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NUMERICAL SOLUTION OF N-TH ORDER FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS BY INTEGRAL MEAN VALUE THEOREM METHOD

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Abstract: In this paper, a new and robust semi-analytical method for solving n-th order Fredholm integro-differential equations is proposed. The main idea in this method is applying the mean value theorem for integrals. This method changing the problems to system of algebraic equations so by solving this system we obtain approximate solution. By present some examples and plot the error function and comparison between exact and approximate solution, we show the ability, simplicity and effectiveness of this method.

AMS Subject Classification: 65R20, 45B05
Key Words: integro-differential equation, integral mean value theorem method, system of algebraic equations

1. Introduction

In recent years, numerous works have been focusing on the development of more advanced and efficient methods for integral equations and integro-differential equations such as the linearization method [2] differential transform method [3], RF-pair method [4], product integration method [5], Hermite-type collocation method [6] and semi-analytical-numerical techniques such as the Adomian decomposition method [10].

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In this paper, we consider the general n-th order integro-differential equations of the type:

$$y^{(n)}(x) = f(x) + \int_a^b k(x,t) y^{(m)}(t)dt, \quad a < x < b, \tag{1}$$

with initial conditions
$y(a) = \alpha_0,\ y'(a) = \alpha_1,\ y''(a) = \alpha_2,\ \ldots,\ y^{(n-1)}(a) = \alpha_{n-1}$,
where $\alpha_i, i = 0, 1, ..., n-1$, are real constants, m and n are integers and $m < n$. In Eq. (1) the functions f, g and k are given, and y is the solution to be determined.

In this paper, we solve n-th order integro-differential equations by using integral mean value theorem method [1, 7]. By using this method, we transform the Fredholm integro-differential equations to a system of algebraic equations and then by solving the obtained system we find approximate solution. The mean value method produce the closed form solution also this method lead to solutions fast and accurately but other iterative methods have not this ability. This advantage can be the most important privilege of this method than the traditional methods.

This paper organized as follow : In Section 2, we present the integral mean value theorem method. Several examples are presented in Section 3. In this section by plotting the graph of error function and comparison between exact and approximate solutions we show accurate of this method. Finally, Section 4 is conclusion.

2. Integral Mean Value Theorem Method

Theorem 1. (Mean Value Theorem for Integrals, see [9]) *If $w(x)$ is continuous in $[a, b]$, then there is a point $c \in [a, b]$, such that*

$$\int_a^b w(x)dx = (b-a)w(c). \tag{2}$$

Now, we illustrate the main idea of our method.

By using theorem (1) for solving Eq. (1), we have

$$y^{(n)}(x) = f(x) + (b-a)\ K(x,c) y^{(m)}(c), \tag{3}$$

where $c \in [a, b]$, and initial conditions are $u^{(i)}(a) = \alpha_i$ and $0 \le i \le (n-1)$. We define differential operator

$$L = \frac{d^n}{dX^n},$$

the inverse operator is an integral operator given by

$$L^{-1}(.) = \int_0^x \int_0^x \ldots \int_0^x (.)\ dX.$$

By applying the inverse operator L^{-1} on both sides of Eq. (3) and using the initial condition, we obtain

$$\begin{aligned} y(x) &= \alpha_0 + \alpha_1\ x + \frac{1}{2!}\ \alpha_2\ x^2 + \ldots + \frac{1}{(n-1)!}\ \alpha_{n-1}\ x^{n-1} \\ &+ L^{-1}(f(x)) + (b-a)\ L^{-1}\ \left(K(x,c)y^{(m)}(c)\right), \end{aligned} \tag{4}$$

by replacement of c for x into Eq. (4), we have

$$\begin{aligned} y(c) &= \alpha_0 + \alpha_1\ c + \frac{1}{2!}\ \alpha_2\ c^2 + \ldots + \frac{1}{(n-1)!}\ \alpha_{n-1}\ c^{n-1} \\ &+ L^{-1}(f(c)) + (b-a)\ L^{-1}\ \left(K(c,c)y^{(m)}(c)\right). \end{aligned} \tag{5}$$

Applying $(n-m)$-time the inverse operator L^{-1} on both sides of Eq. (3) and using the initial condition, we obtain

$$\begin{aligned} y^{(m)}(x) &= \alpha_0 + \alpha_1\ x + \frac{1}{2!}\ \alpha_2\ x^2 + \ldots + \frac{1}{(n-1)!}\ \alpha_{n-1}\ x^{n-1} \\ &+ L^{-1}(f(x)) + (b-a)\ L^{-1}\ \left(K(x,c)y^{(m)}(c)\right). \end{aligned} \tag{6}$$

Now by replacement of c for x into Eq. (6), we have

$$\begin{aligned} y^{(m)}(c) &= \alpha_0 + \alpha_1\ c + \frac{1}{2!}\ \alpha_2\ c^2 + \ldots + \frac{1}{(n-1)!}\ \alpha_{n-1}\ c^{n-1} \\ &+ L^{-1}(f(c)) + (b-a)\ L^{-1}\ \left(K(c,c)y^{(m)}(c)\right). \end{aligned} \tag{7}$$

On the other hand, by Eqs. (1)-(3) and substitute Eq. (6) in Eq. (1) we have

$$\begin{aligned} f(x) + (b-a)\ K(x,c)y^{(m)}(c) &= f(x) + \int_a^b K(x,t) \\ \times F\left[\sum_{i=0}^{n-1} \frac{\alpha_i}{i!} t^{n-1} + L^{-1}(f(t)) + (b-a)\ L^{-1}(K(t,c))y^{(m)}(c)\right]\ dt, \end{aligned} \tag{8}$$

by eliminating of $f(x)$ and substituting $x = c$ into Eq. (8), we can rewrite it as follows:

$$
\begin{aligned}
&(b-a)\ K(c,c)y^{(m)}(c) = \int_a^b K(c,t) \\
&\times F\left[\sum_{i=0}^{n-1} \frac{\alpha_i}{i!} t^{n-1} + L^{-1}(f(t)) + (b-a)\ L^{-1}(K(t,c))y^{(m)}(c)\right]\ dt.
\end{aligned} \tag{9}
$$

Now, by solving system of equations (5), (7) and (9) we obtain c, $y(c)$ and $y^{(m)}(c)$ so by substitute in Eq. (4) approximate solution will be obtained.

3. Numerical Results

In this section, several examples of Fredholm integro-differential equations are chosen to illustrate the presented method. We solve this examples by using the integral mean value theorem method then we plot the comparative curve of exact and approximate solution and error function for this examples.

Example 1. Consider the Fredholm integro-differential equation [8]

$$
\begin{cases}
y'''(x) = \sin x - x - \displaystyle\int_0^{\frac{\pi}{2}} xty'(t)\ dt, \\
y(0) = 1, y'(0) = 0, y''(0) = -1.
\end{cases} \tag{10}
$$

The exact solution for this problem is $y(x) = \cos x$.

Algorithm 1:
Step 1: Applying theorem (1) for Eq. (10)

$$
y'''(x) = \sin x - x - (\frac{\pi}{2} - 0)xcy'(c). \tag{11}
$$

Step 2: Applying 3-th the inverse operator L^{-1} on both sides of Eq. (11) and using the initial condition

$$
y(x) - 1 + \frac{x^2}{2} = \frac{1}{48}\left[-48 + 24x^2 - x^4(2 + c\pi y'(c))\right] + \cos x. \tag{12}
$$

Step 3: Substitute $x = c$ in Eq. (12)

$$
y(c) - 1 + \frac{c^2}{2} = \frac{1}{48}\left[-48 + 24c^2 - c^4(2 + c\pi y'(c))\right] + \cos c. \tag{13}
$$

Step 4: Applying 2-th the inverse operator L^{-1} on both sides of Eq. (11) and using the initial condition

$$y'(x) + x = x - \frac{1}{12}x^3(2 + c\pi y'(c)) - \sin x. \tag{14}$$

Step 5: Substitute $x = c$ in Eq. (14)

$$y'(c) + c = c - \frac{1}{12}c^3(2 + c\pi y'(c)) - \sin c. \tag{15}$$

Step 6: By Eqs. (10)-(11) and substitute (14) in (10) so eliminating of $f(x)$ we have

$$\frac{\pi}{2}xcy'(c) = \int_0^{\frac{\pi}{2}} xt\left[-t + t - \frac{1}{12}t^3(2 + c\pi y'(c)) - \sin t\right] dt. \tag{16}$$

Step 7: Substitute $x = c$ in Eq. (16)

$$\frac{\pi}{2}ccy'(c) = \int_0^{\frac{\pi}{2}} ct\left[-t + t - \frac{1}{12}t^3(2 + c\pi y'(c)) - \sin t\right] dt. \tag{17}$$

Step 8: Solve system of three equations and three unknowns (13) and (15) and (17) simultaneously and find c and $y(c)$ and $y'(c)$ we get

$$\begin{aligned} &c = 0.8484992823467628, \\ &y(c) = 0.6611098620367774, \\ &y'(c) = -0.7502891111549691. \end{aligned}$$

Step 9: By substitute c and $y(c)$ and $y'(c)$ in Eq. (12) find approximate solution.

Hence, we have $y(x) = \cos x - 1.11022 \times 10^{-16}$.

Figure 1 show the comparison between exact solution and approximate solution and error function is demonstrated in Figure 2.

Example 2. Consider the Fredholm integro-differential equation [8]

$$y''(x) = e^x - x + \int_0^1 xty(t)dt,$$

with initial condition

$$y(0) = 1, y'(0) = 1,$$

for which the exact solutions is $y(x) = e^x$.

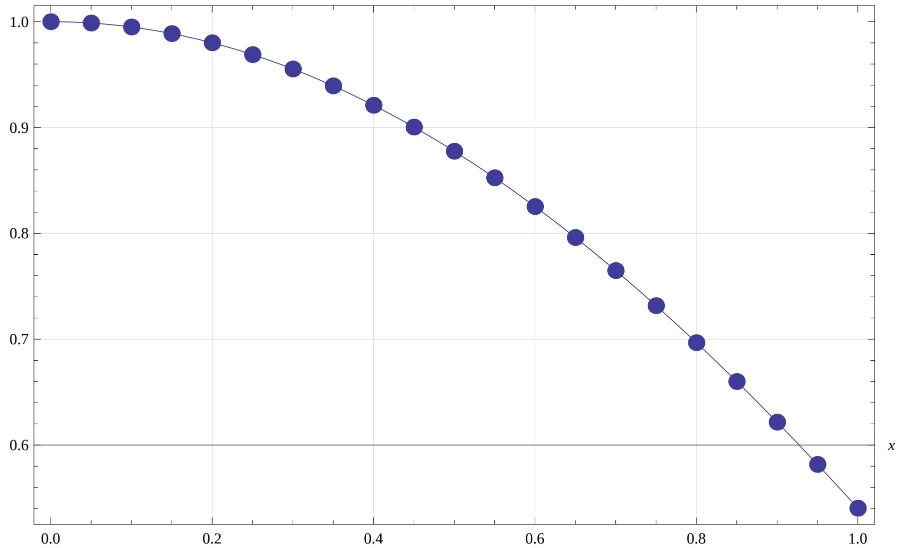

Figure 1

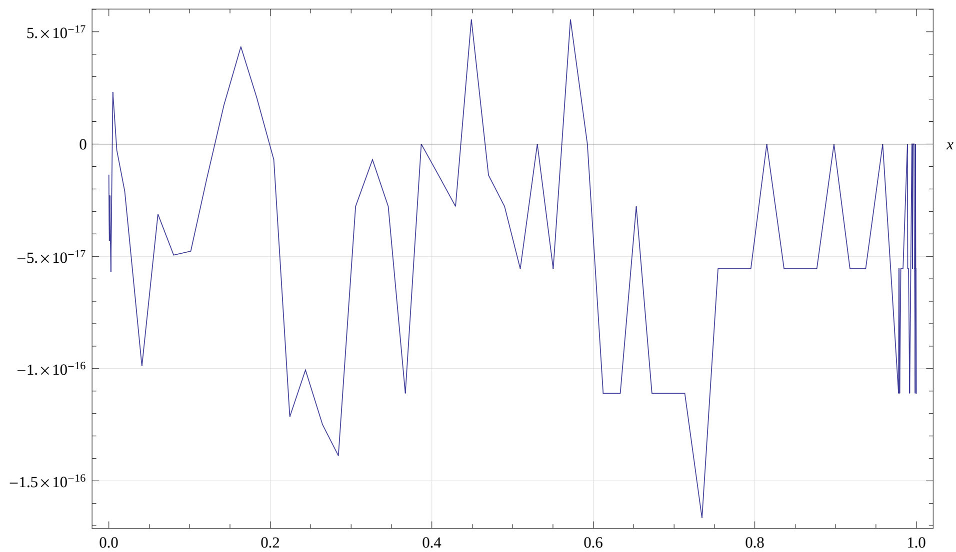

Figure 2

Using the presented method leads to the following system of two equations

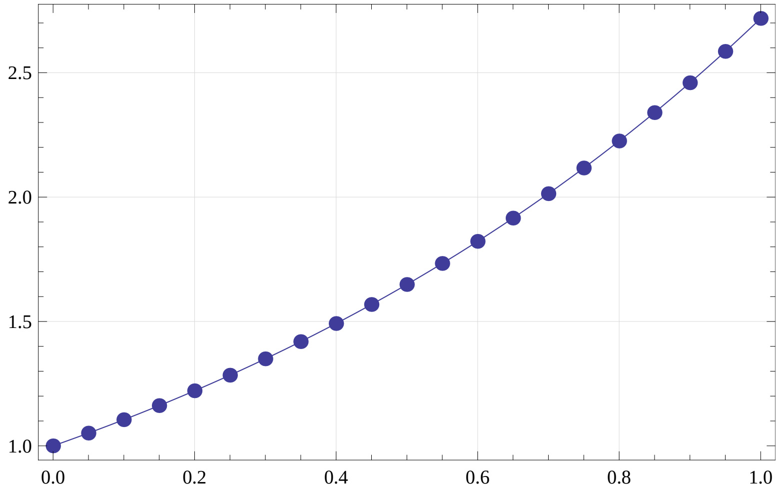

Figure 3

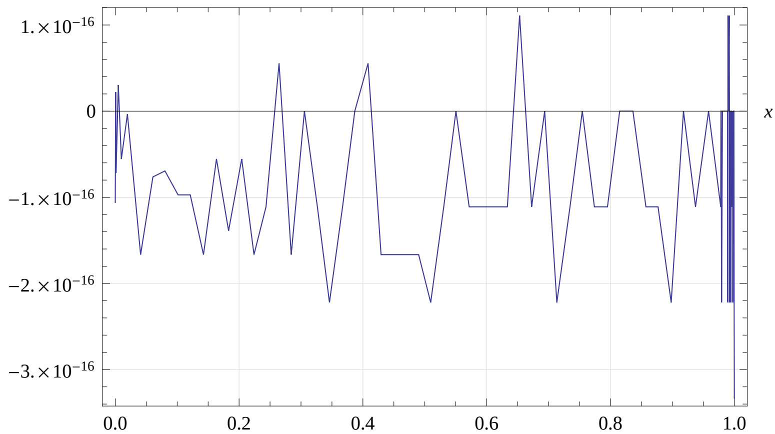

Figure 4

and two unknowns:

$$\begin{cases} ccy(c) = \dfrac{1}{30}c(29 + cy(c)), \\ \\ y(c) = 1 + c + e^c + \dfrac{1}{6}(-6 - 6c + c^3(-1 + cy(c))). \end{cases}$$

Solving the obtained system leads to

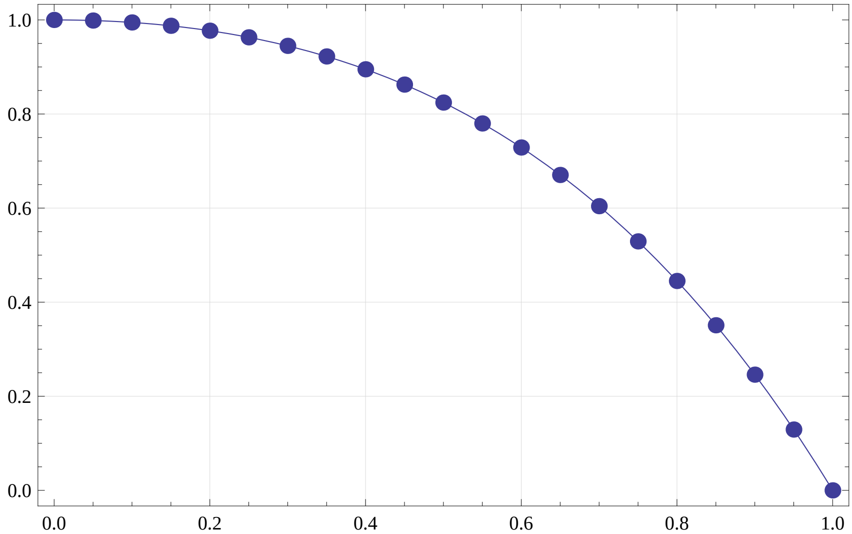

Figure 5

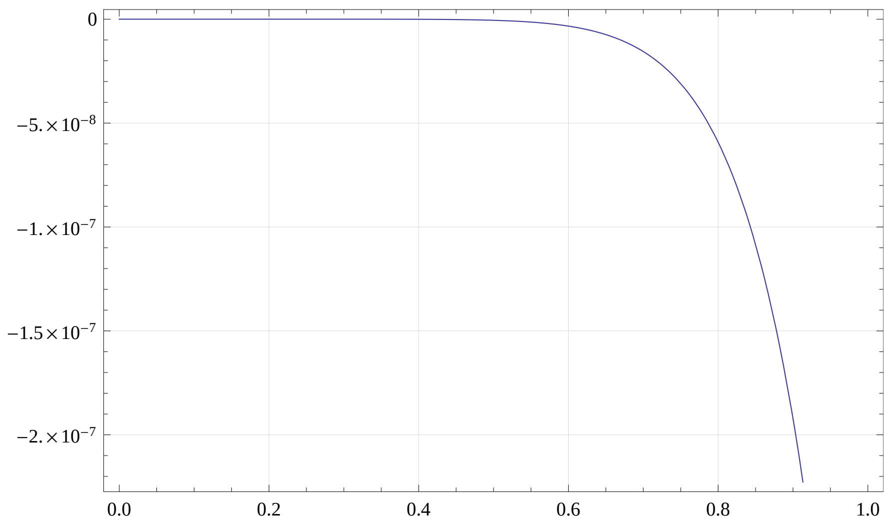

Figure 6

$$c = 0.5671432904097838,$$
$$y(c) = 1.7632228343518968.$$

Hence, approximate solution is

$$y(x) = e^x - 2.22045 \times 10^{-16}.$$

Figure 3 show the comparison between exact solution and approximate solution and error function is demonstrated in Figure 4.

Example 3. Consider the Fredholm integro-differential equation [8]

$$y^{(8)}(x) = -8e^x + x^2 + e^x(1-x) + \int_0^1 x^2 y'(t)\, dt,$$

with initial condition

$$y(0) = 1,\ y'(0) = 0,\ y''(0) = -1,\ y'''(0) = -2,$$

$$y^{(4)}(0) = -3,\ y^{(5)}(0) = -4,\ y^{(6)}(0) = -5,\ y^{(7)}(0) = -6,$$

with the exact solutions are $y(x) = (1-x)e^x$.

Using the presented method leads to the following system of equations:

$$\begin{cases} c^2 y'(c) = \dfrac{c^2}{1814400} + \dfrac{(c^2 y'(c))}{1814400}, \\[2ex] y(c) = -e^c(-1+c) + \dfrac{c^{10}(1+y'(c))}{1814400}, \\[2ex] y'(c) = \dfrac{c^9(1+y'(c))}{181440}. \end{cases}$$

By using the method, values of c and $y(c)$ and $y'(c)$ are found as follows:

$$\begin{aligned} &c = 0.9927542410318262, \\ &y(c) = 0.01955599432612305, \\ &y'(c) = 5.16208818536712 \times 10^{-6}. \end{aligned}$$

Hence, the absolute error is -5.45662×10^{-7}. Figure 5 show the comparison between exact solution and approximate solution and error function is demonstrated in Figure 6.

4. Conclusions

In this paper, By using integral mean value theorem we present the new method for solving Fredholm n-th order integro-differential equations. In this method, by solving the system of equations we find unknowns and then by replace them in main relation we find solution. This method have advantages than other methods. Simplicity of this method, produce the closed form solutions, accurate of obtained solutions are some advantages of this method. Examples that were presented show the ability of the model.

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