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ON TRANSLATION SURFACES WITH ZERO GAUSSIAN CURVATURE IN $H^2 \times R$

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Abstract: In this paper, we study translation surfaces in the product space $H^2 \times R$ and classify translation surfaces with zero Gaussian curvature in $H^2 \times R$.

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1. Introduction

In the three dimensional Euclidean space R^3, a surface Σ is called a *translation surface* if it is given by an immersion

$$\phi : U \subset R^2 \longrightarrow R^3 : \phi(x, y) = (x, y, z(x, y)),$$

where $z(x, y) = f(x) + g(y)$, for f and g smooth functions of a single variable, that is, ϕ is obtained as an Euclidean translation of the smooth curve, $\alpha(s) = (x, 0, f(x))$, along the curve $\beta(y) = (0, y, g(y))$. One of the famous examples of minimal surfaces in R^3 is a Scherk's minimal translation surface. In fact, Scherk [5] showed that except the planes, the only minimal translation surfaces are the surfaces given by

$$z = \frac{1}{a} \log \left| \frac{\cos(ax)}{\cos(ay)} \right| = \frac{1}{a} \log |\cos(ax)| - \frac{1}{a} \log |\cos(ay)|,$$

where c is a non-zero constant.

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On the other hand, Liu [2] was generalized to translation surfaces with constant mean curvature and constant Gaussian curvature in R^3, in particular, he proved that a translation surface with constant Gaussian curvature is congruence to a cylinder, that is, it has a zero Gaussian curvature.

The concept of translation surfaces in R^3 can be generalized the surfaces in the three dimensional Lie group, in particular, homogeneous manifolds. Inoguchi et al. [1] was defined translation surfaces in the 3-dimensional Heisenberg group Nil_3, and classified minimal translation surfaces in Nil_3. López and Munteanu [3] studied minimal translation surfaces in the Sol space Sol_3. Recently, the present author [6] was defined translation surfaces in the Riemannian product space $H^2 \times R$, and was completely classified minimal translation surfaces.

In this paper, we are going to study translation surfaces with zero Gaussian curvature in the product space $H^2 \times R$.

2. Preliminaries

Along this work, we will consider the upper half-plane model for the hyperbolic plane, that is, $H^2 = \{(x, y) \in R^2 \mid y > 0\}$ endowed with the metric

$$g_H = \frac{dx^2 + dy^2}{y^2}.$$

The hyperbolic plane H^2, with the group structure derived by the composition of proper affine maps, is a Lie group and the metric g_H is left invariant. Then the product space $H^2 \times R$ is a Lie group with the product structure (cf. [4])

$$L_{(x,y,z)}(\bar{x}, \bar{y}, \bar{z}) = (x, y, z) * (\bar{x}, \bar{y}, \bar{z}) = (\bar{x}y + x, y\bar{y}, z + \bar{z})$$

and the left invariant metric is given by the product metric

$$g = \frac{dx^2 + dy^2}{y^2} + dz^2.$$

With respect to the metric g, an orthonormal basis of left invariant vector fields on $H^2 \times R$ is

$$e_1 = y\frac{\partial}{\partial x}, \quad e_2 = y\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.$$

From this, the Lie brackets are given by

$$[e_1, e_2] = -e_1, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = 0,$$

and the Levi-Civita connection $\tilde{\nabla}$ of $H^2 \times R$ is expressed as

$$\begin{array}{lll} \tilde{\nabla}_{e_1} e_1 = e_2, & \tilde{\nabla}_{e_1} e_2 = -e_1, & \tilde{\nabla}_{e_1} e_3 = 0, \\ \tilde{\nabla}_{e_2} e_1 = 0, & \tilde{\nabla}_{e_2} e_2 = 0, & \tilde{\nabla}_{e_2} e_3 = 0, \\ \tilde{\nabla}_{e_3} e_1 = 0, & \tilde{\nabla}_{e_3} e_2 = 0, & \tilde{\nabla}_{e_3} e_3 = 0. \end{array}$$

A translation surface in the three dimensional Lie group equipped with a left invariant metric is a surface in the group parametrized as a product of two curves. Since the group operation $*$ of $H^2 \times R$ is not commutative, we give two definitions of translation surfaces in $H^2 \times R$.

Definition 1. ([6]) A surface Σ in the product space $H^2 \times R$ is a translation surface if it is parametrized by

$$\Sigma_1 := x(s,t) = (0, s, f(s)) * (g(t), t, 0) = (sg(t), st, f(s)) \tag{2.1}$$

or

$$\Sigma_2 := x(s,t) = (g(t), t, 0) * (0, s, f(s)) = (g(t), st, f(s)), \tag{2.2}$$

where $f(s)$ and $g(t)$ are smooth functions and $s, t > 0$, which is called a translation surface of type I or II, respectively.

Remarks. 1. If one curve lies in the xz-plane, then the translation surface is a part of xz-plane.

2. The translation surfaces generated by $\alpha(s) = (0, c_1, s)$ and $\beta(t) = (t, c_2, 0)$ $(c_1, c_2 \in R^+)$ are planes.

So, translation surfaces of type I or II except for Remarks 1 and 2 are meaningful for our study, because planes are trivial flat surfaces.

3. Translation Surfaces of Type I with Zero Gaussian Curvature

Let Σ_1 be a translation surface of type I in the product space $H^2 \times R$. Then, Σ_1 is parametrized by

$$x(s,t) = (sg(t), st, f(s)) \tag{3.1}$$

for all $s > 0$ and $t > 0$. In the case, the coefficients of the first fundamental form of Σ_1 are given by ([6])

$$g_{11} = g(x_s, x_s) = \left(\frac{g(t)}{st}\right)^2 + \frac{1}{s^2} + (f'(s))^2,$$
$$g_{12} = g(x_s, x_t) = \frac{g(t)g'(t)}{st^2} + \frac{1}{st},$$
$$g_{22} = g(x_t, x_t) = \left(\frac{g'(t)}{t}\right)^2 + \frac{1}{t^2},$$

and the coefficients of the second fundamental form of Σ_1 are

$$\begin{aligned} h_{11} = g(\tilde{\nabla}_{x_s} x_s, N) = & \frac{1}{ws^2t^3}\left(2tf'(s)g(t) + f'(s)g(t)^2g'(t)\right. \\ & \left. -t^2f'(s)g'(t) + stf''(s)g(t) - st^2f''(s)g'(t)\right), \\ h_{12} = g(\tilde{\nabla}_{x_s} x_t, N) = & \frac{1}{wst^3}\left(f'(s)g(t) + f'(s)g(t)g'(t)^2\right), \\ h_{22} = g(\tilde{\nabla}_{x_t} x_t, N) = & \frac{1}{wt^3}\left(-tf'(s)g''(t) + 2f'(s)g'(t) + f'(s)g'(t)^3\right. \\ & \left. -f'(s)g'(t)\right), \end{aligned}$$

where $N = \frac{1}{w}(x_s \times x_t)$ is a unit normal vector field of Σ_1 and $w = ||x_s \times x_t||$.

We suppose that the translation surface Σ_1 of type I has zero Gaussian curvature. Then we obtain

$$\begin{aligned} & sf'(s)f''(s)[t^3g'(t)g''(t) - t^2g(t)g''(t) - t^2g'(t)^4 + tg(t)g'(t)^3 \\ & - t^2g'(t)^2 + tg(t)g'(t)] - f'(t)^2[2t^2g(t)g''(t) + tg(t)^2g'(t)g''(t) \\ & - t^3g'(t)g''(t) + t^2g'(t)^4 - 2tg(t)g'(t)^3 + g(t)^2g'(t)^2 + t^2g'(t)^2 \\ & - 2tg(t)g'(t) + g(t)^2] = 0. \end{aligned} \tag{3.2}$$

Thus, we deduce the existence of a real number $a \in R$ such that

$$\begin{aligned} & sf'(s)f''(s) = af'(s)^2, \\ & 2t^2g(t)g''(t) + tg(t)^2g'(t)g''(t) - t^3g'(t)g''(t) + t^2g'(t)^4 \\ & - 2tg(t)g'(t)^3 + g(t)^2g'(t)^2 + t^2g'(t)^2 - 2tg(t)g'(t) + g(t)^2 \\ & = a[t^3g'(t)g''(t) - t^2g(t)g''(t) - t^2g'(t)^4 + tg(t)g'(t)^3 \\ & - t^2g'(t)^2 + tg(t)g'(t)]. \end{aligned} \tag{3.3}$$

Let us distinguish the following cases:

1. If $a = 0$, then $f(s)$ is constant and $g(t)$ is a solution of the following ODE:

$$\begin{aligned} & g''(t)[2t^2g(t) + tg(t)^2g'(t) - t^3g'(t)] \\ & + g'(t)^2[t^2g'(t)^2 - 2tg(t)g'(t) + g(t)^2 + t^2] \\ & + g(t)[g(t) - 2tg'(t)] = 0. \end{aligned} \tag{3.4}$$

   In the case, the surface Σ_1 is parametrized as

$$x(s,t) = (sg(t), st, b),$$

   where b is constant and $g(t)$ is any function satisfying (3.4) and it is a plane.

2. Suppose now $a \neq 0$. From the first equation in (3.3), we obtain $f'(s) = 0$ or $f'(s) = c_1 s^a$. If $f(s)$ is constant, Σ_1 is a plane. If $a = -1$, $f(s) = c_1 \ln s + c_2$, and if $a \neq -1$, $f(s) = \frac{c_1}{a+1}s^{a+1} + c_2$ with $c_1, c_2 \in R$. From the second equation in (3.3), $g(t)$ is a solution of the following ODE:

$$\begin{aligned} & (a+2)tg(t)[tg''(t) - g'(t)^3 - g'(t)] \\ & + (a+1)t^2g'(t)[-tg''(t) + g'(t)^3 + g'(t)] \\ & + g(t)^2[tg'(t)g''(t) + g'(t)^2 + 1] = 0. \end{aligned} \tag{3.5}$$

Thus we have

Theorem 2. *Let Σ_1 be a translation surface of type I in $H^2 \times R$. If Σ_1 has zero Gaussian curvatre, then Σ_1 is a plane or parametrized as*

$$x(s,t) = (sg(t), st, f(s)),$$

where $f(s) = c_1 \ln s + c_2$ or $f(s) = \frac{c_1}{a+1}s^{a+1} + c_2 (a \neq -1)$ and $g(t)$ is the function satisfying equation (3.5).

As the special case to compute (3.5), we assume that $g(t)$ is polynomial of degree n and a is positive integer. Then, it can be expressed as

$$g(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_0,$$

where a_n is a non-zero constant.

We consider two cases in order to solve it:

1. Let $n \geq 2$. Then the coefficient of the highest degree t^{4n-2} of the left hand side of (3.5) comes from

$$-(a+2)tg(t)g'(t)^3 + (a+1)t^2g'(t)^4 + g(t)^2[tg'(t)g''(t) + g'(t)^2]$$

and is given by

$$(a+1)a_n^4n^3(n-1).$$

This cannot vanish since $a_n \neq 0$, $n \geq 2$ and a is positive integer. It is a contradiction.

2. Let $n = 1$. In the case, we put $g(t) = b_1t + b_0$ with real constants $b_1 \neq 0$ and b_0. Then, equation (3.5) can be rewritten as follows:

$$(b_1^2+1)[-(a+1)b_1tg(t) + (a+1)b_1^2t^2 + g(t)^2] = 0,$$

it follows that $b_0 = 0$.

From Theorem 2, we thus have

Theorem 3. *Let Σ_1 be a translation surface of type I with zero Gaussian curvature in $H^2 \times R$. If $f(s)$ and $g(t)$ are polynomial, then Σ_1 is parametrized as*

$$x(s,t) = (b_1st, st, \frac{a_1}{m}s^m + a_2),$$

where $b_1 \neq 0, a_1, a_2 \in R$ and m is positive integer.

4. Translation Surfaces of Type II with Zero Gaussian Curvature

Let Σ_2 be a translation surface of type II in the product space $H^2 \times R$. Then, Σ_2 is parametrized by

$$x(s,t) = (g(t), st, f(s)) \tag{4.1}$$

for all $s > 0$ and $t > 0$.

On the other hand, the coefficients of the first fundamental form of Σ_2 are given by ([6])

$$g_{11} = \frac{1}{s^2} + f'(s)^2, \quad g_{12} = \frac{1}{st}, \quad g_{22} = \frac{g'(t)^2}{s^2t^2} + \frac{1}{t^2}$$

and the coefficients of the second fundamental form are

$$\begin{aligned} h_{11} &= -\frac{g'(t)}{ws^3t}(f'(s)+sf''(s)), \\ h_{12} &= \frac{1}{ws^2t^2}f'(s)g'(t), \\ h_{22} &= \frac{1}{ws^3t^3}[f'(s)g'(t)(g'(t)^2-s^2)-s^2f'(s)(tg''(t)-2g'(t))]. \end{aligned}$$

Suppose that Σ_2 has zero Gaussian curvature. Then we have

$$\begin{aligned} &f'(s)g'(t)[-f'(s)g'(t)^3-2s^2f'(s)g'(t)+s^2tf'(s)g''(t) \\ &-sf''(s)g'(t)^3-s^3f''(s)g'(t)+s^3tf''(s)g''(t)]=0. \end{aligned} \tag{4.2}$$

If $f'(s)=0$ or $g'(t)=0$, the surface Σ_2 is a plane.

Now, we assume that $f'(s)g'(t)\neq 0$ on an open interval. Then from (4.2) we have the following equation:

$$\begin{aligned} &tg''(t)[s^2f'(s)+s^3f''(s)]-g'(s)^3[f'(s)+sf''(s)] \\ &-g'(t)[2s^2f'(s)+s^3f''(s)]=0. \end{aligned} \tag{4.3}$$

Dividing (4.3) by $g'(t)$ and taking the derivative with respect to t we have

$$\frac{d}{dt}\left(\frac{tg''(t)}{g'(t)}\right)(s^2f'(s)+s^3f''(s))=\frac{d}{dt}\left(g'(t)^2\right)(f'(s)+sf''(s)).$$

Therefore, there exists a real number $a\in R$ such that

$$\begin{aligned} f'(s)+sf''(s)&=a(s^2f'(s)+s^3f''(s)) \\ \frac{d}{dt}\left(\frac{tg''(t)}{g'(t)}\right)&=a\frac{d}{dt}\left(g'(t)^2\right). \end{aligned} \tag{4.4}$$

Let us distinguish the following cases:

1. Suppose that $a=0$. Then the second equation of (4.4) leads to $tg''(t)=bg'(t)$ ($b\in R$). It follows that $g'(t)=c_1t^b$. If $b\neq -1$, then $g(t)=\frac{c_1}{b+1}t^{b+1}+c_2$ and if $b=-1$, $g(t)=c_1\ln t+c_2$ ($c_1,c_2\in R$). From the first equation of (4.4), we have the ordinary differential equation $sf''(s)=-f'(s)$, and the general solution is given by $f(s)=c_3\ln s+c_4$ ($c_3,c_4\in R$).

2. If $a\neq 0$, then the first equation of (4.4) becomes

$$s(as^2-1)f''(s)+(as^2-1)f'(s)=0,$$

and the general solution is

$$f(s) = c_1 \ln s + c_2,$$

where $c_1, c_2 \in R$.

On the other hand, the second equation of (4.4) writes as

$$g''(t) - \frac{b}{t} g'(t) = \frac{a}{t} g'(t)^3, \tag{4.5}$$

where b is a constant of integration. We put $g'(t) = \varphi(t)$. Then we can obtain the Bernoullis equation as follows:

$$\frac{d\varphi}{dt} - \frac{b}{t}\varphi = \frac{a}{t}\varphi^3$$

and its solution is given by

$$\varphi^{-2} = \frac{-2a}{t^{2b}} \left(\int t^{2b-1} dt - \frac{c_1}{2a} \right), \tag{4.6}$$

where c_1 is a constant of integration.

(i) If $b = 0$, then the general solution of (4.5) appears in the form

$$g(t) = \int \frac{1}{\sqrt{|2a \ln t + 2d_1|}} dt, \tag{4.7}$$

where $d_1 \in R$.

(ii) If $b = 1$, then from (4.6) the function $g(t)$ is given by

$$g(t) = -\frac{1}{a} \sqrt{|c_1 - at^2|} + c_2,$$

where $c_2 \in R$.

(iii) If $b \notin R - \{0, 1\}$, then the general solution of (4.6) is

$$g(t) = \sqrt{|b|} \int \frac{t^b}{\sqrt{|at^{2b} - bc_1|}} dt.$$

Thus, we have the following:

Theorem 4. *Let Σ_2 be a translation surface of type II in $H^2 \times R$. If Σ_2 has zero Gaussian curvature, then Σ_2 is a plane or parametrized as*

$$x(s,t) = (g(t), st, f(s)),$$

where $f(s) = c_1 \ln s + c_2$ and $g(t)$ is one of the following functions:
(1) $g(t) = c_3 \ln t + c_4$,
(2) $g(t) = \frac{c_3}{b+1} t^{b+1} + c_4 \ (b \neq -1)$,
(3) $g(t) = \int \frac{1}{\sqrt{|2a \ln t + 2c_3|}} dt$,
(4) $g(t) = -\frac{1}{a}\sqrt{|c_3 - at^2|} + c_2$,
(5) $g(t) = \sqrt{|b|} \int \frac{t^b}{\sqrt{|at^{2b} - bc_3|}} dt \ (b \neq 0, 1)$.

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