] * 0 = 0$, and by hypothesis, it follows that $((x * 0) * 0) * x = 0$. From KK-3, we conclude that $(x * 0) * 0$. $\square$

Theorem 2.8 Given a KK-algebra X, then the following conditions are equivalent: for all $x, y, z, u \in X$,

(1) X is a p-semisimple;

(2) $(x * y) * (z * u) = (u * z) * (y * x)$;

(3) $(x * y) * 0 = y * x$;

(4) $(x * y) * (z * y) = z * x$;

(5) $x * z = y * z$ implies $x = y$;

(6) $x * y = 0$ implies $x = y$;

Proof. Let X be a KK-algebra and $x, y, z, u \in X$.

(1) $\Rightarrow$ (2). Suppose that X is a p-semisimple. From proposition 2.2 and (P-5), we get $(x*y)*(z*u) = ((z*u)*(x*y))*0 = ((z*u)*0)*((x*y)*0) = (u*z)*(y*x)$.

(2) $\Rightarrow$ (3). Suppose that (2) holds, it follows that $(x*y)*0 = (x*y)*(0*0) = (0*0)*(y*x) = y*x$.

(3) $\Rightarrow$ (4). From (KK-1), (P-5) and (P-6), we can write $(z*x)*((x*y)*(z*y)) = 0$ and $((x*y)*(z*y))*(z*x) = (((y*x)*0)*((y*z)*0))*(z*x) = (((y*x)*(y*z))*0)*((x*z)*0) = (((y*x)*(y*z))*(x*z))*0 = (x*z)*((y*x)*(y*z)) = 0$.

(4) $\Rightarrow$ (5). Assume that $x*z = y*z$, Substituting y for x and z for y and and x for z in (4), we obtain $x*y = (y*z)*(x*z) = 0$. And replacing z by y and y by z in (4), we have $y*x = (x*z)*(y*z) = 0$. Thus $x = y$ because (KK-3).

(5) $\Rightarrow$ (6). Assume that $x*y = 0$, it follows that $x*y = y*y$. By (5), we have $x = y$.

(6) $\Rightarrow$ (1). For $x \in X$, by (P-5), we get $x*((x*0)*0) = (x*0)*(x*0) = 0$. Using (6), it yields $x = (x*0)*0$, proving that X is a p-semisimple. □

Theorem 2.9 Let X be a KK-algebra. Then X is the p-semisimple if and only if one of the following conditions holds: for all $x, y, z \in X$,

(1) $z*x = z*y$ implies $x = y$.

(2) $(x*y)*(x*z) = y*z$.

(3) $(y*x)*(z*x) = (y*z)*0$.

(4) $(x*y)*z = (z*y)*x$.

Proof. Let X be a KK-algebra and $x, y, z \in X$.

(1) From (P-6), $(z*x)*(z*y) \leq x*y$. Then we have $0 \leq x*y$, it follows that $(x*y)*0 = 0$. By theorem 2.8, we obtain $y*x = 0$, so $x = y$. Conversely, since $x*x = 0 = x*((x*0)*0)$. From (1), then $x = (x*0)*0$, proving that X is the p-semisimple.

(2) Now, we will show $(x*y)*(x*z) = y*z$. We see that $(y*z)*((x*y)*(x*z)) = 0$ and $((x*y)*(x*z))*(y*z) = ((z*x)*(y*x))*(y*z) = (y*z)*(y*z) = 0$, these imply that $(x*y)*(x*z) = y*z$. On the other hand, by (2), we get $(x*0)*0 = (x*0)*(x*x) = 0*x = x$.

(3) By theorem 2.8, $(y*x)*(z*x) = z*y = (y*z)*0$. Conversely, by (3), we obtain $(x*0)*0 = (x*x)*(0*x) = 0*(0*x) = x$.

(4) By proposition 2.2 and (P-5), we obtain $(x*y)*z = (z*0)*((x*y)*0) = (z*(x*y))*0 = (x*(z*y))*0$. Then theorem 2.8 implies $(x*y)*z = (z*y)*x$.

On the other hand, we see that $(x*0)*0 = ((0*x)*0)*0 = (0*0)*(0*x) = x$. This completes the proof. □

Proposition 2.10 Assume that X is a KK-algebra. Then B and P are closed of X, where B is the set of all positive of X, and P is the set of all minimal.

Proof. Since 0 is a positive, then B is non-empty set. Let $x, y \in B$, it follows that $x * 0 = 0$ and $y * 0 = 0$. By (P-5), $(x * y) * 0 = (x * 0) * (y * 0) = 0 * 0 = 0$, it means that $x * y \in B$. And similarly, $y * x \in B$, proving that B is a closed of X.

Next, we will show that P is closed of X. Since 0 is a minimal, then P is non-empty set. Let $a, b \in P$ and $x \leq a*b$. Then $b*x \leq b*(a*b) = a*(b*b) = a*0$. Thus $b * x \leq a * 0$ implies $0 = (a * 0) * (b * x) = b * ((a * 0) * x)$. It follows that $(a * 0) * x \leq b$ and since b is minimal, so $(a * 0) * x = b$. We get that $x*b = x*((a*0)*x) = (a*0)*(x*x) = (a*0)*0 \leq a$. By Corollary 2.10 [2], $a*b \leq x$ and $x \leq a*b$, so $x = a*b$, i.e., $a*b$ is a minimal. Therefore $a*b \in P$, proving that P is a closed. □

Define the set P of all minimal element of X is called the *p-semisimple part* of X. From theorem 2.3 and proposition 2.10, we get the following proposition.

Proposition 2.11 Assume that X is a KK-algebra. Then the p-semisimple part P of X is a p-semisimple closed of X, and $P = \{x * 0 : x \in X\}$.

Proposition 2.12 If X is a p-semisimple KK-algebras, then every closed A of X is an ideal of X.

Proof. Let X be a p-semisimple KK-algebras and A be a closed of X. Since A is a closed of X, then $0 \in A$. Now, to show that A satisfies (I-2), which assume that $x, y \in X$ such that $x * y \in A$ and $x \in A$. By closeness of A, it follows that $x * 0 \in A$ and $(x * 0) * (x * y) \in A$. Then $y * ((x * 0) * (x * y)) = (x * 0) * (y * (x * y)) = (x * 0) * (x * (y * y)) = 0$, so $(x * 0) * (x * y) \leq y$. and since y is a minimal of X, $(x * 0) * (x * y) = y$. Hence $y \in A$, this shows that A is an ideal of X. □

For a KK-algebra X and a minimal element a of X, defined the set $V(a) := \{x \in X : x * a = 0\}$, which is called the *branch* of X generated by a.

Proposition 2.13 For any $a \in P$ and $x \in X$, if $x \in V(a)$, then $x*0 = a*0$.

Proof. Since $x \in V(a)$, it follows that $x * a = 0$. Then $(x * 0) * (a * 0) = (x*a)*0 = 0$. Since a is a minimal, so $(a*0)*(x*0) = x*a = 0$. From (KK-3), proving that $x * 0 = a * 0$. □

The next theorem is interesting and useful.

Theorem 2.14 Assume that P is a p-semisimple part of KK-algebra X. Then

(1) $X = \bigcup_{a\in P} V(a)$ and $V(a) \cap V(b) = \emptyset$ whenever $a \neq b$ and $a, b \in P$;

(2) if $x \in V(a)$ and $y \in V(b)$, then $x * y \in V(a * b)$;

(3) if $a \in P$, then $x * a \in P$, for any $x \in X$;

(4) if $a \in P$ and $x \in V(b)$, then $x * a = b * a$.

Proof. Let P be a p-semisimple part of KK-algebra X.

(1) For any $x \in X$, put $a' = (x*0)*0$, then a' is a minimal element of X, and so $a' \in P$. Since $x*a' = x*((x*0)*0) = 0$, it follows that $x \in V(a') \subseteq \bigcup_{a\in P}(a)$. Accordingly, $X = \bigcup_{a\in P}(a)$.

Suppose that a, b are minimals of X such that $a \neq b$. Now, assume that $V(a) \cap V(b) \neq \emptyset$, then there exists $x \in V(a) \cap V(b)$, it means that, $x * a = 0$ and $x * b = 0$. By proposition 2.13, we obtain that $a * 0 = x * 0 = b * 0$. Then $b * a = (a * 0) * (b * 0) = 0$. Likewise, $a * b = 0$. Therefore $a = b$, contradiction with $a \neq b$. This show that $V(a) \cap V(b) = \emptyset$ whenever $a \neq b$.

(2) Assume that $x \in V(a), y \in V(b)$ and a, b are minimals of X. Then $x * a = 0 = y * b$, it follows that $x * 0 = a * 0$ and $y * 0 = b * 0$. Since $a * b = (b * 0) * (a * 0) = (y * 0) * (x * 0) \leq x * y$, so $a * b \leq x * y$. And since $a * b \in P$, thus $x * y \in V(a * b)$.

(3) Assume that $a \in P$ and $x \in X$. By proposition 2.2, it follows that $x * a = (a * 0) * (x * 0)$. Since $a * 0, x * 0 \in P$ and P has closeness property, $x * a \in P$.

(4) Assume that $a \in P$ and $x \in V(b)$. From proposition 2.13, we get that $b * 0 = x * 0$ and $x * a = (a * 0) * (x * 0) = (a * 0) * (b * 0) = b * a$, we conclude that if $a \in P$ and $x \in V(b)$, then $x * a = b * a$. □

For a KK-algebra X, we define a binary operation $\wedge$ by $x \wedge y = (x * y) * y$, for each $x, y \in X$. In particular $a_x = (x * 0) * 0$, and $L_p(X) := \{a \in X : a * x =$

$0 \Rightarrow a = x, \forall x \in X\}$. We call the elements of $L_p(X)$ the $p-atoms$ of X. For any $a \in X$. Note that $a_x \in L_p(X)$, i.e., $(a_x * 0) * 0 = a_x$.

Definition 2.15 *A KK-algebras X is said to be commutative if $x = x \wedge y$ whenever $x \leq y$ for all $x, y \in X$.*

Note that, it can easily be checked that every p-semisimple KK-algebra is commutative.

Definition 2.16 *A KK-algebras X is said to be branchwise commutative if $x \wedge y = y \wedge x$ for all $x, y \in V(a)$ and all $a \in L_p(X)$.*

Proposition 2.17 If X is a branchwise commutative KK-algebra, then X is a commutative.

Proof. Assume that a KK-algebra X is a branchwise commutative. We have that $x \wedge y = y \wedge x$, for any $x, y \in V(a)$ and $a \in L_p(X)$. Now, let $x, y \in X$ such that $x \leq y$. Since $(x * 0) * 0 \in L_p(X)$ and $(x * 0) * 0 \leq 0 * x \leq y$. Hence $x, y \in V((x * 0) * 0)$, implies that $x \wedge y = y \wedge x = (y * x) * x = 0 * x = x$. Consequently, X is a commutative KK-algebra. □

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