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KÖTHE-TOEPLITZ DUALS OF SOME DIFFERENCE SEQUENCE SPACES AND THEIR MATRIX TRANSFORMATIONS OVER NON-ARCHIMEDEAN FIELDS

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Abstract: In this paper we give Köthe-Toeplitz duals of some difference sequence spaces and their inclusion relations using matrix transformations when the sequences, series and the infinite matrices have their entries in a complete non-trivially valued non-archimedean field K.

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1. Introduction

Let $p = (p_k)$ denote a sequence of strictly positive real numbers with $1 \leq p_k \leq \sup_k p_k < \infty$.

If $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ then the difference sequence spaces of all bounded, convergent and null sequences are defined as follows:

$$\Delta\ell_{\infty}(p) = \left\{ x = (x_k) : \sup_k |\Delta x_k|^{p_k} < \infty \right\}$$

$$\Delta c(p) = \{x = (x_k) : |\Delta x_k - \alpha_k|^{p_k} \to 0 \text{ as } k \to \infty \text{ for some } \alpha_k \in K\} \text{ and}$$

$$\Delta c_0(p) = \{x = (x_k) : |\Delta x_k|^{p_k} \to 0 \text{ as } k \to \infty\}$$

when $p_k = m$, $k = 1, 2, \ldots$, $\Delta\ell_{\infty}(p) = \ell_{\infty}(\Delta)$, $\Delta c(p) = c(\Delta)$ and $\Delta c_0(p) = c_0(\Delta)$.

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2. Köthe-Toeplitz Duals

In this section we give Köthe-Toeplitz duals of some difference sequence spaces.

Definition 1. If E is a space of sequences $x = (x_k)$ in K then its Köthe-Toeplitz dual is defined as

$$E^+ = \left\{ x = (x_k) : \sum_{k=1}^{\infty} |x_k y_k| < \infty, \ \forall \ y = (y_k) \in E \right\}$$

Lemma 2. *(see [10])* $\sup_k |x_k - x_{k+1}|^{p_k} < \infty$ *iff:*

(i) $\sup_k k^{-1} |x_k|^{p_k} < \infty$ *and*

(ii) $\sup_k |x_k - k(k+1)^{-1} x_{k+1}|^{p_k} < \infty$.

Theorem 3. *Let* $U_1 = \left\{ a = (a_k) : \sum_{k=1}^{\infty} k|a_k| N^{1/p_k} < \infty \right\}$ *then for every sequence* $p = (p_k)$ *with* $p > 0$, *we have,* $\Delta\ell_{\infty}^{+}(p) = U_1$.

Proof. First to prove:

$$U_1 \subset \Delta\ell_{\infty}^{+}(p). \tag{i}$$

Let $x = (x_k) \in \Delta\ell_{\infty}(p)$, then for each $k \in \mathbb{N}$, there exists an integer N such that

$$N > \max(1, \sup k^{-1} |x_k|^{p_k}) \tag{1}$$

Now, for $a = (a_k) \in U_1$,

$$\begin{aligned}
\sum_{k=1}^{\infty} |a_k x_k| &= \sum_{k=1}^{\infty} |a_k (x_k - x_{k+1} + x_{k+1})| \\
&= \sum_{k=1}^{\infty} |a_k (x_k - x_{k+1}) + a_k x_{k+1}| \\
&\leq \sum_{k=1}^{\infty} \max\{|a_k||x_k - x_{k+1}|, |a_k||x_{k+1}|\} \\
&\leq \sum_{k=1}^{\infty} |a_k| \max\{|x_k - x_{k+1}|, |x_1 - x_{k+2}|\}
\end{aligned}$$

$$\leq \sum_{k=1}^{\infty} |a_k| \max\{kN^{1/p_k}, N^{1/p_k}\} \text{ \{using (1\}}$$

$$\leq \sum_{k=1}^{\infty} k|a_k|N^{1/p_k},$$

$$\sum_{k=1}^{\infty} |a_k x_k| < \infty.$$

This implies that, $a = (a_k) \in \Delta\ell_{\infty}^{+}(p)$. Hence (i) holds.

Now, let us prove the reverse inclusion. That is to prove:

$$\Delta\ell_{\infty}^{+}(p) \subset U_1 \tag{ii}$$

Let $a = (a_k) \in \Delta\ell_{\infty}^{+}(p)$ for every $x = (x_k) \in \Delta\ell_{\infty}(p)$ and suppose that $a \notin U_1$. Then there exists an integer $N > 1$ such that

$$\sum_{k=1}^{\infty} k|a_k|N^{1/p_k} \text{ is infinite.} \tag{2}$$

Therefore there exists a strictly increasing sequence (k_N) of positive integers such that $k = (k_N)$, $k|a_k|N^{1/p_k} > M$.

We define $x = (x_k)$ by

$$x_k = \begin{cases} 0, & k \neq k_N \\ \frac{kN^{1/p_k}}{M}, & k = k_N \end{cases}$$

Then for every integer $N > 1$ and for $k = k_N$, we have

$$\sum_{k=1}^{\infty} |a_k x_k| > \infty \text{\{using (2)\}}$$

which is a contradiction to that $a = (a_k) \in \Delta\ell_{\infty}^{+}(p)$. Hence (ii) holds.

Therefore we have $\Delta\ell_{\infty}^{+}(p) \subset U_1$.

That is, the Köthe-Toeplitz dual of $\Delta\ell_{\infty}(p)$ is

$$U_1 = \left\{ a = (a_k) : \sum_{k=1}^{\infty} k|a_k|N^{1/p_k} < \infty \right\}$$

□

Theorem 4. *Let*

$$U_2 = \left\{ a = (a_k) : \sum_{k=1}^{\infty} k|a_k| N^{-1/p_k} < \infty \right\},$$

then for every sequence $p = (p_k)$ with $p > 0$, we have $\Delta c_0^+(p) = U_2$.

Proof. First to prove: $U_2 \subset \Delta c_0^+(p)$.

Let $x = (x_k) \in \Delta c_0(p)$. Then for each $k \in \mathbb{N}$, there exists an integer N such that

$$|\Delta x_k|^{p_k} < \frac{k}{N}, \quad \text{since} \quad |\Delta x_k|^{p_k} \to 0 \quad \text{as } k \to \infty. \tag{3}$$

Now, for $a = (a_k) \in U_2$,

$$\begin{aligned}
\sum_{k=1}^{\infty} |a_k x_k| &= \sum_{k=1}^{\infty} |a_k(x_k - x_{k+1}) + a_k x_{k+1}| \\
&\le \sum_{k=1}^{\infty} \max\{|a_k||x_k - x_{k+1}|, |a_k||x_{k+1}|\} \\
&\le \sum_{k=1}^{\infty} |a_k| \max\{|x_k - x_{k+1}|, |x_1 - x_{k+2}|\} \\
&\le \sum_{k=1}^{\infty} |a_k| \max\{|\Delta x_k|, |\Delta x_1|\} \\
&\le \sum_{k=1}^{\infty} |a_k| \max\{kN^{-1/p_k}, N^{-1/p_k}\} \ \{\text{using (3)}\} \\
&\le \sum_{k=1}^{\infty} k|a_k| N^{-1/p_k}, \\
\sum_{k=1}^{\infty} |a_k x_k| &< \infty.
\end{aligned}$$

This implies that, $a \in \Delta c_0^+(p)$.

Hence we have $U_2 \subset \Delta c_0^+(p)$. The converse part can easily be proved as in the above theorem. $\square$

3. Inclusion Theorems

In this section we give inclusion theorems on Matrix transformations between the difference sequence spaces over K.

Let $A = (a_{nk})$, $a_{nk} \in K$, $n, k = 1, 2, \ldots$, be an infinite matrix, let (X, Y) denote the set of all matrices $A = (a_{nk})$; $n, k = 1, 2, \ldots$, that transform a sequence $x = (x_k) \in X$ in to a sequence $Ax = (A_n(x)) = y = y_n \in Y$ defined by

$$y_n = \sum_{k=1}^{\infty} a_{nk} x_k, \quad n = 1, 2, \ldots .$$

Theorem 5. *Let $0 < p_k \leq 1$ for every k. Then $A \in (c_0(p), \Delta\ell_\infty(p))$ iff*

$$\sup_n \sup_k |a_{nk}|^{p_k} N^{-p_k} < \infty$$

Proof. Let $x = (x_k) \in c_0(p)$. Then for each $k \in \mathbb{N}$, there exists an integer $N > 1$ such that

$$|x_k|^{p_k} < \frac{1}{N} \tag{4}$$

Now to prove $Ax \in \Delta\ell_\infty(p)$, we take

$$\begin{aligned}
|\Delta Ax|^{p_k} = |\Delta A_n(x)|^{p_k} &= |A_n(x) - A_{n+1}(x)|^{p_k} \\
&\leq \max\{|A_n(x)|^{p_k}, |A_{n+1}(x)|^{p_k}\} \\
&\leq \sup_n |A_n(x)|^{p_k} \\
&\leq \sup_n \left| \sum_{k=1}^{\infty} a_{nk} x_k \right|^{p_k} \\
&\leq \sup_n \left[\max\{|a_{n1}x_1|^{p_k}, |a_{n2}x_2|^{p_k}, \ldots, |a_{nk}x_k|^{p_k}, \ldots\}\right] \\
&\leq \sup_n \left[\sup_k |a_{nk}|^{p_k} |x_k|^{p_k}\right] \\
&\leq \sup_n \sup_k |a_{nk}|^{p_k} N^{-p_k} \quad \{\text{using (4)}\}, \\
|\Delta Ax|^{p_k} &< \infty.
\end{aligned}$$

This implies that $Ax \in \Delta\ell_\infty(p)$.

Therefore, we have $A \in (c_0(p), \Delta\ell_\infty(p))$.

Conversely, suppose that $A \in (c_0(p), \Delta\ell_\infty(p))$, but $\sup_n \left[\sup_k |a_{nk}|^{p_k} N^{-p_k}\right]$ is infinite.

Then for each $x = (x_k) \in c_0(p)$ as $|x_k|^{p_k} \to 0$ we have

$$|x_k|^{p_k} < \epsilon < 1$$

implies that $\sup_n \left[\sup_k |a_{nk}|^{p_k} N^{-p_k} |x_k|^{p_k}\right]$ is infinite, i.e.

$$\sup_n \left[\sup_k |a_{nk} x_k|^{p_k} N^{-p_k}\right]$$

is infinite.

This implies that $Ax \notin \Delta\ell_\infty(p)$ which is a contradiction. This completes the proof of the converse part and hence the theorem. $\square$

Theorem 6. *Let $p = (p_k)$ with $1 < p_k \le \sup_k p_k < \infty$ and $p_k^{-1} + q_k^{-1} = 1$. Then $A \in (\ell(p), \Delta c(p))$ iff there exists an integer $N > 1$ such that:*

(i) $\Delta a_{nk} \to a_k$, *as* $n \to \infty$,

(ii) $\sup_k |a_k|^{p_k} N^{1/q_k}$ *converges.*

Proof. Sufficiency. Suppose that (i) & (ii) hold.

Then for each $x = (x_k) \in \ell(p)$, since $\sum_{k=1}^{\infty} |x_k|^{p_k} < \infty$ there exists an integer $N > 1$ such that $(|x_k|^{p_k})^{q_k} < N$.
Let us now prove that $\Delta Ax \in c(p)$. For, we take,

$$\begin{aligned}
|\Delta Ax|^{p_k} = |A_n(x) - A_{n+1}(x)|^{p_k} &= \left|\sum_{k=1}^{\infty} (a_{n,k} - a_{n+1,k}) x_k\right|^{p_k} \\
&\le \sup_k |a_{nk} - a_{n+1,k}|^{p_k} |x_k|^{p_k} \\
&\le \sup_k |\Delta a_{nk}|^{p_k} |x_k|^{p_k} \\
&\le \lim_n \sup_k |\Delta a_{nk}|^{p_k} |x_k|^{p_k} \\
&\le \sup_k |a_k|^{p_k} N^{1/q_k} \quad [\text{by } (i)] \\
&< \epsilon \quad [\text{by } (ii)].
\end{aligned}$$

Hence we have $\Delta Ax \in c(p)$, that is $Ax \in \Delta c(p)$. Therefore $A \in (\ell(p), \Delta c(p))$.

Necessity. Let $A \in (\ell(p), \Delta c(p))$ and suppose that $\sup_k |a_k|^{p_k} N^{1/q_k}$ diverges. Then there exists an integer $M_1 > 1$ and for some $k \in \mathbb{N}$, $\sup_k |a_k|^{p_k} N^{1/q_k} > M_1$ and also that $\Delta a_{nk} \not\to a_k$, as $n \to \infty$ for some $k \in \mathbb{N}$, then there exists an integer $M_2 > 1$ such that $|\Delta a_{nk} - a_k|^{p_k} > M_2$.

Now for each $x = (x_k) \in \ell(p)$, we have that

$$\begin{aligned}
|\Delta A x|^{p_k} = |\Delta A_n(x)|^{p_k} &= \left|\sum_{k=1}^{\infty}(a_{n,k} - a_{n+1,k})x_k\right|^{p_k} \\
&= \left|\sum_{k=1}^{\infty}(a_{n,k} - a_{n+1,k})x_k + a_k x_k - a_k x_k\right|^{p_k} \\
&\leq \max\{|\Delta a_{nk} - a_k|^{p_k}|x_k|^{p_k}, |a_k|^{p_k}|x_k|^{p_k}\} \\
&\leq |x_k|^{p_k} \max\{|\Delta a_{nk} - a_k|^{p_k}, |a_k|^{p_k}\} \\
&> N^{1/q_k} \max\{M_1, M_2\} \\
&> M N^{1/q_k} \quad \text{where } M = \max\{M_1, M_2\} \\
\textit{i.e.}, |\Delta A_n(x)|^{p_k} &> M_0 > \epsilon \quad \text{where } M_0 = M N^{1/q_k},
\end{aligned}$$

therefore $\Delta A x \notin c(p)$.

That is $Ax \notin \Delta c(p)$, which is a contradiction. This completes the proof of the theorem. □

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