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# FRACTIONAL POLYNOMIAL METHOD FOR SOLVING FRACTIONAL ORDER RICCATI DIFFERENTIAL EQUATION

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**Abstract:** The aim of this article is to present the fractional shifted Legendre polynomial method to solve the Riccati differential equation of fractional order. The properties of shifted Legendre polynomials together with the Caputo fractional derivative are used to reduce the problem to the solution of algebraic equations. A new theoretical analysis such as convergence analysis and error bound for the proposed technique has been demonstrated. The obtained results reveal that the performance of the proposed method is very accurate and reliable.

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**Key Words:** fractional Riccati differential equation, fractional shifted Legendre polynomial method, Caputo fractional derivative, nonlinear differential equation

## 1. Introduction

General form of fractional Riccati differential equation [11] is

$$\frac{d^\alpha y}{dt^\alpha} = A(t) + B(t)y + C(t)y^2, \quad t > 0,\ m-1 < \alpha \le m, \tag{1}$$

with initial conditions (in Caputo sense)

$$y^j(0) = s_j, \quad j = 0, 1, ..., m-1,$$

where A(t),B(t) and C(t) are given functions, $s_j, j = 0, 1, ..., m-1$, are arbitrary constants and $\alpha$ is a parameter describing the order of the fractional derivative.

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The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. The importance of Riccati equation usually arises in the optimal control problems, network synthesis, robust stabilization, random process, control process and diffusion problems [2, 10]. Many authors have studied on the solutions of Riccati equations and the references can be found in [1, 3, 5, 7, 8].

In this paper, we present the fractional shifted Legendre polynomial method (FSLPM) based solutions for Eq.(1) with the theoretical analysis. Basic definitions of fractional calculus theory are available in [4, 9]. In this article we use the following definition in the Caputo sense.

**Definition 1.** The fractional derivative of *f(x)* in the Caputo sense is defined as

$$D_{*}^{\alpha} f(x) = J^{m-\alpha} D^{m} f(x) = \frac{1}{\Gamma m - \alpha} \int_{0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t) dt,$$

for $m-1<\alpha\le m, m\in\mathbb{N}, x>0, f\in C_{-1}^{m}$.

Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative. Some properties of the operator $D^{\alpha}$ are as follows:

(i) $D^{\alpha}D^{\beta}f(x) = D^{\alpha+\beta}f(x)$,

(ii) $D^{\alpha}C = 0$ ($C$ is a constant),

(iii) $$D^{\alpha}x^{\beta} = \begin{cases} 0, & \text{for } \beta\in\mathbb{N}_0 \text{ and } \beta<\lceil\alpha\rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha} & \text{for } \beta\in\mathbb{N}_0 \text{ and } \beta\ge\lceil\alpha\rceil \\ & \text{or } \beta\notin\mathbb{N} \text{ and } \beta>\lfloor\alpha\rfloor. \end{cases}$$

Similar to the integer-order derivative, the Caputo fractional derivative is a linear operation:

$$D^{\alpha}\left(\sum_{i=1}^{n} c_i f_i(t)\right) = \sum_{i=1}^{n} c_i D^{\alpha} f_i(t),$$

where $\{c_i\}_{i=1}^{n}$ are constants.

## 2. Method and Analysis

The fractional-order Legendre functions(FLFs)[6] are defined by introducing the change of variable $t = x^{\alpha}$ and $\alpha > 0$ on shifted Legendre polynomials.

$FL_i^{\alpha}(x) = \frac{(2i+1)(2x^{\alpha}-1)}{i+1} FL_i^{\alpha}(x) - \frac{i}{i+1} FL_{i-1}^{\alpha}(x), i = 1, 2, ...,$ where $FL_0^{\alpha}(x) = 1$ and $FL_1^{\alpha}(x) = 2x^{\alpha} - 1$.

Note that$FL_i^{\alpha}(0) = (-1)^i$ and $FL_i^{\alpha}(1) = 1$.The FLFs are orthogonal with respect to the weight function $w(x) = x^{\alpha-1}$ in the interval $(0, 1]$ with the orthogonality property $\int_0^1 FL_n^{\alpha}(x)FL_m^{\alpha}(x)w(x)dx = \frac{1}{(2n+1)\alpha}\delta_{nm}$.

A function y(x) defined over the interval $(0, 1]$ may be expressed in terms of fractional shifted Legendre polynomials as

$$y(x) = \sum_{i=0}^{\infty} c_i FL_i^{\alpha}(x), \tag{2}$$

where the coefficients $c_i$ are given by

$$c_i = \alpha(2i+1) \int_0^1 FL_i^{\alpha}(x)y(x)w(x)dx, \quad i = 0, 1, 2, \ldots.$$

In practice, only the first $m$ terms of Eq. (2) are considered. Then we have

$$y(x) \simeq y_m(x) = \sum_{i=0}^{m-1} c_i FL_i^{\alpha}(x) = C^T FL^{\alpha}(x), \tag{3}$$

with

$$C = [c_0, c_1, ..., c_{m-1}]^T,$$
$$FL^{\alpha}(x) = [FL_0^{\alpha}(x), FL_1^{\alpha}(x), ..., FL_{m-1}^{\alpha}(x)]^T.$$

The $m$ unknown coefficients $c_i$ can be obtained from the equations and initial boundary conditions to solve Eq. (1). In case the exact solution to a problem is known, the accuracy and efficiency of the proposed method based on maximum absolute error $e_m$ defined as $e_m = max\{|y_{exact}(t) - y_m(t)|\}, \ 0 < t < \tau$. The convergence analysis and error estimation for the proposed technique have been given through the following theorem.

**Theorem 2.** *Suppose that the function given in Eq.(3) is the best approximation to the exact solution y(t) then the error bound is presented as follows* $\epsilon \leq \sum_{i=m}^{\infty} \frac{M}{(2i-3)^{1/2}(2i-1)} \frac{1}{\sqrt{(2i+1)\alpha}}$
*where*

$$\epsilon = \left( \int_0^1 \left[ \sum_{i=0}^{\infty} c_i FL_i^{\alpha}(t) - \sum_{i=0}^{m-1} c_i FL_i^{\alpha}(t) \right]^2 w(t)dt \right)^{\frac{1}{2}}$$

*and*

$$\left| y''(t) \right| < M.$$

The above theorem can be proved through Cauchy sequence.
Next, we examine the performance of FSLPM with some examples.

**Example 1.** Consider the fractional Riccati equation [7]

$$D_{\ast}^{\alpha} y(t) = -y(t)^2 + 1, \quad 0 < \alpha \leq 1,$$

subject to the initial condition y(0)=0.
By FSLPM with m=6, the approximate solution is

$$y(t) = \frac{1}{\Gamma(1+\alpha)} - \frac{t^{3\alpha}\Gamma(1+3\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} + \frac{2t^{5\alpha}\Gamma(1+2\alpha)\Gamma(1+4\alpha)}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)\Gamma(1+5\alpha)} - \frac{t^{7\alpha}\Gamma\left((1+2\alpha)\right)^2\Gamma(1+6\alpha)}{\left(\Gamma(1+\alpha)\right)^4\left(\Gamma(1+3\alpha)\right)^2\Gamma(1+7\alpha)}.$$

The exact solution of Example 1, when $\alpha = 1$ is $y(t) = \frac{e^{2t}-1}{e^{2t}+1}$.

**Example 2.** Consider the fractional Riccati equation [7]

$$D_{\ast}^{\alpha} y(t) = 2y(t) - y^2(t) + 1, \quad 0 < \alpha \leq 1,$$

subject to the initial condition y(0)=0.
We apply the FSLPM with m=5, and we get

$$y(t) = \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{4t^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{t^{3\alpha}\Gamma(1+2\alpha)}{\left(\Gamma(1+\alpha)\right)^2\Gamma(1+3\alpha)} + \frac{8t^{4\alpha}}{\Gamma(1+4\alpha)} - \frac{2t^{4\alpha}\Gamma(1+2\alpha)}{\left(\Gamma(1+\alpha)\right)^2\Gamma(1+4\alpha)}.$$

The exact solution of Example 2, when $\alpha = 1$ is

$$y(t) = 1 + \sqrt{2}tanh\left(\sqrt{2}t + \frac{1}{2}log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right).$$

The numerical values of y(t) with $\alpha$= 0.5 to 1.0 are plotted in Fig 1(a) for m= 10 and in Fig 1(b) for Example 1 and 2 respectively. It also shows that the convergence of the FSLPM for different values of $\alpha$.

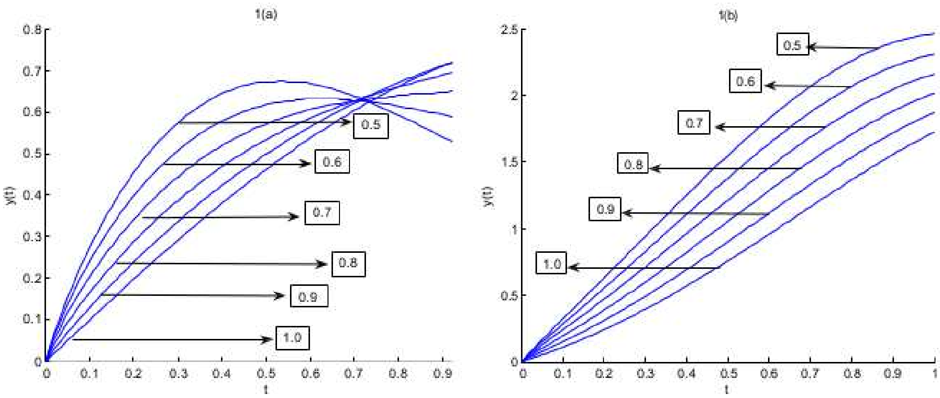


Figure 1: Fig 1(a). Comparison of y(t) for m = 10 and with $\alpha$= 0.5 to 1.0 for Example 1.(b) Comparison of y(t) for m = 10 and with $\alpha$= 0.5 to 1.0 for Example 2.

## 3. Conclusion

In this work, we have proposed the FSLPM for solving Riccati differential equation based on fractional order Legendre function. The proposed method has been tested, for validity and applicability for some problems. The theoretical analysis has also been discussed. The proposed technique can be extended to solve fractional PDE.

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