International Journal of Pure and Applied Mathematics
Volume 99 No. 3 2015, 257-275
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v99i3.3
ijpam.eu

A GENERALIZATION OF BAER RINGS

K. Paykan[1 §], A. Moussavi[2]

[1]Department of Mathematics
Garmsar Branch
Islamic Azad University
Garmsar, IRAN

[2]Department of Pure Mathematics
Faculty of Mathematical Sciences
Tarbiat Modares University
P.O. Box 14115-134, Tehran, IRAN

Abstract: A ring R is called generalized right Baer if for any non-empty subset S of R, the right annihilator $r_R(S^n)$ is generated by an idempotent for some positive integer n. Generalized Baer rings are special cases of generalized PP rings and a generalization of Baer rings. In this paper, many properties of these rings are studied and some characterizations of von Neumann regular rings and PP rings are extended. The behavior of the generalized right Baer condition is investigated with respect to various constructions and extensions and it is used to generalize many results on Baer rings and generalized right PP-rings. Some families of generalized right Baer-rings are presented and connections to related classes of rings are investigated.

AMS Subject Classification: 16S60, 16D70, 16S50, 16D20, 16L30
Key Words: generalized p.q.-Baer ring, generalized PP-ring, PP-ring, quasi Baer ring, Baer ring, Armendariz ring, annihilator

1. Introduction

Throughout this paper all rings are associative with identity and all modules are unital. Recall from [15] that R is a *Baer* ring if the right annihilator of

Received: September 21, 2014

§Correspondence author

every nonempty subset of R is generated by an idempotent. In [15] Kaplansky introduced Baer rings to abstract various properties of AW^*-algebras and von Neumann algebras. The class of Baer rings includes the von Neumann algebras. In [10] Clark defines a ring to be *quasi-Baer* if the left annihilator of every ideal is generated, as a left ideal, by an idempotent. He then uses the quasi-Baer concept to characterize when a finite-dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra.

Closely related to Baer rings are PP-rings. A ring R is called *right (left) PP* if every principal right (left) ideal is projective (equivalently, if the right (left) annihilator of any element of R is generated (as a right (left) ideal) by an idempotent of R). R is called a *PP-ring* (also called a Rickart ring [4, p.18]), if it is both right and left PP. The concept of PP-ring is not left-right symmetric by Chase [8]. A right PP-ring R is Baer (so PP) when R is orthogonally finite by Small [22], and a right PP-ring R is PP when R is *abelian* (idempotents are central) by Endo [11]. A ring R is called π-*regular* if for each $a \in R$ there exist a positive integer $n = n(a)$, depending on a, such that $a^n \in a^nRa^n$ [14]. A π-regular ring is called (*von Neumann*) *regular* when $n = 1$. According to Huh et al. [14], a ring R is called a *generalized left PP-ring* if for any $x \in R$ the left ideal Rx^n is projective for some positive integer n, depending on x, or equivalently, if for any $x \in R$ the left annihilator of x^n is generated by an idempotent for some positive integer n, depending on x. Von Neumann regular rings are right (left) PP by Goodearl [12, Theorem 1.1], and π-regular rings are generalized PP in the same sense as von Neumann regular rings.

Birkenmeier, Kim and Park in [7] introduced a principally quasi-Baer ring and used them to generalize many results on *reduced* (i.e., it has no nonzero nilpotent elements) PP.-rings. A ring R is called *right principally quasi-Baer* (or simply *right p.q.-Baer*) if the right annihilator of a principal right ideal is generated by an idempotent. Equivalently, R is right p.q.-Baer if R modulo the right annihilator of any principal right ideal is projective. The class of p.q.-Baer rings include any domain, any semisimple ring, any biregular ring, any Baer, and any quasi-Baer ring. Some examples were given in [7] to show that the class of left p.q.-Baer rings is not contained in the class of right PP-rings and the class of right PP-rings is not contained in the class of left p.q.-Baer rings. From [20], a ring R is called *generalized right* (*principally*) *quasi-Baer* if for any (principal) right ideal I of R, the right annihilator of I^n is generated by an idempotent for some positive integer n, depending on I.

We say a ring R is *generalized right Baer* if for any non-empty subset S of R, the right annihilator $r_R(S^n)$ is generated by an idempotent for some positive integer n, where S^n is a set that contains elements $a_1a_2...a_n$ such that $a_i \in S$ for $1 \leq i \leq n$. Given a fixed positive integer n, we say a ring R is *n-generalized right Baer* if for any non-empty subset S of R, the right annihilator of S^n is generated by an idempotent. Left cases may be defined analogously. A ring is called a *generalized Baer ring* if it is both generalized right and left Baer ring. Baer rings are clearly generalized right (left) Baer. Also, the class of generalized right (left) Baer rings is obviously included in the classes of generalized right (left) quasi Baer rings and generalized right (left) PP rings. Using Examples 2.2 and 2.1, various classes of generalized right (left) quasi Baer rings and generalized right (left) PP rings are provided which are not generalized right (left) Baer. On the other hand, in Example 2.3, we give rich classes of generalized right (left) Baer rings which are not Baer.

These classes of rings arise naturally and play a substantial role in the theory of operator algebras in functional analysis. In section 2 we provide several basic results. In section 3 we discuss various constructions and extensions under which the class of generalized right (left) Baer rings is closed.

2. Generalized Baer Rings

Given a ring R, for a nonempty subset X of R, $r_R(X)$ and $\ell_R(X)$ denote the right and left annihilators of X in R respectively. For notation we use $Z_r(R)$ and $C(R)$ for the right singular ideal and the center of the ring R, respectively.

The following examples show that there is rich classes of generalized PP ring which are not generalized Baer ring.

Example 2.1. (i) For a field F, take $F_n = F$ for $n = 1, 2, \cdots$, let

$$R = \begin{pmatrix} \Pi_{n=1}^{\infty} F_n & \bigoplus_{n=1}^{\infty} F_n \\ \bigoplus_{n=1}^{\infty} F_n & < \bigoplus_{n=1}^{\infty} F_n, 1 > \end{pmatrix}$$

which is a subring of the 2×2 matrix ring over the ring $\Pi_{n=1}^{\infty} F_n$, where $< \bigoplus_{n=1}^{\infty} F_n, 1 >$ is the F-algebra generated by $\bigoplus_{n=1}^{\infty} F_n$ and $1_{\Pi_{n=1}^{\infty} F_n}$. Then by [7, Example1.6], the ring R is a semiprime PP ring (and hence generalized PP ring) which is not p.q.-Baer. Thus by [20, Proposition 2.2(i)] the ring R is not generalized p.q.-Baer (and hence not generalized Baer).

(ii) For a field F, let $R =< \bigoplus_{n=1}^{\infty} F_n, 1 >$ be F-algebra generated by $\bigoplus_{n=1}^{\infty} F_n$ and $1_{\Pi_{n=1}^{\infty} F_n}$. Then by [19, Example 1 (2)], the ring R is commutative von Neumann regular (hence reduced PP) which is not Baer ring. Thus by Proposition 2.4, we will show that R is not generalized Baer ring.

(*iii*) Let R be a reduced PP ring which is not Baer ring (e.g., Example (ii)). Then by [14, Proposition 3] the ring $S(R, n)$ (as Lemma 3.1) is generalized PP for each $n \geq 2$, but we will show in Corollary 3.4 that the ring $S(R, n)$ is not generalized right Baer ring.

The following examples show that there are various classes of generalized quasi-Baer ring which are not generalized Baer. We denote $\mathbb{Z}$ and $\mathbb{Z}_n$ the ring of integers and the integers modulo n, respectively.

Example 2.2. (*i*) Let $M_2(\mathbb{Z})$ denote the 2-by-2 full matrix ring over $\mathbb{Z}$. Then by [14, Example 4], $M_2(\mathbb{Z})[x]$, the polynomial ring over $M_2(\mathbb{Z})$, is not generalized right PP (and hence it is not generalized right Baer) but it is quasi-Baer (and hence is generalized right quasi-Baer), by [21, Proposition 16] and [6, Theorem 1.2].

(*ii*) Let $R = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{Z},\ a \equiv d,\ b \equiv 0 \text{ and } c \equiv 0 (mod 2)\}$. Since R is a prime ring, so it is quasi-Baer and hence generalized right quasi-Baer. For each positive integer k, we have $2e_{22} \in r_R(2e_{11})^k$, where e_{11}, e_{12} denote the matrix units. Also the idempotents of R are 0 and 1. Thus R is not generalized right Baer.

(*iii*) Assume that R is a abelian generalized right quasi-Baer ring which is not generalized right Baer (e.g., Example (ii)). Then by [20, Theorem 3.2] the rings $S(R, n)$ and $R[x]/\langle x^n \rangle$ are abelian generalized right quasi-Baer for each $n \geq 2$, but we will show in Theorem 3.2 that the rings $S(R, n)$ and $R[x]/\langle x^n \rangle$ are not generalized right Baer.

(*iv*) Let I be a nonempty finite index set and for each $i \in I$, R_i be an abelian generalized right quasi-Baer ring and assume at least for one $i \in I$, R_i is not generalized right Baer (e.g., Example(ii),(iii)). Then $R = \prod_{i \in I} R_i$ is a generalized right quasi-Baer ring which is not generalized right Baer, by Proposition 3.11.

The following examples show that there are rich classes of generalized right Baer rings which are not Baer.

Example 2.3. (*i*) Let R be a reduced Baer ring. Then we will show in Theorems 3.2 and 3.7 that for each $n \geq 2$ the rings $S(R, n)$ and $T(R, n)$ (as Theorem 3.7) are abelian generalized right Baer rings but they are not Baer, since by [5, Proposition 1.5] every abelian Baer ring is reduced but the rings $S(R, n)$ and $T(R, n)$ are not reduced.

(*ii*) Let R be a reduced Baer ring. Then the rings $S(R, 2)$ and $S(R, 3)$ are abelian generalized right Baer by Theorems 3.2. Thus the rings $S(R, 2)[x]$, $S(R, 2)[[x]]$, $S(R, 3)[x]$ and $S(R, 3)[[x]]$ by Theorems 3.14, 3.20 are generalized right Baer which are not Baer.

(*iii*) Let R be a reduced Baer ring. Then we will show in Theorem 3.7, the ring $T(R,n)$ for each $n \geq 2$ is abelian generalized right Baer ring. Thus the rings $T(R,n)[x]$ and $T(R,n)[[x]]$ by Theorems 3.14, 3.20 are generalized right Baer ring but are not Baer ring.

(*iv*) For each positive integer n and prime number p, it is easy to show that the ring $\mathbb{Z}_{p^n}$ is n-generalized Baer and the ring $\mathbb{Z}_n$ is generalized Baer but they are not Baer.

(*v*) Let I be a nonempty finite index set and for each $i \in I$, R_i be an abelian generalized right Baer ring and assume at least for one $i \in I$, R_i is not Baer. Then we will show in Proposition 3.11, that the ring $R = \prod_{i \in I} R_i$ is a generalized right Baer ring which is not Baer.

Recall from [3] that a ring R satisfies *the IFP* (*insertion of factors property*) or is *semicommutative* if $r_R(x)$ is an ideal of R for all $x \in R$ (equivalently, $ab = 0$ implies $arb = 0$, for each $a, b, r \in R$).

We include the following two results to indicate conditions under which the notions Baer ring, generalized Baer ring and generalized quasi-Baer ring coincide.

Proposition 2.4. *Let R be a ring, then:*

(1) *A reduced ring R is generalized right Baer if and only if R is Baer;*

(2) *A ring R satisfying IFP is generalized right Baer if and only if it is generalized right quasi-Baer;*

(3) *A semiprime generalized right Baer ring R is quasi-Baer.*

Proof. (1) Let R be a reduced generalized right Baer ring and S a subset of R. Then $r_R(S^n) = eR$, for some element idempotent e of R and positive integer n. Since R is reduced, $r_R(S) = r_R(S^n)$, which implies that R is a Baer ring.

(2) Suppose that R be generalized right quasi-Baer and S an arbitrary subset of R. Then $r_R((< S >)^n) = eR$ for some idempotent e in R and positive integer n, where $< S >$ is the right ideal generated by S. Since R satisfies IFP we have $r_R(S^n) = r_R((< S >)^n)$. Hence $r_R(S^n) = eR$, which implies that the ring R is generalized right Baer.

(3) Since every generalized right Baer ring is generalized right quasi-Baer, R is a quasi-Baer ring by [20, Proposition 2.2(i)]. □

Proposition 2.5. *Let R be a right Noetherian ring with IFP. Then the following conditions are equivalents:*

(1) *R is generalized right Baer;*

(2) *R is generalized right quasi-Baer;*

(3) *R is generalized right p.q-Baer;*

(4) *R is generalized right PP.*

Proof. $(1) \Rightarrow (2), (2) \Rightarrow (3)$ are clearly true by definitions. $(3) \Leftrightarrow (4)$ follows from [20, Proposition 2.2(ii)]. $(3) \Rightarrow (1)$ By Preposition 2.4(ii) it is sufficient to show that R is generalized right quasi-Baer. Let I be a non-zero right ideal of R, since R is right Noetherian, $r_R(I^k) = eR$ for some idempotent $e \in R$ and positive integer k, by [20, Proposition 2.8 (ii)]. Thus R is generalized right quasi-Baer. □

Proposition 2.6. *Every prime ideal of a generalized right Baer ring R is either generated by an idempotent or it is a right essential ideal.*

Proof. Let P be a prime ideal of R not essential as a right ideal. Then there exists a non-zero right ideal I of R such that $P \cap I = 0$. Since R is generalized right Baer, there exists a positive integer n such that $r_R(I^n) = eR$ for some idempotent $e \in R$. It is clear that $P \subseteq r_R(I^n) = eR$. Let $x \in r_R(I^n)$. So $I^n xR = 0$. Since P is a prime ideal, $I^n \subseteq P$ or $xR \subseteq P$. If $I^n \subseteq P$ then $I \subseteq P$. Thus $I \cap P = I = 0$, which is a contradiction. Hence $x \in P$ and it implies that $P = eR$. □

As it is well know, every Baer or right PP ring is right non-singular. For generalized right Baer rings we have:

Proposition 2.7. *The right singular ideal $Z_r(R)$ of any generalized right Baer ring R is nil.*

Proof. Let $x \in Z_r(R)$. Then $r_R(x)$ is right essential in R. Since R is generalized right Baer, there exists an idempotent $e \in R$ such that $r_R(x^n) = eR$ for some positive integer n. We show that $x^n = 0$. Assume to the contrary that $x^n \neq 0$. So $e \neq 1$. Therefore $r_R(x^n) \cap (1-e)R = eR \cap (1-e)R = 0$ and $(1-e)R \neq 0$, which is a contradiction, since $r_R(x)$ is right essential. Therefore $x^n = 0$. □

The following example shows that a subring of a generalized right Baer ring need not be generalized right Baer.

Example 2.8. (i) Let $R = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \equiv b(modp)\}$, where p is a prime number. Then R is a commutative reduced ring. Note that the only idempotents of R are $(0, 0)$ and $(1, 1)$. One can show that $r_R((p, 0)) = (0, p)R$. So that $r_R((p, 0)R)$ does not contain a nonzero idempotent of R. Hence R is not generalized right Baer, but $\mathbb{Z} \oplus \mathbb{Z}$ is generalized right Baer ring.

(ii) Let F be the quotient field of the commutative domain $\mathbb{Z}[x]$ where $\mathbb{Z}$ is the ring of integers. Letting S be the 2-by-2 full matrix ring over F, then since S is right noetherian, S is orthogonally finite, so it is Baer by [9, Lemma 8.4]. Hence S is a generalized right Baer ring. But the 2-by-2 full matrix ring over $\mathbb{Z}[x]$, which is a subring of S, is not a generalized right PP ring by [14, Example 7], and hence it not general generalized right Baer.

In [15, Theorem 7], Kaplansky proved that the center of a Baer ring is also Baer. By [14, Proposition11] the center of PP (generalized PP) ring is also PP (generalized PP). We prove by use of Lemma 2.9, that the center of a generalized Baer ring is also generalized Baer.

Lemma 2.9. *Let S be a subset of $C(R)$ and m, n be positive integers. Suppose that $r_R(S^m) = eR$ and $\ell_R(S^n) = Rf$, for some non-zero idempotents $e, f \in R$ such that $S^h e \neq 0$ for all h with $0 < h < m$ and $fS^k \neq 0$ for all k with $0 < k < n$. Then $m = n$ and $e = f \in C(R)$.*

Proof. First assume $m < n$, then $fS^m \neq 0$ by the condition. Since $m < n$ and $e \in r_R(S^m)$, $S \subseteq C(R)$ hence $eS^n = 0$ and $e \in \ell_R(S^n) = Rf$. Thus we have $ef = e$. Since $fS^n = 0$ and S is central so $S^m fS^{n-m} = 0$ and $fS^{n-m} \subseteq r_R(S^m) = eR$. Now, we prove that $fS^{n-m} \subseteq eS^{n-m}$(**). Let $z \in fS^{n-m}$ then $z = fa_1a_2...a_{n-m}$ where $a_i \in S$. Since $fS^{n-m} \subseteq eR$, $z = ey$ for some $y \in R$. But $ez = ey = z$ hence $z = ez = efa_1a_2...a_{n-m} = ea_1a_2...a_{n-m} \in eS^{n-m}$.

Now, if $n \geq 2m$ then $n-m \geq m$ and so $eS^{n-m} = eS^m S^{n-2m} = S^m eS^{n-2m} = 0$. Hence by using (**), we have $fS^{n-m} = 0$, a contradiction, as $n > n - m$. Consequently $m < n < 2m$ and so by using (**), $fS^m = fS^{n-m}S^{2m-n} \subseteq eS^{n-m}S^{2m-n} = eS^m = S^m e = 0$. Thus $fS^m = 0$ which is a contradiction, as $m < n$. It follows that $m \geq n$. Next assume $m > n$. Then by symmetry of the preceding case we also obtain a contradiction, hence we have $m = n$. Then $eR = r_R(S^m) = \ell_R(S^n) = Rf$ implies that $e = ef = f$. Since Re is a two-sided ideal of R so $e = f \in C(R)$. ☐

Proposition 2.10. *If R is a generalized Baer ring (n-generalized Baer), then $C(R)$ is generalized Baer ring (n-generalized Baer).*

Proof. Let S be a subset of $C(R)$. Since R is generalized Baer, so $r_R(S^m) = eR$ and $\ell_R(S^n) = Rf$ for some idempotents $e, f \in R$ and positive integer

m, n. We may assume that m, n are the smallest such ones. Then $m = n$ and $e = f \in C(R)$ by Lemma 2.9. Now, we claim that $r_{C(R)}(S^n) = r_R(S^n) \cap C(R) = eC(R)$. Since $S^n e = 0$, so $eC(R) \subseteq r_{C(R)}(S^n)$. Conversely if $a \in r_{C(R)}(S^n)$ then $a = ea \in eC(R)$. Consequently, $r_{C(R)}(S^n) = eC(R)$ and thus $C(R)$ is generalized Baer. □

Example 2.8(ii) show that the concept of generalized right Baer ring is not a Morita invariant property. Because $\mathbb{Z}[x]$ is Baer (and hence is generalized Baer) but the 2-by-2 full matrix ring over $\mathbb{Z}[x]$ is not generalized right Baer. But we may find a kind of subring of generalized right Baer rings which may be generalized right Baer as follows.

Proposition 2.11. *Let R be a generalized right Baer (resp. n-generalized right Baer) ring. Then the ring eRe is generalized right Baer (resp. n-generalized right Baer), for every idempotent $e \in R$.*

Proof. Suppose that S is a subset of eRe. Since R is generalized Baer, there is an idempotent $f \in R$ such that $r_R(S^n) = fR$, for some positive n. Note that $r_{eRe}(S^n) = r_R(S^n) \cap eRe$. So $r_{eRe}(S^n) = fR \cap eRe$. Now, we show that $1-e \in r_R(S^n)$. Let $a_i \in S \subseteq eRe$ for every $1 \leq i \leq n$. Thus there exist elements $r_i \in R$ such that $a_i = er_ie$. Hence $a_1a_2...a_n(1-e) = (er_1e)(er_2e)...(er_ne)(1-e) = 0$. It follows that $1-e \in r_R(S^n) = fR$. Thus $1-e = f(1-e)$ and $ef = efe$. Let $g = ef$. Then clearly $g^2 = g \in eRe$. Now we prove that $r_{eRe}(S^n) = g(eRe)$. Since $S^n = S^n e$, $S^n g = S^n(ef) = (S^n e)f = S^n f = 0$. It follows that $g(eRe) \subseteq r_{eRe}(S^n)$. Conversely, if $y \in r_{eRe}(S^n)$ then $y = ey$. Since $y = ey \in r_{eRe}(S^n) = fR \cap eRe$ thus $y = ey = fy = eye$. Consequently $y = ey = e(fy) = gy \in g(eRe)$. Hence $r_{eRe}(S^n) \subseteq g(eRe)$. Therefore the ring eRe is a generalized right Baer ring. □

We will use the following lemma in the sequel.

Lemma 2.12. *Assume that S is a subset of R and $r_R(S^n) = eR$ for some positive integer n and a central idempotent $e \in R$. Then $r_R(S^n) = r_R(S^m)$ for each positive integer $m \geq n$.*

Proof. It is enough to show that $r_R(S^n) = r_R(S^{n+1})$. Let $x \in r_R(S^{n+1})$ then $S^{n+1}x = S^n Sx = 0$. Hence $Sx \subseteq eR$. It follows that for every $a \in S$ there exists $r \in R$ such that $ax = er$. Now we show that $Sx = Sxe$. Let $y = ax \in Sx$. So $y = ax = er = ere = axe \in Sxe$. Thus $Sx \subseteq Sxe$. Conversely if $t = axe \in Sxe$, then $t = axe = ere = er = ax \in Sx$, so $Sx = Sxe$. Hence $S^n x = S^{n-1}Sx = S^{n-1}Sxe = S^n xe = S^n ex = 0$. Thus $x \in r_R(S^n)$. □

In the next result we investigate a condition for which a ring R being generalized Baer implies $R/Nil(R)$ is a generalized Baer ring where $Nil(R)$ is the set of nilpotent elements of R.

Proposition 2.13. *Let R be a commutative generalized Baer ring. Then $R/Nil(R)$ is a commutative PP ring.*

Proof. Let $\overline{R} = R/Nil(R)$ and $\overline{0} \neq \overline{x} \in \overline{R}$. Since $R/Nil(R)$ is a reduced ring, $\ell_{\overline{R}}(\overline{x}) = \ell_{\overline{R}}(\overline{x}^k)$ for every positive integer k. Since R is generalized Baer, there exists a positive integer n such that $\ell_R(x^n) = Re$ for some idempotent $e \in R$. We show that $\ell_{\overline{R}}(\overline{x}) = \overline{R}\overline{e}$. Let $\overline{r} \in \ell_{\overline{R}}(\overline{x})$, then $rx \in Nil(R)$. So there is a positive integer m such that $(rx)^m = 0$. So $r^m \in \ell_R(x^{mn}) = \ell_R(x^n)$ by Lemma 2.12. Since $\ell_R(x^n) = Re$, $r^m e = r^m$ implies $r^m(1-e) = 0$. Then $(r(1-e))^m = 0$. So $r(1-e) \in Nil(R)$ thus $(r - re) \in Nil(R)$. Therefore $\overline{r}\ \overline{e} = \overline{r}$. It follows that $\ell_{\overline{R}}(\overline{x}) \subseteq \overline{R}\overline{e}$. Also since $ex^n = 0$, it implies $(ex)^n = 0$. So $ex \in Nil(R)$ and this means $\overline{e} \in \ell_{\overline{R}}(\overline{x})$, so $\overline{R}\overline{e} \subseteq \ell_{\overline{R}}(\overline{x})$ and the proof is complete. □

A ring R is called *orthogonally finite* if there are no infinite sets of orthogonal idempotents in R. By [18, Proposition 6.59] for any ring R, the following are equivalent:

(1) R satisfies ACC on right direct summands;

(2) satisfies DCC on left direct summands;

(3) R has no infinite set of nonzero orthogonal idempotents.

Proposition 2.14. *Let R be a orthogonally finite generalized left Baer ring. Then for every right annihilator L there exists an idempotent $e \in R$ such that $L = eR \oplus (L \cap (1-e)R)$ and $L \cap (1-e)R$ is nil.*

Proof. If L is nil then there is nothing to prove. Assume to the contrary that L is not nil. So there exists some $x \in L$ which is not nilpotent. Since R is generalized left Baer, so $\ell_R(x^n) = Rf$ for some positive integer n and some idempotent $f \neq 1$. Thus $\ell_R(L) \subseteq \ell_R(x^n) = Rf$ and hence $(1-f)R = r_R(Rf) \subseteq r_R(\ell_R(L)) = L$. Thus L has a non-zero idempotent element. Now let e be a non-zero idempotent in L such that $r_R(e)$ is minimal among all right annihilators of idempotent elements of L. Then, by a similar argument as that in [18, Theorem 7.55], one can show that $L \cap (1-e)R$ is nil. Since $R = eR \oplus (1-e)R$ and $eR \subseteq L$, it is easy to see that $L = eR \oplus (L \cap (1-e)R)$. □

3. Extensions of Generalized Baer Rings

For a ring R and (R,R)-bimodule M, let $T(R,M)=\{(a,x)\mid a\in R, x\in M\}$ with the multiplication defined by $(a_1,x_1)(a_2,x_2)=(a_1a_2, a_1x_2+x_1a_2)$. Then $T(R,M)$ is a ring which is called the trivial extension of R by M. Notice that $T(R,M)$ is isomorphism to the ring of matrices $\begin{pmatrix} a & x \\ 0 & a \end{pmatrix}$, where $a\in R, x\in M$ and the usual matrix operations are used.

Lemma 3.1. *[14, Lemma 2] Let R be an abelian ring and define*

$$S(R,n):=\left\{\begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij}\in R\right\},$$

with n a positive integer $n\geq 2$. Then every idempotent in $S(R,n)$ is of the form $\begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \end{pmatrix}$, with $f^2=f\in R$ and so $S(R,n)$ is an abelian matrix ring.

We now prove the following result which enables us to generate examples of generalized right Baer rings which are not Baer.

Theorem 3.2. *Let R be an abelian ring. Then R is a generalized right Baer ring if and only if $S(R,n)$ is a generalized right Baer ring, for each positive integer $n\geq 2$.*

Proof. We proceed by induction on n. First, we claim that the trivial extension $S(R,2)$ of R by R is a generalized right Baer ring. Let S be a subset of $S(R,2)$ and J be the set of entries of main diagonal of the elements of S. Since R is generalized right Baer, $r_R(J^m)=fR$ for some idempotent $f\in R$ and positive integer m. Since R is abelian, $r_R(J^m)=r_R(J^{m+1})=\cdots=r_R(J^{2m})=fR$, by Lemma 2.12. For any $\begin{pmatrix} a_i & b_i \\ 0 & a_i \end{pmatrix}\in S$ where, $1\leq i\leq m$, $\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix}\cdots\begin{pmatrix} a_m & b_m \\ 0 & a_m \end{pmatrix}=\begin{pmatrix} a_1a_2...a_m & b \\ 0 & a_1a_2...a_m \end{pmatrix}$ with b has m terms, and any term of it contains $m-1, a_i$'s and one b_i (*). Assume

that $e = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$ then $e^2 = e \in S(R,2)$. Now we show that $r_{S(R,2)}(S^{2m}) = eS(R,2)$. On the other hand $r_R(J^m) = fR$, so using by (*), we have
$\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} \cdots \begin{pmatrix} a_{2m} & b_{2m} \\ 0 & a_{2m} \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} = 0$. Thus $eS(R,2) \subseteq r_{S(R,2)}(S^{2m})$.
Conversely, if $\begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \in r_{S(R,2)}(S^{2m})$ then $\begin{pmatrix} a_1a_2...a_{2m} & b \\ 0 & a_1a_2...a_{2m} \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = 0$, for each $\begin{pmatrix} a_1a_2...a_{2m} & b \\ 0 & a_1a_2...a_{2m} \end{pmatrix} \in S^{2m}$. It follows that $a_1a_2...a_{2m}c = 0 = a_1a_2...a_{2m}d + bc$. Consequently $c \in fR$, and $c = fc$, since $r_R(J^m) = fR$. Therefore by using (*), $bc = bcf = 0$, hence $a_1a_2...a_{2m}d = 0$. Thus $d = fd$, as $r_R(J^m) = fR$. Consequently $\begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c \end{pmatrix}$, hence $r_{S(R,2)}(S^{2m}) \subseteq eS(R,2)$, and $S(R,2)$ is generalized right Baer ring. Therefore if $r_R(J^m) = fR$ where $f^2 = f \in R$, then we have $r_{S(R,2)}(S^{2m}) = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} S(R,2)$.
Now, assume that S is a subset of $S(R,n)$. Consider the set S_1 of elements B in $S(R,n-1)$ such that B is obtained by deleting n-th row and n-th column of a matrix in S, and the set S_2 of elements in $S(R,n-1)$ such that B is obtained by deleting 1-th row and 1-th column of a matrix in S.
Then by the induction hypothesis and Lemma 3.1, there exists $e_i^2 = e_i$ in $S(R,n-1)$, $f_i^2 = f_i \in R$ and positive integers k_i for $i = 1,2$ such that

$$r_{S(R,n-1)}(S_i^{(n-1)k_i}) = e_iS(R,n-1),\ e_i = f_iI_{n-1} \text{ and } r_R(J^{k_i}) = f_iR.$$

Put $k = max\{k_1, k_2\}$. Then $r_R(J^k) = r_R(J^{k_1}) = r_R(J^{k_2})$, by Lemma 2.12. Hence $f_1 = f_2$, $e_1 = e_2$ and $e_1S(R,n-1) = e_2S(R,n-1)$. Since $S(R,n-1)$ is abelian, by Lemma 3.1, so by using again Lemma 2.12, we have:

$$\begin{aligned} r_{S(R,n-1)}(S_1^{(n-1)k}) = r_{S(R,n-1)}(S_1^{(n-1)k_1}) &= r_{S(R,n-1)}(S_2^{(n-1)k_2}) \\ &= r_{S(R,n-1)}(S_2^{(n-1)k}). \end{aligned}$$

Now, suppose that

$$\begin{pmatrix} x & x_{12} & \cdots & x_{1n} \\ 0 & x & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{pmatrix} \in r_{S(R,n)}(S^{nk}),$$

$$\begin{pmatrix} a_1\cdots a_{nk} & y_{12} & \cdots & y_{1n} \\ 0 & a_1\cdots a_{nk} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_1\cdots a_{nk} \end{pmatrix}$$

be in S^{nk}. Since $r_{S(R,n-1)}(S_1^{(n-1)k}) = r_{S(R,n-1)}(S_2^{(n-1)k}) = e_1S(R,n-1)$, x and $x_{ij}^{,}$s are in f_1R for each i and j except x_{1n}. We have $a_1\cdots a_{nk}x_{1n} + y_{12}x_{2n} + \cdots + y_{1n}x = 0$. Since $x_{ij}^{,}$s except x_{1n} are in f_1R and f_1 is central, $a_1\cdots a_{nk}x_{1n}+y_{1n}x = 0$. Now, we know that $x = f_1x = xf_1$ and $r_R(J^{nk}) = f_1R$, so $a_1\cdots a_{nk}x_{1n}f_1 + y_{1n}xf_1 = a_1\cdots a_{nk}f_1x_{1n} + y_{1n}x = y_{1n}x = 0$. It follows that $a_1\cdots a_{nk}x_{1n} = 0$, then $x_{1n} \in f_1R$. Hence $r_{S(R,n)}(S^{nk}) \subseteq e_1S(R,n)$. Conversely, since e_1 is central and $f_1 \in r_R(J^k)$, $S^{nk}e_1 = (S^ke_1)^n = 0$. It implies that $e_1S(R,n) \subseteq r_{S(R,n)}(S^{nk})$, hence $S(R,n)$ is a generalized right Baer ring. Now, let $S(R,n)$ be a generalized right Baer ring, we show that R is also generalized right Baer. Let S be a subset of R. Put $B = \{aI_n | a \in S$ where I_n is the identity $n\times n$ matrix$\}$. Since $S(R,n)$ is generalized right Baer, $r_{S(R,n)}(B^k) = eS(R,n)$ for some $e^2 = e = fI_n$ and positive integer k, where $f^2 = f \in R$, by Lemma 3.1. Hence for any $a_i \in S$ where $1 \le i \le k$ we have $a_1\cdots a_kfI_n = 0$, since $B^ke = 0$. It follow that $a_1\cdots a_kf = 0$. Thus $fR \subseteq r_R(S^k)$. Conversely if $b \in r_R(S^k)$ then for any $a_i \in S$ we have $a_1\cdots a_kb = 0$. Hence $a_1\cdots a_kbI_n = 0$. Thus $bI_n \in r_{S(R,n)}(B^k) = eS(R,n)$. It follows that $b \in fR$. Therefore $r_R(S^k) = fR$, and R is generalized right Baer. □

Corollary 3.3. *Let R be an abelian Baer ring. Then $S(R,n)$ is a n-generalized right Baer ring, for each $n \ge 2$.*

Corollary 3.4. *Let R be a reduced ring. If $S(R,n)$ is a generalized right Baer ring, then R is a Baer ring.*

Proof. By Theorem 3.2, the ring R is generalized right Baer. Since R is reduced, so R is Baer by Proposition 2.4. □

Since every n-generalized right Baer ring is n-generalized PP, we have:

Corollary 3.5. *[14, Proposition 6] Let D be a domain. Then $S(R,n)$ is an n-generalized PP ring.*

Corollary 3.6. *[20, Corollary 3.5] Let D be a domain. Then $S(R,n)$ is an n-generalized right p.q-Baer ring.*

Let R be a ring. Consider the following set of triangular. matrices

$$T(R,n) := \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \mid a_i \in R \right\},$$

with $n \geq 2$. It is easy to see that $T(R,n)$ is a subring of the triangular matrix ring, with matrix addition and multiplication. We can denote elements of $T(R,n)$ by $(a_1, a_2, \cdots, a_n)$, then $T(R,n)$ is a ring with addition pointwise and multiplication given by
$(a_1, a_2, \cdots, a_n)(b_1, b_2, \cdots, b_n) = (a_1b_1, a_1b_2 + a_2b_1, \cdots, a_1b_n + a_2b_{n-1} + \cdots + a_nb_1)$, for each $a_i, b_j \in R$. On the other hand, there is a ring isomorphism $\varphi : R[x]/\langle x^n\rangle \to T(R,n)$, given by, $\varphi(a_1+a_2x+\cdots+a_nx^{n-1}) = (a_1, a_2, \cdots, a_n)$, with $a_i \in R$, $1 \leq i \leq n$. So $T(R,n) \cong R[x]/\langle x^n\rangle$, where $R[x]$ is the ring of polynomials in an indeterminant x, and $\langle x^n\rangle$ is the ideal generated by x^n.

Theorem 3.7. *Let R be an abelian ring. Then R is a generalized right Baer ring if and only if $T(R,n)$ is a generalized right Baer ring, for each positive integer $n \geq 2$.*

Proof. The proof is similar to that of Theorem 3.2. □

Corollary 3.8. *Let R be an abelian ring. Then R is a generalized right Baer ring if and only if $R[x]/\langle x^n\rangle$ is a generalized right Baer ring, for each positive integer $n \geq 2$.*

Corollary 3.9. *Let R be an abelian Baer ring. Then $R[x]/\langle x^n\rangle$ is a n-generalized right Baer ring, for each $n \geq 2$.*

Corollary 3.10. *Let R be a reduced ring. If $R[x]/\langle x^n\rangle$ is a generalized right Baer ring, then R is a Baer ring.*

Proposition 3.11. *Let R_i, $i \in I$ be a ring for a nonempty index set I.*

(*i*) *R_i is an n-generalized right Baer ring for each $i \in I$ if and only if $R = \prod_{i\in I} R_i$ is an n-generalized right Baer ring;*

(*ii*) *If $R = \prod_{i\in I} R_i$ is a generalized right Baer ring, then R_i is a generalized right Baer ring for each $i \in I$.*

(*iii*) *If $\mid I \mid< \infty$ and for each $i \in I$, R_i is abelian generalized right Baer ring then $R = \prod_{i\in I} R_i$ is generalized right Baer ring.*

Proof. (i) Suppose that every R_i is n-generalized right Baer. Let S be a subset of R. Hence $S = \prod_{i \in I} S_i$ for some subsets S_i of R_i. Since R_i is n-generalized right Baer, $r_{R_i}(S_i^n) = e_i R_i$ for some idempotents $e_i \in R_i$. Since $r_R(S^n) = \prod_{i \in I} r_{R_i}(S_i^n)$, $r_R(S^n) = \prod_{i \in I} e_i R_i$. Put $e = (e_i)_{i \in I} \in R = \prod_{i \in I} R_i$, it is clearly that e is idempotent. Thus $r_R(S^n) = eR$. The converse is clear.

(ii)The proof is obvious.

(iii) Let for every $i = 1, 2, ..., k$ the rings R_i are abelian generalized right Baer ring. We show that the ring $R = \prod_{i=1}^{k} R_i$ is generalized right Baer. Let S be a subset of R. Hence there exist subsets S_i of R_i such that $S = \prod_{i=1}^{k} S_i$. Since for every i the rings R_i are generalized right Baer, $r_{R_i}(S_i^{n_i}) = e_i R_i$ for some $e_i^2 = e_i \in R_i$ and positive integer n_i. Put $n = max\{n_1, n_2, ..n_k\}$. Since R_i is abelian, by Lemma 2.12, $r_{R_i}(S_i^n) = e_i R_i$ for every i. Hence $r_R(S^n) = \prod_{i=1}^{k} e_i R_i$. Put $e = (e_i)_{i=1}^{k} \in R = \prod_{i=1}^{k} R_i$, it is clear that e is idempotent. Hence $r_R(S^n) = eR$. Therefore R is generalized right Baer ring. □

The following example shows that the direct product of abelian generalized right Baer rings, when the index set is infinite, may not be generalized right Baer.

Example 3.12. Let D be a domain. Then $S(D, n)$ is abelian generalized right Baer for each $n \geq 2$, by Lemma 3.1 and Theorem 3.2. Put $R = \prod_{n=2}^{\infty} S(D, n)$. Then by [14, Example 5] the ring R is not generalized right PP, so R is not a generalized right Baer ring.

From [1], a ring R is called an *Armendariz* ring if whenever two polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ satisfy $f(x)g(x) = 0$ we have $a_i b_j = 0$ for each i and j. Following [13], for a ring R and for each positive integer n, put

$$rAnn_R(2^R)_{(n)} = \{r_R(U^n) | U \subseteq R\}$$

and

$$rAnn_{R[x]}(2^{R[x]})_{(n)} = \{r_{R[x]}(V^n) | V \subseteq R[x]\}.$$

Also for a polynomial $f(x) \in R[x]$, let C_f denotes the set of coefficients of $f(x)$ and for a subset S of $R[x]$, let C_S denotes the set $\bigcup_{f \in S} C_f$.

Proposition 3.13. *Let R be an Armendariz ring. Then for every positive integer n; the map $\varphi : rAnn_R(2^R)_{(n)} \to rAnn_{R[x]}(2^{R[x]})_{(n)}; A \to AR[x]$ is bijective.*

Proof. Let U be a subset of R. Since $r_{R[x]}(U^n) = r_R(U^n)R[x]$ so φ is a well-defined mapping. Obviously φ is injective. Now we prove that φ is surjective. First we show that for every $f_1, f_2, ..., f_n \in R[x]$ then

$$r_{R[x]}(f_1 f_2 ... f_n) = r_{R[x]}(C_{f_1} C_{f_2} ... C_{f_n}).$$

Let $g(x) = \sum_{j=0}^{m} b_j x^j \in r_{R[x]}(f_1 f_2 ... f_n)$. Then $(f_1 f_2 ... f_n)g = 0$. Since R is Armendariz, by [1, Proposition 1] $(a_1 a_2 ... a_n)b_i = 0$ for every $a_i \in C_{f_i}$ and $b_i \in C_g$. Thus $C_{f_1} C_{f_2} ... C_{f_n} g(x) = \sum_{j=0}^{m} C_{f_1} C_{f_2} ... C_{f_n} b_j = 0$. It follows that $g(x) \in r_{R[x]}(C_{f_1} C_{f_2} ... C_{f_n})$. Conversely we can prove similarly. Now, let S be a subset of $R[x]$. Hence we have $r_{R[x]}(S^n) = r_{R[x]}(\bigcup_{f_i \in S} \{f_1 f_2 ... f_n\}) = \bigcap_{f_i \in S} r_{R[x]}(f_1 f_2 ... f_n) = \bigcap_{f_i \in S} r_{R[x]}(C_{f_1} C_{f_2} ... C_{f_n}) = r_{R[x]}(\bigcup_{f_i \in S} C_{f_1} C_{f_2} ... C_{f_n})$. On the other hand, it is clear that $(C_S)^n = \bigcup_{f_i \in S} C_{f_1} C_{f_2} ... C_{f_n}$. Since for every subset U of R, $r_{R[x]}(U) = r_R(U)R[x]$, thus $r_{R[x]}(S^n) = r_{R[x]}((C_S)^n) = r_R((C_S)^n)R[x]$. It implies that φ is surjective and the proof is complete. □

Theorem 3.14. *Let R be an Armendariz ring. If R is a generalized right Baer (resp. n-generalized right Baer) ring, then $R[x]$ is generalized right Baer (resp. n-generalized right Baer).*

Proof. Assume that S is a subset of $R[x]$. Since R is generalized right Baer, $r_R((C_S)^n) = eR$ for some idempotent $e \in R$ and positive integer n. Hence by Proposition 3.13, $r_{R[x]}(S^n) = r_R((C_S)^n)R[x]$. It follows that $r_{R[x]}(S^n) = eR[x]$, and $R[x]$ is a generalized right Baer ring. □

Theorem 3.15. *Let R be a ring. If $R[x]$ is a generalized right Baer (resp. n-generalized right Baer) ring, then R is a generalized right Baer (resp. n-generalized right Baer) ring.*

Proof. Let S be a subset of R. Since $R[x]$ is generalized right Baer, $r_{R[x]}(S^n) = e(x)R[x]$ for some $(e(x))^2 = e(x) \in R[x]$ and positive integer n. Assume e_0 is the constant coefficient of $e(x)$. Then $e_0^2 = e_0$. Since $S^n e(x) = 0$, $S^n e_0 = 0$. Hence $e_0 R \subseteq r_R(S^n)$. Conversely, if $b \in r_R(S^n)$ then $S^n b = 0$, $b \in r_R(S^n) \cap R = e(x)R[x] \cap R$. So $b = e(x)h(x)$ for some $h(x) = h_0 + h_1 x + ... + h_k x^k \in R[x]$. It follows that $b = e_0 h_0$ hence $b \in e_0 R$. Therefore $r_R(S^n) = e_0 R$, and R is generalized right Baer ring. □

Since every reduced ring is Armendariz, by Theorems 3.14 and 3.15, for some $n = 1$ immediately implies the following corollary.

Corollary 3.16. *[2, Theorem B] Let R be a reduced ring. Then R is a Baer ring if and only if $R[x]$ is a Baer ring.*

Corollary 3.17. *[16, Theorem 10] Let R be an Armendariz ring. Then R is a Baer ring if and only if $R[x]$ is a Baer ring.*

The following example shows that there is a large class of Armendariz and generalized right Baer rings which are not Baer and satisfying in Theorem 3.14.

Example 3.18. (i) Let R be a reduced Baer ring. Then by [16, Proposition 2] the rings $S(R, 2)$ and $S(R, 3)$ are Armendariz. Also the rings $S(R, 2)$ and $S(R, 3)$ are generalized right Baer by Theorem 3.2. Thus the rings $S(R, 2)[x]$ and $S(R, 3)[x]$ are generalized right Baer which are not Baer, since the rings $S(R, 2)$ and $S(R, 3)$ are not reduced.

(ii) Since by [1, Theorem 2] the ring R is Armendariz if and only if $R[x]$ is Armendariz. Hence the rings $S(R, 2)[x]$ and $S(R, 3)[x]$ are also Armendariz. Thus the rings $(S(R, 2)[x])[y]$ and $(S(R, 3)[x])[y]$ are generalized right Baer by Theorem 3.14.

(iii) Let R be a reduced Baer ring. Then by [1, Theorem 5], the ring $R[x]/\langle x^n\rangle$, (and hence the ring $T(R, n)$) is Armendariz for each $n \geq 2$. Also by Corollary 3.9, the ring $T(R, n)$ is n-generalized right Baer. Thus the ring $T(R, n)[x]$ is generalized right Baer which not Baer.

From [17], a ring R is called an *power-serieswise Armendariz* ring if whenever power series $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$ satisfy $f(x)g(x) = 0$ we have $a_i b_j = 0$ for each i and j. Power-serieswise Armendariz rings are Armendariz.

Following [17], for a ring R and for each positive integer n, put

$$rAnn_R(2^R)_{(n)} = \{r_R(U^n) | U \subseteq R\}\ ,\ rAnn_{R[[x]]}(2^{R[[x]]})_{(n)} = \{r_{R[[x]]}(V^n) | V \subseteq R[[x]]\}.$$

Proposition 3.19. *Let R be a power-serieswise Armendariz ring. Then for every positive integer n; the map*

$$\varphi : rAnn_R(2^R)_{(n)} \to rAnn_{R[[x]]}(2^{R[x]]})_{(n)}; A \to AR[[x]]$$

is bijective.

Proof. The proof is similar to that of Proposition 3.13. □

Theorem 3.20. *Let R be a power-serieswise Armendariz ring. If R is a generalized right Baer (resp. n-generalized right Baer) ring, then $R[[x]]$ is also a generalized right Baer (resp. n-generalized right Baer) ring.*

Proof. The proof is similar to that of Theorem 3.14. □

Theorem 3.21. *Let R be a ring. If $R[[x]]$ is a generalized right Baer (resp. n-generalized right Baer) ring, then R is a generalized right Baer (resp. n-generalized right Baer) ring.*

Proof. The proof is similar to that of Theorem 3.15. □

Since every abelian Baer ring is reduced and by [16, Lemma 2.3 (1)], reduced rings are power-serieswise Armendariz, for $n = 1$ by Theorems 3.20 and 3.21, it immediately implies the following:

Corollary 3.22. *[16, Corollary 2.7] Let R be an abelian ring. Then R is a Baer ring if and only if $R[[x]]$ is a Baer ring.*

The following example shows that there is a large class of power-serieswise Armendariz ring and generalized right Baer which are not Baer ring and satisfying in Theorem 3.20.

Example 3.23. (i) Let R be Baer reduced ring. Then the rings $S(R, 2)$ and $S(R, 3)$ are power-serieswise Armendariz ring and generalized right Baer by [17, Proposition 3.3 , Corollary 3.6] and Theorem 3.2. Thus the rings $S(R, 2)[[x]]$ and $S(R, 3)[[x]]$ by Theorem 3.20 are generalized right Baer but those are not Baer rings, since the rings $S(R, 2)$ and $S(R, 3)$ are not reduced.

(ii) Let R be a reduced Baer ring. Then by [17, Proposition 3.3] the ring $R[x]/\langle x^n \rangle$, (and hence the ring $T(R, n)$) is power-serieswise Armendariz for each $n \geq 2$. Thus the ring $T(R, n)[[x]]$ is generalized right Baer, by Theorem 3.20.

The following example shows that condition "Armendariz" in the Theorem 3.14 and condition "power-serieswise Armendariz " in Theorem 3.20 are not superfluous.

Example 3.24. Let R be the 2-by-2 full matrix ring over $\mathbb{Z}$. Then R is Baer and hence is generalized right Baer but by [14, Example 4], $R[x]$ is not generalized right PP (and hence it is not generalized right Baer). But R is not Armendariz (and hence it is not power-serieswise Armendariz). This is because, if we take $f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} x$ and $g(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x$, then $f(x)g(x) = 0$. But $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \neq 0$.

Acknowledgments

The authors would like to express their deep gratitude to the referee for a very careful reading of the article, and many valuable comments, which have greatly improved presentation of the article.

References

[1] D. D. Anderson and V. Camillo, Armendariz rings and Gaussian rings, *Comm. Algebra*, **26** (7) (1998), 2265-2272.

[2] E. P. Armendariz, A note on extensions of Baer and PP-rings, *J.Austral. Math. Soc.*, **18** (1974), 470-473.

[3] H.E. Bell, Near-rings in which each element is a power of itself, *Bull. Aust. Math. Soc.* **2** (1970), 363-368.

[4] S.K. Berberian, Baer*-Rings, *Springer-verlag (Berlin-Heidelberg-New York*, 1972).

[5] G.F. Birkenmeier, J.Y. Kim, J.K. Park, Quasi-Baer Extensions and Biregular Rings. *Bull. Austral. Math. Soc.* , **61** (2000)39-52.

[6] G.F. Birkenmeier, J.Y. Kim, J.K. Park, Polynomial extensions of Baer and quasi-Baer rings, *J. Pure. Algebra* **159** (2001), 25-42.

[7] G.F. Birkenmeier, J.Y. Kim, J.K. Park, Principally quasi-Baer rings, *Comm. Algebra*, **29** (2)(2001), 639-660.

[8] S.A. Chase, Generalization of triangular matrices, *Nagoya Math. J.*, **18** (1961), 13-25.

[9] A.W. Chatters and C. R. Hajarnavis, Rings with chain conditions, *Pitman. Boston* 1980.

[10] W.E. Clark, Twisted matrix units semigroup algebras, *Duke Math. J.*, **34** (1967), 417-424.

[11] S. Endo, Note on PP-rings, *Nagoya Math. J.*, **17** (1960), 167-170.

[12] K.R. Goodearl, Von neumann regular rings, *Krieger: Malabar,* 1991.

[13] Y. Hirano, On annihilator ideals of a polynomial ring over a noncommutative ring, *J. Pure Appl. Algebra*, **168** (2002), 45-52.

[14] C. Huh, H. K. Kim and Y. Lee, PP-rings and generalized PP rings, *J. Pure Appl. Algebra*, **167** (2002), 37-52.

[15] I. Kaplansky, Rings of operators, *Benjamin New York* 1965.

[16] N.K. Kim, K.H. Lee, Y. Lee, Armendariz rings and reduced rings, *J. Algebra*, **223** (2) (2000), 477-488.

[17] N.K. Kim, K.H. Lee, Y. Lee, Power series rings satisfying a zero divisor property, C*omm. Algebra*, **34** (6) (2006), 2205-2218.

[18] T.Y. Lam, Lectures on Modules and Rings, *Graduate Texts in Mathematics*, **189** Springer, New York (1999).

[19] Y. Lee, N.K. Kim, C.Y. Hong, Counterexamples on Baer rings. *Comm. Algebra*, **25** (2) (1997), 497-507.

[20] A. Moussavi, H. Haj Seyyed Javadi, E. Hashemi, Generalized quasi-Baer rings, *Comm. Algebra* **33** (7)(2005), 2115-2129.

[21] P. Pollingher and A. Zaks, On Baer and quasi-Baer rings, *Duke Math. J.*, **37** (1970), 127-138.

[22] L.W. Small, Semihereditary rings, *Bull. Amer. Math. Soc.*, **73** (1967), 656-658.