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NUMERICAL SOLUTION OF TWO-DIMENSIONAL PDES BY AN EFFECTIVE MODIFICATION OF THE HOMOTOPY PERTURBATION METHOD

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Abstract: In this paper, we present the exact solutions of two dimensional partial differential equations(PDEs) by using an efficient modification of the homotopy perturbation method. The feature of this method is its flexibility and ability to solve linear and nonlinear PDEs without the calculation of complicated. Some illustrative examples are given to demonstrate the efficiency and reliability of the method. From the presented examples, we found that the proposed method can be applied to a wide class of linear and nonlinear PDEs.

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1. Introduction

It is well known that there are many physical and mechanical problems can

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be described by PDEs in mathematical physics, and other areas of science and engineering. PDEs arise in a wide variety of problems such as fluid dynamics, quantum field theory and plasma physics to describe the various phenomena. These problems, except for a limited numbers, do not have a precise analytical solution, so these equations should be solved using approximate methods. The homotopy perturbation method (HPM), first proposed by He in 1998 [1 - 3]. Using the homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0, 1]$ which is considered as a "small parameter". The HPM deforms a difficult problem into a simple problem which can be easily solved. In [4- 6] He gave a very lucid as well as elementary discussion of why the HPM works so well for both linear and nonlinear equations. Homotopy perturbation method is a novel and effective method, and can solve various PDEs, for example nonlinear oscillators with discontinuities [7], nonlinear wave equations [8], limit cycle and bifurcations [9], boundary value problems [10] and many other problems [11 -15]. These applications also verified that the HPM offers certain advantages over other conventional numerical methods.

The goal of this work is to extend an efficient modification of the homotopy perturbation method to solve two dimensional PDEs. Several illustrative examples are given to reveal the efficiency and reliability of the modified homotopy perturbation method.

The rest of this paper is organized as follows: In Section 2, we give basic idea of homotopy perturbation method. The modification of the homotopy perturbation method is introduced in Sections 3. In Section 4, we present numerical results to demonstrate the efficiency of the modified homotopy perturbation method for some PDEs.

2. Basic Idea of Homotopy Perturbation Method

To illustrate the basic idea of this method, we consider the following nonlinear differential equation[1-2] :

$$A(u) = f(r), \qquad r \in \delta, \tag{1}$$

with the following boundary conditions :

$$B(u, \frac{\partial u}{\partial n}) = 0, \qquad r \in \tau, \tag{2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function and τ is the boundary of the domain δ.

The operator A can be decomposed into two operators L and N, where L is a linear operator and N is a nonlinear operator. We can rewrite (1) as follows:

$$L(u) + N(u) - f(r) = 0 \qquad r \in \delta. \tag{3}$$

Using the homotopy technique, we construct a homotopy $v(r,p) : \delta \times [0,1] \longrightarrow R$, which satisfies:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0, \tag{4}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{5}$$

where $r \in \delta$ and $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation for the solution of (1), which satisfies the boundary conditions. Obviously, from (4) and (5), we have:

$$H(v,0) = L(v) - L(u_0) = 0, \tag{6}$$

$$H(v,1) = L(v) + N(v) - f(r) = 0. \tag{7}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v) - L(u_0)$ and $L(v) + N(v) - f(r)$ are homotopic. The basic assumption is that the solution of (4) and (5) can be expressed as a power series in p :

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + ... \tag{8}$$

The approximate solution of (1), therefore, can be readily obtained:

$$u = lim_{p\to 1}v = v_0 + v_1 + v_2 + v_3 + ... \tag{9}$$

The convergence of the series (9) has been proved in [1-2].

3. The Modified form of the HPM

In this section, we present the algorithm of the new modification of the homotopy perturbation method. To achieve our goal,we can construct the following homotopy:

$$H(v,p) = (1-p)(L(v) - L(u_0)) - f(r) + p[L(v) + R(v) + N(v)] = 0, \tag{10}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) - f(r) + p[R(v) + N(v)] = 0, \tag{11}$$

where L is a linear operator of lowest order, R is a linear operator of the remaining linear terms. Substituting (8) into (11), and equating coefficients of like powers of p, we obtain:

$$\begin{aligned}
p^0 &: L(v_0) - L(u_0) = f(r), \quad v_0(X,0) = g_0(X),\\
p^1 &: L(v_1) + L(u_0) + R(v_0) + N(v_0) = 0, \quad v_1(X,0) = 0,\\
p^2 &: L(v_2) + R(v_1) + N(v_0, v_1) = 0, \quad v_2(X,0) = 0,\\
p^3 &: L(v_3) + R(v_2) + N(v_0, v_1, v_2) = 0, \quad v_3(X,0) = 0,\\
&.\\
&.\\
&.
\end{aligned} \tag{12}$$

where $X = (x, y)$ and the nonlinear $N(v)$ satisfy the relation:

$$N(v) = N(v_0) + pN(v_0, v_1) + p^2 N(v_0, v_1, v_2) + \dots.$$

It is obvious that the algorithm of the new modification of the HPM, base on the homotopy given in the (10) , (11), reduces the number of the terms involved in each component and hence the size of calculations is minimized compared to the standard HPM. Moreover this reduction of the terms in each component facilitates the construction of the homotopy perturbation solution.

4. Test Problems

In order to assess the advantages and the accuracy of this method for solving two-dimensional PDEs, we will consider the following examples. **Example1.** Consider the following linear equation:

$$u_t - u_{xx} + u_{yy} + u = (1 + t)\sinh(x + y), \tag{13}$$

with the initial condition:

$$u(x, y, 0) = \sinh(x + y). \tag{14}$$

In view of the modified homotopy perturbation method (11), we construct the following homotopy:

$$u_t - (1 + t)\sinh(x + y) = p(u_{xx} - u_{yy} - u). \tag{15}$$

Substituting (8) and the initial condition (14) into the homotopy (15) and equating the terms with identical powers of p, we obtain the following set of linear differential equations:

$$\begin{aligned}
p^0 &: (u_0)_t = (1+t)\sinh(x+y), \qquad u_0(x,y,0) = \sinh(x+y),\\
p^1 &: (u_1)_t = (u_0)_{xx} - (u_0)_{yy} - (u_0), \qquad u_1(x,y,0) = 0,\\
p^2 &: (u_2)_t = (u_1)_{xx} - (u_1)_{yy} - (u_1), \qquad u_2(x,y,0) = 0,\\
p^3 &: (u_3)_t = (u_2)_{xx} - (u_2)_{yy} - (u_2), \qquad u_3(x,y,0) = 0,\\
&\;.\\
&\;.
\end{aligned}$$

Consequently, solving the above equations, the first few components of the modified HPM solution for (13) are derived as follows:

$$\begin{aligned}
u_0(x,y,t) &= (1+t+\frac{t^2}{2!})\sinh(x+y),\\
u_1(x,y,t) &= -(t+\frac{t^2}{2!}+\frac{t^3}{3!})\sinh(x+y),\\
u_2(x,y,t) &= (\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!})\sinh(x+y),\\
u_3(x,y,t) &= -(\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!})\sinh(x+y),\\
&\;.\\
&\;.
\end{aligned}$$

In this manner the other components can be easily obtained. The solution of (13), when $p \to 1$ will be as follows:

$$u(x,y,t) = (t+1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+...)\sinh(x+y).$$

Therefore

$$u(x,y,t) = (t+e^{-t})\sinh(x+y),$$

is the exact solution of the equation.

Example2.

Consider the following nonlinear equation:

$$iu_t + \frac{1}{2}(u_{xx}+u_{yy}) - (1-\sin^2(x)\sin^2(y))u - |u|^2 u = 0, \tag{16}$$

with the initial condition:

$$u(x,y,0) = \sin(x)\sin(y). \tag{17}$$

In order to solve this equation by using the modified homotopy perturbation method (11), we construct a homotopy as following:

$$iu_t + p(\frac{1}{2}(u_{xx} + u_{yy}) - (1 - \sin^2(x)\sin^2(y))u - |u|^2 u) = 0. \tag{18}$$

Substituting (8) and the initial condition (17) into the homotopy (18) and equating the terms with identical powers of p, we obtain the following set of linear differential equations:

$$\begin{aligned}
p^0 \quad &: \quad i(u_0)_t = 0, \qquad u_0(x,y,0) = \sin(x)\sin(y),\\
p^1 \quad &: \quad i(u_1)_t = -\frac{1}{2}((u_0)_{xx} + (u_0)_{yy}) + (1 - \sin^2(x)\sin^2(y))u_0 + u_0^2\overline{u_0}\\
&, \qquad u_1(x,y,0) = 0,\\
p^2 \quad &: \quad i(u_2)_t = -\frac{1}{2}((u_1)_{xx} + (u_1)_{yy}) + (1 - \sin^2(x)\sin^2(y))u_1 + u_0^2\overline{u_1}\\
&+ \quad 2u_0u_1\overline{u_0}, \qquad u_2(x,y,0) = 0,\\
p^3 \quad &: \quad i(u_3)_t = -\frac{1}{2}((u_2)_{xx} + (u_2)_{yy}) + (1 - \sin^2(x)\sin^2(y))u_2 + u_0^2\overline{u_2}\\
&+ \quad \overline{u_0}(u_1^2 + 2u_0u_2) + 2u_0u_1\overline{u_1}, \qquad u_3(x,y,0) = 0,\\
&.\\
&.
\end{aligned}$$

With solving the above equations, we obtain:

$$\begin{aligned}
u_0(x,y,t) &= \sin(x)\sin(y),\\
u_1(x,y,t) &= (-2it)\sin(x)\sin(y),\\
u_2(x,y,t) &= \frac{(-2it)^2}{2!}\sin(x)\sin(y),\\
u_3(x,y,t) &= \frac{(-2it)^3}{3!}\sin(x)\sin(y),\\
&.\\
&.
\end{aligned}$$

Hence,the solution of (16), when $p \to 1$ will be as follows:

$$u(x,y,t) = (1 + (-2it) + \frac{(-2it)^2}{2!} + \frac{(-2it)^3}{3!} + ...)\sin(x)\sin(y)$$

$$= \quad e^{-2it}\sin(x)\sin(y),$$

which is an exact solution.

Example3.
Consider the following equation:

$$u_t = u_{xx} + u_{yy} + a\sin(ax)\sin(ay)u_x + a\cos(ax)\cos(ay)u_y + u, \tag{19}$$

with the initial condition:

$$u(x, y, 0) = \sin(ax)\cos(ay). \tag{20}$$

According to the modified homotopy perturbation method (11), we construct the following homotopy:

$$u_t = p(u_{xx} + u_{yy} + a\sin(ax)\sin(ay)u_x + a\cos(ax)\cos(ay)u_y + u). \tag{21}$$

Substituting (8) and the initial condition (20) into the homotopy (21) and equating the terms with identical powers of p, we obtain the following set of linear differential equations:

$$\begin{aligned}
p^0 \quad &: \quad (u_0)_t = 0, \qquad u_0(x, y, 0) = \sin(ax)\cos(ay),\\
p^1 \quad &: \quad (u_1)_t = (u_0)_{xx} + (u_0)_{yy} + a\sin(ax)\sin(ay)(u_0)_x\\
&+ \quad a\cos(ax)\cos(ay)(u_0)_y + u_0, \qquad u_1(x, y, 0) = 0,\\
p^2 \quad &: \quad (u_2)_t = (u_1)_{xx} + (u_1)_{yy} + a\sin(ax)\sin(ay)(u_1)_x\\
&+ \quad a\cos(ax)\cos(ay)(u_1)_y + u_1, , \quad u_2(x, y, 0) = 0,\\
p^3 \quad &: \quad (u_3)_t = (u_2)_{xx} + (u_2)_{yy} + a\sin(ax)\sin(ay)(u_2)_x\\
&+ \quad a\cos(ax)\cos(ay)(u_2)_y + u_2, , \qquad u_3(x, y, 0) = 0,\\
&.\\
&.
\end{aligned}$$

The solution of the above equations, can be obtained as follows:

$$\begin{aligned}
u_0(x, y, t) &= \sin(ax)\cos(ay),\\
u_1(x, y, t) &= (1 - 2a^2)t\sin(ax)\cos(ay),\\
u_2(x, y, t) &= (1 - 2a^2)^2\frac{(t)^2}{2!}\sin(ax)\cos(ay),
\end{aligned}$$

$$
\begin{aligned}
u_3(x,y,t) &= (1-2a^2)^3\frac{(t)^3}{3!}\sin(ax)\cos(ay),\\
&\ .\\
&\ .
\end{aligned}
$$

Thus, we can obtain:

$$u(x,y,t) = (1+(1-2a^2)t+(1-2a^2)^2\frac{(t)^2}{2!}+(1-2a^2)^3\frac{(t)^3}{3!}+...)\sin(ax)\cos(ay).$$

Therefore

$$u(x,y,t) = e^{(1-2a^2)t}\sin(ax)\cos(ay),$$

is the exact solution of the equation.

5. Conclusions

In this work, we successfully apply an efficient modification of HPM for solving two-dimensional PDEs. A clear conclusion that can be drawn from our results is that this method provides fast convergence to exact solutions. It is also worth noting that this modified HPM is an effective, simple and quite accurate tool to handle and solve linear and nonlinear PDEs, having wide applications in engineering and sciences. Overall, the reliability of the method and the reduction of the size of computational give this method wide applications.

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