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POSITIVE SOLUTIONS FOR NONLINEAR EIGENVALUE PROBLEMS

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Abstract: In this paper, by using the Krasnoselskii fixed point theorem, we investigate the existence of positive solutions for the following nonlinear eigenvalue problem:

$$\begin{cases} u''(t) + a(t)u'(t) + b(t)u(t) + \lambda h(t)f(t,u(t)) = 0, & t \in [t_1, t_2], \\ u(t_1) = 0, \quad \alpha u(\eta) = u(t_2), \end{cases}$$

where $t_1 < \eta < t_2$ and $0 < \alpha\phi_1(\eta) < 1$ and ϕ_1 is the unique solution of the linear boundary value problem:

$$\begin{cases} u''(t) + a(t)u'(t) + b(t)u(t) = 0, & t \in [t_1, t_2], \\ u(t_1) = 0, \quad u(t_2) = 1. \end{cases}$$

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1. Introduction

In the past few years, there has been much attention focused on the existence of positive solutions for three-point boundary value problems. Il'in and Moiseev [1] studied the existence of solutions for a linear second order differential

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equation. Gupta [2] investigated three-point boundary value problems for nonlinear ordinary differential equations. Ma [3, 4] and Marano [7] established the existence of positive solutions for nonlinear three-point boundary-value problems. Recently, Ma and Wang [5] used the Krasnoselskii fixed point theorem to prove the existence of at least one positive solution for the following nonlinear three-point boundary-value problem:

$$\begin{cases} u''(t) + a(t)u'(t) + b(t)u(t) + h(t)f(u) = 0, \ t \in (0,1), \\ u(0) = 0, \ \alpha u(\eta) = u(1), \end{cases}$$

where $0 < \eta < 1$, $h \in C([0,1],[0,\infty))$ satisfies that there exists $x_0 \in [0,1]$ such that $h(x_0) > 0$ and $f \in C([0,1],[0,\infty))$, $0 < \alpha\phi_1(\eta) < 1$ and ϕ_1 is the unique solution of the linear boundary value problem:

$$\begin{cases} u''(t) + a(t)u'(t) + b(t)u(t) = 0, t \in (0,1), \\ u(0) = 0, u(1) = 1. \end{cases}$$

Motivated and inspired by the results in [2]-[5] and [7], in this paper, by using the Krasnoselskii fixed point theorem, we study the existence of positive solutions for the following nonlinear eigenvalue problem:

$$\begin{cases} u''(t) + a(t)u'(t) + b(t)u(t) + \lambda h(t)f(t,u(t)) = 0, t \in [t_1,t_2], \\ u(t_1) = 0, \alpha u(\eta) = u(t_2), \end{cases} \tag{1}$$

where $t_1 < \eta < t_2$, $0 < \alpha\phi_1(\eta) < 1$ and ϕ_1 is the unique solution of the linear boundary value problem:

$$\begin{cases} u''(t) + a(t)u'(t) + b(t)u(t) = 0, t \in [t_1,t_2], \\ u(t_1) = 0, u(t_2) = 1 \end{cases}$$

and a, b, f and h satisfy the following conditions:

(H_1) $f \in C([t_1,t_2] \times [0,+\infty),[0,+\infty))$;

(H_2) $h \in C([t_1,t_2],[0,+\infty))$ and there exists $x_0 \in [t_1,t_2]$ such that $h(x_0) > 0$;

(H_3) $a \in C[t_1,t_2]$, $b \in C([t_1,t_2],(-\infty,0))$.

2. Preliminaries

In order to prove our main results, we need the following lemmas.

Lemma 1. ([6]) *Let E is a Banach space and let $K \subset E$ be a cone. Assume that Ω_1, Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and $A : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ is a completely continuous operator such that*

(a) $\|Au\| \leq \|u\|,\ u \in K \cap \partial\overline{\Omega}_1$ *and* $\|Au\| \geq \|u\|,\ u \in K \cap \partial\overline{\Omega}_2$; *or*

(b) $\|Au\| \geq \|u\|,\ u \in K \cap \partial\overline{\Omega}_1$ *and* $\|Au\| \leq \|u\|,\ u \in K \cap \partial\overline{\Omega}_2$.

Then A has at least one fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

The arguments of Lemmas 2-6 below are similar to that of the correspondent results in [5], we omit them here.

Lemma 2. *Assume that (H_3) hold. Let ϕ_1 and ϕ_2 be the solutions of*

$$\begin{cases} u''(t) + a(t)u'(t) + b(t)u(t) = 0, t \in [t_1, t_2], \\ u(t_1) = 0, u(t_2) = 1 \end{cases} \tag{2}$$

and

$$\begin{cases} u''(t) + a(t)u'(t) + b(t)u(t) = 0, t \in [t_1, t_2], \\ u(t_1) = 1, u(t_2) = 0, \end{cases} \tag{3}$$

respectively. Then

(a) ϕ_1 *is strictly increasing on* $[t_1, t_2]$;

(b) ϕ_2 *is strictly decreasing on* $[t_1, t_2]$.

Lemma 3. *Assume that (H_3) hold. Then (2) and (3) have a unique solution, respectively.*

(H_4) $0 < \alpha\phi_1(\eta) < 1$.

Lemma 4. *Let (H_3), (H_4) hold and $y \in C[t_1, t_2]$. Then the problem*

$$\begin{cases} u''(t) + a(t)u'(t) + b(t)u(t) + y(t) = 0,\ t \in [t_1, t_2], \\ u(t_1) = 0,\ \alpha u(\eta) = u(t_2), \end{cases} \tag{4}$$

is equivalent to the integral equation

$$u(t) = \int_{t_1}^{t_2} G(t, s)p(s)y(s)ds + A\phi_1(t),$$

where

$$A = \frac{\alpha}{1 - \alpha\phi_1(\eta)} \int_{t_1}^{t_2} G(\eta, s)p(s)y(s)ds, \quad p(t) = \exp\left(\int_{t_1}^{t} a(s)ds\right),$$

$$\rho = \phi_1'(t_1), \quad G(t,s) = \frac{1}{\rho}\begin{cases} \phi_1(t)\phi_2(s), s > t, \\ \phi_1(s)\phi_2(t), s \le t. \end{cases}$$

Moreover, $u(t) \ge 0$ *for* $t \in [t_1, t_2]$ *provided that* $y(t) \ge 0$.

Lemma 5. *Let* (H_3),(H_4) *hold,* $y \in C[t_1, t_2]$ *and* y *be nonnegative. Then the unique solution of* (4) *satisfies that*

$$u(t) \ge \|u\| q(t), \quad t \in [t_1, t_2],$$

where $q(t) = \min\{\frac{\phi_1(t)}{\|\phi_1\|}, \frac{\phi_2(t)}{\|\phi_2\|}\},\ t \in [t_1, t_2]$.

Lemma 6. *Assume that* (H_3), (H_4) *hold. Let* $y \in C[t_1, t_2]$ *and be non-negative. Then for any constants* l_1 *and* l_2 *with* $t_1 < l_1 < l_2 < t_2$*, there exists* $\gamma = \min_{t \in [l_1, l_2]} q(t)$ *such that the unique solution of the problem* (4) *satisfies that*

$$u(t) \ge \gamma \|u\|, \quad t \in [l_1, l_2],$$

where $q(t)$ *is as in Lemma 5.*

3. Main Results

In this section, we will use Lemma 1 to prove the existence of positive solutions for the nonlinear eigenvalue problem (1). For fixed l_1, l_2 such that $t_1 < l_1 < l_2 < t_2$, set

$$\begin{aligned} \underline{f}_0 &= \liminf_{u \to 0^+} \frac{1}{u} \min\{f(t,u) : t \in [l_1, l_2]\}, \\ \overline{f}_0 &= \limsup_{u \to 0^+} \frac{1}{u} \max\{f(t,u) : t \in [t_1, t_2]\}, \\ \underline{f}_\infty &= \liminf_{u \to \infty} \frac{1}{u} \min\{f(t,u) : t \in [l_1, l_2]\}, \\ \overline{f}_\infty &= \limsup_{u \to \infty} \frac{1}{u} \max\{f(t,u) : t \in [t_1, t_2]\}. \end{aligned}$$

Our main results are as follows:

Theorem 7. *Let* (H_1)*-*(H_4) *hold. Then for each* λ *satisfying*

$$\frac{1}{\underline{f}_0 \gamma \int_{l_1}^{l_2} G(x_0, s) p(s) h(s) ds} < \lambda < \frac{1}{\overline{f}_\infty B}, \tag{5}$$

where $B = \int_{t_1}^{t_2} G(s,s)p(s)h(s)ds + \frac{\alpha}{1-\alpha\phi_1(\eta)} \int_{t_1}^{t_2} G(\eta,s)p(s)h(s)ds$ *and* γ *is defined in Lemma 6, the nonlinear eigenvalue problem (1) has at least one positive solution.*

Proof. It follows from Lemma 4 that the nonlinear eigenvalue problem (1) has a solution $u = u(t)$ if and only if u solves the operator equation

$$\begin{aligned} Tu(t) =&\lambda[\int_{t_1}^{t_2} G(t,s)p(s)h(s)f(s,u(s))ds \\ &+ \frac{\alpha\phi_1(t)}{1-\alpha\phi_1(\eta)} \int_{t_1}^{t_2} G(\eta,s)p(s)h(s)f(s,u(s))ds], t \in [t_1,t_2], \end{aligned} \tag{6}$$

where p, G and ρ are defined as in Lemma 4. Put

$$K = \{u : u \in C[t_1,t_2], u \geq 0, \min_{t\in[l_1,l_2]} u(t) \geq \gamma\|u\|\}.$$

It is clear that K is a cone in $C[t_1,t_2]$. Moreover, by Lemma 3 and Lemma 5, we get that $TK \subseteq K$. It is also easy to prove that $T : K \to K$ is completely continuous.

Let λ be given as in (5) and choose $\varepsilon > 0$ such that

$$\frac{1}{(\underline{f}_0 - \varepsilon)\gamma \int_{l_1}^{l_2} G(x_0,s)p(s)h(s)ds} \leq \lambda \leq \frac{1}{(\overline{f}_\infty + \varepsilon)B}. \tag{7}$$

It follows from $\underline{f}_0 = \liminf_{u\to 0^+} \frac{1}{u} \min\{f(t,u) : t \in [l_1,l_2]\}$ that there exists an $H_1 > 0$ such that $f(t,u) \geq (\underline{f}_0 - \varepsilon)u$ for $0 < u \leq H_1$ and $t \in [l_1,l_2]$. Pick $u \in K$ with $\|u\| = H_1$. Using (6) and (7), we have

$$\begin{aligned} Tu(x_0) \geq&\lambda \int_{t_1}^{t_2} G(x_0,s)p(s)h(s)f(s,u(s))ds \\ \geq&\lambda \int_{l_1}^{l_2} G(x_0,s)p(s)h(s)f(s,u(s))ds \\ \geq&\lambda(\underline{f}_0 - \varepsilon)\gamma\|u\| \int_{l_1}^{l_2} G(x_0,s)p(s)h(s)ds \geq \|u\|. \end{aligned}$$

Therefore, $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$, where $\Omega_1 = \{x \in C[t_1,t_2] : \|x\| < H_1\}$.

In view of $\overline{f}_\infty = \limsup_{u\to\infty} \frac{1}{u} \max\{f(t,u) : t \in [t_1,t_2]\}$, we derive that there exists an $\overline{H}_2 > H_1$ such that $f(t,u) \leq (f_\infty+\varepsilon)u$ for $u \geq \overline{H}_2$ and $t \in [t_1,t_2]$. We consider the following cases:

Case 1. Suppose that f is bounded. It is clear that there exists $M > 0$ with $f(t,u) \le M$ for all $u \in [0,+\infty)$, $t \in [t_1,t_2]$. Let $H_2 = \max\{2\overline{H}_2, \lambda MB\}$. Take $u \in K$ with $\|u\| = H_2$. Since $G(t,s) \le G(s,s)$ for $t,s \in [t_1,t_2]$, we infer that

$$\begin{aligned} Tu(t) \le& \lambda\Big[\int_{t_1}^{t_2} G(s,s)p(s)h(s)f(s,u(s))ds \\ &+ \frac{\alpha\phi_1(t)}{1-\alpha\phi_1(\eta)}\int_{t_1}^{t_2} G(\eta,s)p(s)h(s)f(s,u(s))ds\Big] \\ \le& \lambda MB, \quad t \in [t_1,t_2]. \end{aligned}$$

As a result, $\|Tu\| \le \|u\|$ for $u \in K \cap \partial\Omega_2$, where $\Omega_2 = \{x \in C[t_1,t_2] : \|x\| < H_2\}$.

Case 2. Suppose that f is unbounded. It follows that there exist $H_2 \ge \max\{\overline{H}_2, 2H_1\}$ and $t_0 \in [t_1,t_2]$ such that $f(t,u) \le f(t_0,H_2)$ for $0 \le u \le H_2$ and $t \in [t_1,t_2]$. Choose $u \in K$ with $\|u\| = H_2$, by (7), we have

$$\begin{aligned} Tu(t) \le& \lambda\Big[\int_{t_1}^{t_2} G(s,s)p(s)h(s)f(s,u(s))ds \\ &+ \frac{\alpha}{1-\alpha\phi_1(\eta)}\int_{t_1}^{t_2} G(\eta,s)p(s)h(s)f(s,u(s))ds\Big] \\ \le& \lambda f(t_0,H_2)B \le \lambda(\overline{f}_\infty + \varepsilon)H_2B \le H_2 = \|u\|, \quad t \in [t_1,t_2]. \end{aligned}$$

That is, $\|Tu\| \le \|u\|$ for all $u \in K \cap \partial\Omega_2$, where $\Omega_2 = \{x \in C[t_1,t_2] : \|x\| < H_2\}$.

Consequently, Lemma 1 guarantees that the nonlinear eigenvalue problem (1) has at least one positive solution in K. This completes the proof. □

Theorem 8. *Assume that* (H_1)-(H_4) *hold. Let* B *and* γ *be as in Theorem 7. Then for each* λ *satisfying*

$$\frac{1}{\underline{f}_\infty\gamma\int_{l_1}^{l_2} G(x_0,s)p(s)h(s)ds} < \lambda < \frac{1}{\overline{f}_0 B}, \tag{8}$$

the nonlinear eigenvalue problem (1) *has at least one positive solution.*

Proof. Let K and T be as in the proof of Theorem 7. Let λ be given as in (8) and choose $\varepsilon > 0$ satisfying

$$\frac{1}{(\underline{f}_\infty - \varepsilon)\gamma\int_{l_1}^{l_2} G(x_0,s)p(s)h(s)ds} \le \lambda \le \frac{1}{(\overline{f}_0 + \varepsilon)B}. \tag{9}$$

Considering $\overline{f}_0$, there exists an $H_1 > 0$ such that $f(t,u) \leq (\overline{f}_0 + \varepsilon)u$ for $0 < u \leq H_1$, $t \in [t_1, t_2]$. Pick $u \in K$ with $\|u\| = H_1$. In view of (9), we have

$$\begin{aligned} Tu(t) \leq & \lambda\Big[\int_{t_1}^{t_2} G(s,s)p(s)h(s)f(s,u(s))ds \\ & + \frac{\alpha}{1-\alpha\phi_1(\eta)}\int_{t_1}^{t_2} G(\eta,s)p(s)h(s)f(s,u(s))ds\Big] \\ \leq & \lambda(\overline{f}_0 + \varepsilon)B\|u\| \leq \|u\|, \quad t \in [t_1, t_2]. \end{aligned}$$

Consequently, $\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$, where $\Omega_1 = \{x \in C[t_1, t_2] : \|x\| < H_1\}$.

Note that $\underline{f}_\infty = \liminf_{u\to\infty} \frac{1}{u} \min\{f(t,u) : t \in [l_1, l_2]\}$ ensures that there exists an $\overline{H}_2 > 0$ satisfying $f(t,u) \geq (\underline{f}_\infty - \varepsilon)u$ for all $u \geq \overline{H}_2$ and $t \in [l_1, l_2]$. Let $H_2 = \max\{2H_1, \gamma^{-1}\overline{H}_2\}$ and $\Omega_2 = \{x \in C[t_1, t_2] : \|x\| < H_2\}$. For $u \in K$ with $\|u\| = H_2$, it follows from $\min_{l_1 \leq t \leq l_2} u(t) \geq \overline{H}_2$ that

$$\begin{aligned} Tu(x_0) \geq & \lambda \int_{t_1}^{t_2} G(x_0,s)p(s)h(s)f(s,u(s))ds \\ \geq & \lambda \int_{l_1}^{l_2} G(x_0,s)p(s)h(s)f(s,u(s))ds \\ \geq & \lambda(\underline{f}_\infty - \varepsilon)\gamma\|u\| \int_{l_1}^{l_2} G(x_0,s)p(s)h(s)ds \geq \|u\|. \end{aligned}$$

This is, $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$. Lemma 1 ensures that T has a fixed point $u(t) \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$. Hence the nonlinear eigenvalue problem (1) possesses at least one positive solution in K. This completes the proof. □

Theorem 9. *Let (H_1)-(H_4) hold, and B and γ be as in Theorem 7. Assume that there exist positive constants p_1, p_2, q_1 and q_2 with $p_1 < q_1 < q_2 < p_2$ satisfying*

$$f(t,u) \leq \overline{f}_\infty p_1, \quad (t,u) \in [t_1, t_2] \times [0, q_1], \tag{10}$$

and

$$f(t,u) \geq \underline{f}_0 p_2, \quad (t,u) \in [l_1, l_2] \times [q_2, +\infty). \tag{11}$$

Then for each λ satisfying (5), *the nonlinear eigenvalue problem* (1) *has at least two positive solutions.*

Proof. Let K and T be as in the proof of Theorem 7. Set $P_1 = \{x \in C[t_1,t_2] : \|x\| < p_1\}$, $P_2 = \{x \in C[t_1,t_2] : \|x\| < p_2\}$ and take $u \in K$ with $\|u\| = p_1$. It follows from (6) and (10) that

$$\begin{aligned} Tu(t) \leq& \lambda\left[\int_{t_1}^{t_2} G(s,s)p(s)h(s)f(s,u(s))ds \right.\\ &\left. + \frac{\alpha}{1-\alpha\phi_1(\eta)}\int_{t_1}^{t_2} G(\eta,s)p(s)h(s)f(s,u(s))ds\right]\\ \leq& \lambda p_1 \overline{f}_\infty B \leq p_1. \end{aligned}$$

That is, $\|Tu\| \leq \|u\|$ for $u \in K \cap \partial P_1$.

By $\underline{f}_0 = \liminf_{u\to 0^+} \frac{1}{u}\min\{f(t,u) : t \in [l_1,l_2]\}$, we infer that there exists an H_1 such that $0 < H_1 < \frac{1}{2}p_1$ and $f(t,u) \geq (\underline{f}_0 - \varepsilon)u$ for $0 < u \leq H_1$, $t \in [l_1,l_2]$. As in the proof of Theorem 7, we get that $\|Tu\| \geq \|u\|$ for all $u \in K \cap \partial\Omega_1$, where $\Omega_1 = \{x \in C[t_1,t_2] : \|x\| < H_1\}$. It follows from Lemma 1 that T has a fixed point $u_1 \in \overline{P}_1 \backslash \Omega_1$ with $H_1 \leq \|u_1\| \leq p_1$.

Now choose $u \in K$ with $\|u\| = p_2$. In view of (6) and (11), we get that

$$Tu(x_0) \geq \lambda \int_{t_1}^{t_2} G(x_0,s)p(s)h(s)f(s,u(s))ds \geq p_2.$$

Therefore, $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial P_2$.

Since $\overline{f}_\infty = \limsup_{u\to\infty} \frac{1}{u}\max\{f(t,u) : t \in [t_1,t_2]\}$, it follows that there exists $\overline{H}_2 > 2p_2$ such that $f(t,u) \leq (\overline{f}_\infty + \varepsilon)u$ for $u \geq \overline{H}_2$ and $t \in [t_1,t_2]$. As in the proof of Theorem 7, we derive that $\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_2$, where $\Omega_2 = \{x \in C[t_1,t_2] : \|x\| < H_2\}$ and $H_2 > \overline{H}_2$. Lemma 1 ensures that T has a fixed point $u_2 \in \overline{\Omega}_2 \backslash P_2$ with $p_2 \leq \|u_2\| \leq H_2$. As a result, T has two different fixed points u_1, u_2 in K. That is, the nonlinear eigenvalue problem (1) has at least two positive solutions u_1 and u_2. This completes the proof. □

Similarly, we obtain the following result:

Theorem 10. *Let* (H_1)*-*(H_4) *hold, and* B *and* γ *be as in Theorem 7. Assume that there exist positive constants* p_1,p_2, q_1 *and* q_2 *with* $p_1 < q_1 < q_2 < p_2$ *satisfying*

$$f(t,u) \geq \underline{f}_\infty p_1, \quad (t,u) \in [l_1,l_2] \times [0,q_1]$$

and

$$f(t,u) \leq \overline{f}_0 p_2, \quad (t,u) \in [t_1,t_2] \times [q_2,+\infty).$$

Then for each λ *satisfying* (8), *the nonlinear eigenvalue problem* (1) *has at least two positive solutions.*

4. An Example

In this section, we construct an example to explain Theorem 7.

Example 11. Consider the nonlinear eigenvalue problem

$$\begin{cases} u''(t) + \frac{1}{t}u'(t) - \frac{1}{t^2}u(t) + \lambda \frac{576t^2(u^2(t) + 180u(t))}{u(t) + 2} = 0, t \in [\frac{1}{5}, 1], \\ u(\frac{1}{5}) = 0, \quad u(\frac{2}{3}) = 2u(1). \end{cases} \tag{12}$$

It is easy to verify that $\phi_1(t) = \frac{25}{24}t - \frac{1}{24t}$, $\phi_2(t) = -\frac{5}{24}t + \frac{5}{24t}$ and $1 - \frac{1}{2}\phi_1(\frac{2}{3}) = \frac{197}{288}$. Take $l_1 = \frac{1}{3}$, $l_2 = \frac{3}{4}$, $x_0 = \frac{2}{3}$ and $\gamma = \frac{35}{288}$. Let

$$f(t, u) = \frac{(u^2 + 180u)t}{u + 2}, \ (t, u) \in [\frac{1}{5}, 1] \times [0, +\infty), h(t) = 576t, t \in [\frac{1}{5}, 1].$$

It is obvious that $f \in C([\frac{1}{5}, 1] \times [0, +\infty), [0, +\infty))$ and

$$\underline{f}_0 = \liminf_{u \to 0^+} \frac{1}{u} \min\left\{ \frac{(u^2 + 180u)t}{u + 2} : t \in \left[\frac{1}{3}, \frac{3}{4}\right]\right\} = 30,$$
$$\overline{f}_\infty = \limsup_{u \to \infty} \frac{1}{u} \max\left\{ \frac{(u^2 + 180u)t}{u + 2} : t \in \left[\frac{1}{5}, 1\right]\right\} = 1.$$

If $\frac{1}{50} < \lambda < \frac{1}{15}$, then by Theorem 7, the nonlinear eigenvalue problem (12) possesses at least one positive solution.

References

[1] V. A. Il'in, E. I. Moiseev, Nonlocal boundary value problem of the second kind for a Sturm-Liouville operator, *Differential Equations*, **23**, No. 8 (1987), 979-987.

[2] C. P. Gupta, Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equations, *J. Math. Anal. Appl.*, **168**, No. 2 (1992), 540-551, **doi:**10.1016/0022-247X(92)90179-H.

[3] R. Ma, Existence theorems for a second order three-point boundary value problem, *J. Math. Anal. Appl.*, **212**, No. 2 (1997), 430-442, **doi:**10.1006/jmaa.1997.5515.

[4] R. Ma, Positive solutions of a nonlinear three-point boundary value problem, *Electronic Journal of Differential Equations*, **1998**, No. 34 (1998), 1-8.

[5] R. Ma, H. Wang, Positive solutions of nonlinear three-point boundary value problems, *J. Math. Anal. Appl.*, **279**, No. 1 (2003), 216-227, **doi:**10.1016/S0022-247X(02)00661-3.

[6] M. A. Krasnoselskii, *Positive Solutions of Operator Equation*, Noordhoff, Groningen (1964).

[7] S. A. Marano, A remark on a second order three-point boundary value problem, *J. Math. Anal. Appl.*, **183**, No. 3 (1994), 518-522, **doi:**10.1006/jmaa.1994.1158.