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ON GRAPHS WITH BINDING NUMBER ONE

H.B. Walikar[1], B.B. Mulla[2 §]

[1]Department of Mathematics
Karnatak University
Dharwad, 580003
Karnataka State, INDIA

[2]Department of Mathematics
Smt. Indira Gandhi College of Engineering
Sector-16, Koparkhairane, Navi Mumbai-400709
Maharashtra State, INDIA

Abstract: I. Anderson (see [1]) asked to study class of graphs with binding number one. We characterize bipartite graphs with binding number one and prove that graphs with binding number one cannot be characterized in terms of forbidden subgraphs by embedding an arbitrary graph G in a graph H with binding number one. We obtain bounds on the size of a graph with binding number one. I. Anderson [1] has defined any graph G to be binding minimal if

$$bind\,(G-x) < bind\,(G)\,.$$

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[§]Correspondence author

1. Introduction

We consider only finite simple graphs G with vertex set $V(G)$ and edge set $E(G)$. For a graph $G=(V,E)$ and for a set $X \subseteq V(G)$, we denote by $\Gamma_G(X)$ or briefly $\Gamma(X)$ to be the set of vertices joined to each vertex in X. A set of independent edges which cover all the vertices of a graph is called 1-factor of that graph. A set $X \neq \emptyset$ is called admissible set if $\Gamma(X) \neq V(G)$.The binding number of a graph G is defined by D.R. Woodall (see [7]) as $bind(G) = \min_{X\in\sum} \frac{|\Gamma(X)|}{|X|}$, where Σ is the set of all admissible sets of G. Further an admissible set X is said to be a realizing set if $bind(G) = \frac{|\Gamma(X)|}{|X|}$. For concepts not defined here see [2],[3]. We enlist few results from [2], [3], [4], [5], [6] and [7] and are as follows,

Proposition 1.1. (see [7]) *If G is a bipartite graph, then*

$$bind(G) \leq 1.$$

Proposition 1.2. (see [7]) *For any graph G with minimum degree $\delta(G)$, $bind(G) \leq \frac{n-1}{n-\delta(G)}$.*

Proposition 1.3. (see [4]) *If G has a 1-factor then $bind(G) \geq 1$.*

Proposition 1.4. (see [4]) *For any graph G, $bind(G) \leq \frac{n}{\beta_0} - 1$, where β_0 denotes the vertex independence number of G.*

Proposition 1.5. (see [2], p. 171) *For a bipartite graph G,*

$$\alpha_0(G) = \beta_1(G)$$

holds.

Proposition 1.6. (see [3], p. 95) *For any non-trivial connected graph G, $\alpha_0 + \beta_0 = n$ holds.*

Proposition 1.7. (see [5]) *Let G be graph with n vertices. If $bind(G) < 1$, then every realizing set for $bind(G)$ is independent and $|\Gamma(X)| + |X| \leq n-2$ or $= n$.*

Proposition 1.8. **(see [6])** *Let G be a graph with $bind(G) = c$. If $x \in E(G)$, then for any admissible set X in $G - x$ such that $x \cap X = \phi$ and $\frac{|\Gamma(X)|}{|X|} \geq c$.*

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2. Results

Proposition 2.1. *If G is a bipartite graph of order n then, $bind(G) \leq \frac{\beta_1}{n-\beta_1}$, where β_1 denotes the edge independence number of G.*

Proof. Let $bind(G) > \frac{\beta_1}{n-\beta_1}$, then using propositions (see 1.4[4]) and (see 1.6, [3], p. 95), we get $\beta_0\beta_1 < n^2 - n\beta_0 - n\beta_1 + \beta_0\beta_1 \Rightarrow n\{n-(\beta_0+\beta_1)\} > 0$ $\Rightarrow n-(\beta_0+\beta_1) > 0$. Since $n > 0$, we have $\beta_0 + \beta_1 < n$. But as G is bipartite, $\alpha_0(G) = \beta_1(G)$ holds (see 1.5 [2], p. 171) and thereby $\alpha_0 + \beta_0 < n$, a contradiction to the proposition (see 1.6, [3], p. 95). □

Corollary 2.1.1. *If G is a bipartite graph with $bind(G) = 1$, then G has $1 - factor$.*

Proof. It follows from the proposition 2.1 and the fact that

$$\beta_1 \geq \frac{n}{2}$$

holds for all graphs. □

Proposition 2.2. *If G is a bipartite graph, then $\beta_0 = \frac{n}{2}$ if and only if $bind(G) = 1$.*

Proof. Using propositions (see 1.6, [3], p. 95), (see 1.5, [2], p. 171) we get, $\beta_1 = \frac{n}{2}$ which implies that G has 1-factor. Further using propositions (see 1.1[7]) and (see 1.3[4]), we have $bind(G) = 1$. Conversely, using proposition (see 1.4[4]), we get $\beta_0 \leq \frac{n}{2}$. But for any arbitrary graph G, $\beta_0 \geq \frac{n}{2}$ and hence $\beta_0 = \frac{n}{2}$. □

The following theorem rules out the possibility of characterizing the graphs with binding number 1 in terms of forbidden subgraphs.

Theorem 2.3. *Every graph can be embedded in a graph H with $bind(H) = 1$.*

Proof. To achieve the goal we consider two cases.

Case 1. Suppose G contains 1-factor. Let $F = \left\{u_1v_1, u_2v_2, u_{\frac{n}{2}}v_{\frac{n}{2}}\right\}$ be the 1- factor of G and H be the graph obtained from G by taking two new vertices u and v with edge set $E(H) = E(G) \cup \left\{uv, uu_1, uu_2, \ldots, uu_{\frac{n}{2}}\right\}$. Clearly H is a graph with $1 - factor$ $F' = F \cup \{uv\}$ and by proposition (see 1.3[4]), $bind(H) \geq 1$. However H is a graph of order n+2 with minimum degree $\delta(H) = 1$. Thus using proposition (see 1.2[7]), $bind(H) \leq 1$. Hence H contains G as an induced subgraph with $bind(H) = 1$.

Case 2. Suppose G does not contain 1-factor. let M be the maximum matching of G with $|M| = \beta_1$. Let $u_1, u_2, \ldots, u_\beta, u_{\beta+1}, u_{\beta+2}, \ldots, u_{2\beta}$ and $u_1u_{\beta+1}, u_2u_{\beta+1}, u_3u_{\beta+1}, \ldots, u_\beta u_{2\beta}$ be the vertices and edges of M. Label the remaining vertices as $u_{2\beta+1}, u_{2\beta+2}, \ldots, u_n$. Now the graph H is obtained from G by taking new vertices $v_1, v_2, v_{n-2\beta}$ such that

$$E(H) = E(G) \cup \{u_{2\beta+1}v_1, u_{2\beta+2}v_2, \ldots u_nv_{n-2\beta}\}.$$

Clearly the edges $F = M \cup \{u_{2\beta+1}v_1, u_{2\beta+2}v_2, \ldots u_nv_{n-2\beta}\}$ forms 1-factor of H. Lastly result follows from propositions (see 1.3[4]) and (see 1.2[7]). □

Lemma 2.4. *If $bind(G) = 1$, then there exists a realizing set X such that $X \cap \Gamma(X) = \phi$.*

Proof. Let X be the realizing set in G. If possible assume that $X \cap \Gamma(X) \neq \phi$ and $Z = X \cap \Gamma(X)$. Consider $Y = X - \Gamma(X)$ and $U = \Gamma(X) - X$, then $X = Y \cup Z$ and $\Gamma(X) = U \cup Z$. Thus $|Y \cup Z| = |U \cup Z|$ and $|Y| = |U|$. Now claim that $\Gamma(Y) = U$. If $\Gamma(Y) \neq U$ then $\Gamma(Y)$ is proper subset of U. Let $W = U - \Gamma(Y)$, then $\Gamma(X) = \Gamma(Y) \cup W \cap Z$ and $|\Gamma(X)| = |\Gamma(Y)| + |W| + |Z|$. Thus $|U| + |Z| = |\Gamma(Y)| + |W| + |Z|$ gives $|U| = |Y| > |\Gamma(Y)|$ and $1 = bind(G) \leq \frac{|\Gamma(Y)|}{|Y|} < 1$, a contradiction to the fact that $bind(G) = 1$ and hence $|U| = |\Gamma(Y)|$ holds making Y the required set. □

Theorem 2.5. *Let G be a graph of order n with $bind(G) = 1$ and $a = \min\{X : X$ is a realizing set in $G\}$, then*

$$\frac{n}{2} \leq m \leq \binom{n}{2} + \left[\frac{a(\Delta + 2) + 3a(a - n)}{2}\right],$$

where Δ is the maximum degree of G, $[X]$ is the greatest integer not greater than x and m denotes the size of G.

Proof. If $m < \frac{n}{2}$ then G contains at least one isolated vertex so that $bind(G) = 0$. For the upper bound, let X be the realizing set with $|X| = a$ such that $X \cap \Gamma(X) = \phi$ (such a realizing set exists by the above Lemma 2.4). The following observation can be made about the degree of vertices of G.

$$\deg u \leq \begin{cases} a & \text{if } u \in X \\ \Delta & \text{if } u \in \Gamma(X) \\ n - a - 1 & \text{if } u \in V(G) - (X \cup \Gamma(X)) \end{cases}$$

$$2m = \Sigma_{u\in X}\text{deg u} + \Sigma_{u\in\Gamma(X)}\text{deg u} + \Sigma_{u\in V(G)-(X\cup\Gamma(X))}\text{deg u},$$

where the sum is taken over three mutually disjoint sets that exist by the above Lemma 2.4. Clearly, $2m \leq a\times a + \Delta\times a + (n-2a)(n-a-1)$ leads to $m \leq \binom{n}{2} + \left[\frac{a(\Delta+2)+3a(a-n)}{2}\right]$ □

Remark 1. The bounds are the best possible if $G = \frac{n}{2}K_2$, where $bind(G) = 1$ and $m = \frac{n}{2}$.Also there exists a graph G of order n with $bind(G) = 1$ and $m \leq \binom{n}{2} + \left[\frac{a(\Delta+2)+3a(a-n)}{2}\right]$ Which can be seen from the following construction. Consider the graph $H_1 = K_a + \bar{K_a}$ with partite sets $V_1 = \{u_1, u_2, ..., u_a\}$, $V_2 = \{v_1, v_2, ..., v_a\}$ and $H_2 = K_{n-2a}$ with vertex set $W = \{w_1, w_2, ..., w_{n-2a}\}$ where n is even and $n > 2a$. Let G be a graph obtained by joining v_i to w_j for each $i = 1, 2, 3\ldots a$ and $j = 1, 2, \ldots, n-2a$. Clearly G contains 1-factor and hence $bind(G) \geq 1$.Also If $X = \{u_1, u_2, \ldots, u_a\}$,$\Gamma(X) = \{v_1, v_2, \ldots v_a\}$ giving $bind(G) \leq \frac{|\Gamma(X)|}{|X|} = \frac{a}{a} = 1$ and hence $bind(G) = 1$ holds. Further $\Delta(G) = n-1$ and $m = \binom{n-a}{2} + a^2 = \binom{n}{2} + \left[\frac{a(\Delta+2)+3a(a-n)}{2}\right]$. It is not difficult to see that for any odd 'n' the upper bound in the theorem cannot be attained.

Theorem 2.6. *Let G be connected graph with $bind(G) = 1$, then G is binding minimal if and only if $G = K_2$.*

Proof. Suppose $bind(G) = 1$ and G is binding minimal then $bind(G-x) < bind(G) = 1$ for every edge x of G. Let $x = uv$ be an arbitrary edge in G. Here we consider two cases:

Case 1: there exists a realizing set X in $G-x$ with $X\cap\{u,v\} = \phi$.

Case 2: $X\cap\{u,v\} \neq \phi$, for every realizing set X in $G-x$. The Case 1 implies that $\frac{|\Gamma_{G-x}(X)|}{|X|} = \frac{|\Gamma_G(X)|}{|X|} \geq bind(G) = 1$. Then, $1 > bind(G-x) = \frac{|\Gamma_{G-x}(X)|}{|X|} = \frac{|\Gamma_G(X)|-2}{|X|} \geq bind(G) = 1$, a contradiction. In case 2 we have two more possibilities to consider:

(i) both u and $v \in X$,

(ii) either $u \in X$ or $v \in X$. If (i) holds then, $|\Gamma_{G-x}(X)| = |\Gamma_G(X)| - 2$. Since $bind(G-x) < 1$, by proposition(see 1.7[5]), $|\Gamma_{G-x}(X)| + |X| \leq n-2$ or n. As such by case1 there are no vertices of the form u and v outside $X\cup\Gamma_{G-x}(X)$ and hence $|\Gamma_{G-x}(X)| + |X| = n$ holds. Hence we get, $|\Gamma_G(X)| - 2 + |X| = n$ that is, $|\Gamma_G(X)| + |X| = n+2$, a contradiction. Thus either $u \in X$ Or $v \in X$. Without loss of generality, let $u \in X$ or $v \in \Gamma_{G-x}(X)$. If $degu \geq 1$ in $G-x$ then, $\Gamma_{G-x}(X) = \Gamma_G(X)$ and

$1 > bind(G-x) = \frac{|\Gamma_{G-x}(X)|}{|X|} = \frac{|\Gamma_G(X)|-2}{|X|} \geq bind(G)$, a contradiction indicating that $degu = 0$ in $G-x$. Thus 'x' contains only one vertex namely 'u' and $G = K_2$ holds. Converse is obvious. □

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