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ORTHOGONAL STABILITY OF A MIXED TYPE CUBIC-QUARTIC FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN SPACES

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Abstract: Using direct method, we prove the Hyers-Ulam stability of the orthogonally cubic-quartic functional equation

$$f(2x+y)+f(2x-y) = 3f(x+y)+f(-x-y)+3f(x-y)+f(y-x) \\ +18f(x)+6f(-x)-3f(y)-3f(-y), \quad (1)$$

for all x, y with $x \perp y$, in non-Archimedean Banach spaces. Here $\perp$ is the orthogonality in the sense of Rätz.

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Key Words: Hyers-Ulam stability, orthogonally cubic-quartic functional equation, non-Archimedean Banach space, orthogonality space

1. Introduction and Preliminaries

In 1897, Hensel [5] introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [2, 7, 8, 10]).

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A valuation is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r| \cdot |s|$ and the triangle inequality holds, i.e.

$$|r+s| \leq |r| + |s|, \quad \forall r, s \in K.$$

A field K is called a valued field if K carries a valuation. Throughout this paper, we assume that the base field is a valued field, hence call it simply a field. The usual absolute values of R and C are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \leq max\{|r|, |s|\}, \quad \forall r, s \in K,$$

then the function $|\cdot|$ is called a non-Archimedean valuation, and the field is called a non-Archimedean field. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in N$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Definition 1. ([9]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is said to be a non-Archimedean norm if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|rx\| = |r| \, \|x\| \quad (r \in K, x \in X)$;

(iii) the strong triangle inequality

$$\|x+y\| \leq max\{\|x\|, \|y\|\} \quad \forall x, y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a non-Archimedean normed space.

Definition 2. (i) Let Cbe a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called Cauchy if for a given $\epsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \leq \epsilon$$

for all $n, m \geq N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called convergent if for a given $\epsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x_m\| \leq \epsilon$$

for all $n \geq N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} \{x_n\} = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a non-Archimedean Banach space.

Assume that X is a real inner product space and $f : X \to R$ is a solution of the orthogonal Cauchy functional equation $f(x+y) = f(x) + f(y), \langle x, y \rangle = 0$. By the Pythagorean theorem $f(x) = \|x\|^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus orthogonal Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space.

The stability problem of functional equations originated from a question of S.M. Ulam [15] in 1940, concerning the stability of group homomorphisms.

Let $(G_1, \cdot)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

The case of approximately additive functions was solved by D.H. Hyers [6] under the assumption that G_1 and G_2 are Banach spaces. In 1951 and in 1978, a generalized version of the theorem of Hyers for approximately linear mappings was given by T. Aoki [1] and Th.M. Rassias [13]. In 1982, J.M. Rassias [11, 12] provided a generalizations of the Hyers stability theorem which allows the Cauchy difference to be bounded.

There are several orthogonality notations on a real normed space are available. But here, we present the orthogonality concept introduced by J.Rätz[14]. This is given in the following definition.

Definition 3. [14] A vector space X is called an orthogonality vector space if there is a relation $x \perp y$ on X such that

(i) $x \perp 0, 0 \perp x$ for all $x \in X$;

(ii) if $x \perp y$ and $x, y \neq 0$, then x, y are linearly independent;

(iii) $x \perp y$, $ax \perp by$ for all $a, b \in R$;

(iv) if P is a two-dimensional subspace of X;then: (a) for every $x \in P$ there exists $0 \neq y \in P$ such that $x \perp y$; (b) there exists vectors $x, y \neq 0$ such that $x \perp y$ and $x + y \perp x - y$.

Any vector space can be made into an orthogonality vector space if we define $x \perp 0, 0 \perp x$ for all x and for non zero vector x, y define $x \perp y$ iff x, y are linearly independent. The relation $\perp$ is called symmetic if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. The pair $(X, \perp)$ is called anorthogonality space. It becomes orthogonality normed space when the orthogonality space is equipped with a norm.

The orthogonal Cauchy functional equation

$$f(x+y) = f(x) + f(y), x \perp y$$

in which $\perp$ is an abstract orthogonally was first investigated by S.Gudder and D. Strawther [4]. R. Ger and J. Sikorska discussed the orthogonal stability of the equation (1) in [3].

This paper is organized as follows: In Section 2, we prove the Hyers-Ulam stability of the orthogonally cubic-quartic functional equation (1) in non-Archimedean orthogonality spaces for an odd mapping.

In Section 3, we prove the Hyers-Ulam stability of the orthogonally cubic-quartic functional equation (1) in non-Archimedean orthogonality spaces for an even mapping.

Throughout this paper, assume that $(X, \perp)$ is a non-Archimedean orthogonality space and that $(Y, \|\cdot\|)$ is a real non-Archimedean Banach space. Assume that $|2| \neq 1$.

2. Stability of the Orthogonally Cubic-Quartic Functional Equation: An Odd Mapping Case

In the present section, we investigate the stability problem for the orthogonally cubic-quartic functional equation

$$\begin{aligned} Df(x,y) := {} & f(2x+y) + f(2x-y) - 3f(x+y) - f(-x-y) - 3f(x-y) \\ & -f(y-x) - 18f(x) - 6f(-x) + 3f(y) + 3f(-y) = 0 \end{aligned}$$

for all $x, y \in X$ with $x \perp y$ in non-Archimedean Banach spaces: an odd mapping case.

Definition 4. An odd mapping $f : X \to Y$ is called an orthogonally additive mapping if

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x)$$

for all $x, y \in X$ with $x \perp y$.

Theorem 5. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\phi(x,y) := \sum_{i=0}^{\infty} \frac{1}{|2|^{3i}} \varphi(2^i x, 2^i y) < +\infty$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \to Y$ be an odd mapping satisfying

$$\|Df(x,y)\| \leq \varphi(x,y) \tag{2.1}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally cubic mapping $C : X \to Y$ Such that

$$\|f(x) - C(x)\| \leq \frac{1}{|2|^4} \phi(x,0) \tag{2.2}$$

for all $x \in X$.

Proof. Putting $y = 0$ in (2.1), we get

$$\|2f(2x) - 16f(x)\| \leq \varphi(x,0) \tag{2.3}$$

for all $x \in X$, since $x \perp 0$. So

$$\left\| \frac{1}{8} f(2x) - f(x) \right\| \leq \frac{1}{|2|^4} \varphi(x,0)$$

for all $x \in X$. Replacing x by $2^n x$ in (2.3), we get

$$\left\| \frac{1}{8} f(2^{n+1}x) - f(2^n x) \right\| \leq \frac{1}{|2|^4} \varphi(2^n x, 0)$$

for all $n \geq 0$ and all $x \in X$, since $2^n x \perp 0$. So

$$\left\| \frac{1}{2^{n+3}} f(2^{n+1}x) - \frac{1}{2^n} f(2^n x) \right\| \leq \frac{1}{|2|^{n+4}} \varphi(2^n x, 0)$$

for all $n \geq 0$ and all $x \in X$.

Now we define a mapping g such that

$$g(n,x) := \frac{1}{2^{n+3}} f(2^{n+1}x) - \frac{1}{2^n} f(2^n x)$$

for all $n \geq 0$ and all $x \in X$. Then

$$\|g(n,x)\| \leq \frac{1}{|2|^{n+4}} \varphi(2^n x, 0)$$

for all $n \geq 0$ and all $x \in X$. So

$$\begin{aligned}\left\|\frac{1}{2^m}f(2^m x) - \frac{1}{2^n}f(2^n x)\right\| &= \left\|\sum_{i=n}^{m-1} g(i,x)\right\| \\ &\leq max\left\{\|g(n,x)\|, ..., \|g(m-1,x)\|\right\} \\ &\leq \sum_{i=n}^{m-1} \|g(i,x)\| \\ &\leq \sum_{i=n}^{m-1} \frac{1}{|2|^{i+4}}\varphi(2^i x, 0),\end{aligned} \tag{2.4}$$

whichs tends to zero as $n \to \infty$, for all $m > n \geq 0$ and all $x \in X$. Thus the sequence $\left\{\frac{1}{2^n}f(2^n x)\right\}$ is a Cauchy sequence. Since Y is a non-Archimedean Banach space,The sequence $\left\{\frac{1}{2^n}f(2^n x)\right\}$ converges. So we can define a mapping $C : X \to Y$ such that

$$\lim_{n\to\infty} \frac{1}{2^n}f(2^n x) = C(x)$$

for all $x \in X$.

Replacing x, y by $2^n x, 2^n y$ in (2.1), respectively, we get

$$\|Df(2^n x, 2^n y)\| \leq \varphi(2^n x, 2^n y)$$

for all $x, y \in X$ with $x \perp y$, since $2^n x \perp 2^n y$. Then

$$\left\|\frac{1}{2^n}Df(2^n x, 2^n y)\right\| \leq \frac{1}{|2|^n}\varphi(2^n x, 2^n y)$$

for all $x, y \in X$ with $x \perp y$. So

$$\|DC(x,y)\| = \lim_{n\to\infty}\left\|\frac{1}{2^n}Df(2^n x, 2^n y)\right\| = \lim_{n\to\infty}\frac{1}{|2|^n}\varphi(2^n x, 2^n y) = 0$$

for all $x, y \in X$ with $x \perp y$. Thus

$$DC(x,y) = 0$$

for all $x, y \in X$ with $x \perp y$.

Since $f(x)$ is an odd mapping, $C(x)$ is an odd mapping. So the mapping $C : X \to Y$ is an orthogonally cubic mapping.

Let $n = 0$ and $m \to \infty$ in (2.4), we get the inequality (2.2).

To prove the uniquness of C, let $L : X \to Y$ be another orthogonally additive mapping satisfying (2.2).

$$\begin{aligned}
&\|C(x) - L(x)\| \\
&\qquad \le \left\| \frac{1}{2^n} C(2^n x) - \frac{1}{2^n} L(2^n x) \right\| \\
&\qquad \le max \left\{ \left\| \frac{1}{2^n} C(2^n x) - \frac{1}{2^n} f(2^n x) \right\|, \left\| \frac{1}{2^n} f(2^n x) - \frac{1}{2^n} L(2^n x) \right\| \right\} \\
&\qquad \le \frac{1}{|2|^{n+4}} \varphi(2^n x, 0),
\end{aligned}$$

which tends to zero as $n \to \infty$. So $C : X \to Y$ is unique.

Therefore, there exists a unique orthogonally cubic mapping $C : X \to Y$ satisfying (2.2). This completes the proof. □

Corollary 6. *Let $p > 3$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\|Df(x, y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{2.5}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally cubic mapping $A : X \to Y$ such that

$$\|f(x) - C(x)\| \le \frac{\theta}{|2| \left(|2|^3 - |2|^p\right)} \|x\|^p$$

for all $x \in X$.

Theorem 7. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\phi(x, y) := \sum_{i=1}^{\infty} = |2|^{3i} \varphi(\frac{x}{2^i}, \frac{y}{2^i}) < +\infty$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \to Y$ be an odd mapping satisfying (2.1). Then there exists a unique orthogonally cubic mapping $C : X \to Y$ such that

$$\|f(x) - C(x)\| \le \frac{1}{|2|^4} \phi(x, 0)$$

for all $x \in X$.

Proof. It follows from (2.3) that

$$\left\| f(x) - 8f(\frac{x}{2}) \right\| \le \frac{1}{8} \varphi(\frac{x}{2}, 0),$$

for all $x, y \in X$, since $x \perp 0$. The rest of the proof is similar to the proof of Theorem 5

□

Corollary 8. *Let $0 < p < 3$. Let $f : X \to Y$ be an odd mapping satisfying (2.5). Then there exists a unique orthogonally cubic mapping $C : X \to Y$ such that*

$$\|f(x) - C(x)\| \leq \frac{\theta}{|2|\left(|2|^p - |2|^3\right)} \|x\|^p$$

for all $x \in X$.

3. Stability of the Orthogonally Cubic-Quartic Functional Equation: An Even Mapping Case

In the present section, we investigate the stability problem for the orthogonally cubic-quartic functional equation

$$\begin{aligned} Df(x,y) := f(2x+y) + f(2x-y) - 3f(x+y) - f(-x-y) - 3f(x-y) \\ -f(y-x) - 18f(x) - 6f(-x) + 3f(y) + 3f(-y) = 0 \end{aligned}$$

for all $x, y \in X$ with $x \perp y$ in non-Archimedean Banach spaces: an even mapping case.

Definition 9. An odd mapping $f : X \to Y$ is called an orthogonally quartic mapping if

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y)$$

for all $x, y \in X$ with $x \perp y$.

Theorem 10. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\phi(x,y) := \sum_{i=0}^{\infty} \frac{1}{|2|^4{}_i} \varphi(2^i x, 2^i y) < +\infty$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.1) Then there exists a unique orthogonally quartic mapping $Q : X \to Y$ Such that

$$\|f(x) - Q(x)\| \leq \frac{1}{|2|^5} \phi(x, 0)$$

for all $x \in X$.

Proof. Putting $y = 0$ in (2.1), we get

$$\|2f(2x) - 32f(x)\| \le \varphi(x, 0)$$

for all $x \in X$, since $x \perp 0$. So

$$\left\| \frac{1}{16} f(2x) - f(x) \right\| \le \frac{1}{|2|^5} \varphi(x, 0) \tag{3.1}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 5 □

Corollary 11. *Let $p > 4$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and $(ref2 - 5)$. Then there exists a unique orthogonally quartic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{\theta}{|2| \left(|2|^4 - |2|^p\right)} \|x\|^p$$

for all $x \in X$.

Theorem 12. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\phi(x, y) := \sum_{i=1}^{\infty} = |2|^4 \, i\varphi(\frac{x}{2^i}, \frac{y}{2^i}) < +\infty$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.1). Then there exists a unique orthogonally quartic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{|2|^5} \phi(x, 0)$$

for all $x \in X$.

Proof. It follows from (3.1) that

$$\left\| f(x) - 16 f(\frac{x}{2}) \right\| \le \frac{1}{|2|} \varphi(\frac{x}{2}, 0)$$

for all $x, y \in X$, since $x \perp 0$.

The rest of the proof is similar to the proof of Theorem 5

□

Corollary 13. *Let $0 < p < 4$. Let $f : X \to Y$ be an even mapping satisfying (2.5). Then there exists a unique orthogonally quartic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{\theta}{|2|\left(|2|^p - |2|^4\right)} \|x\|^p$$

for all $x \in X$.

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