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OSCILLATORY AND NONOSCILLATORY BEHAVIOUR OF SOLUTIONS OF GENERALIZED MIXED DIFFERENCE EQUATIONS

M. Maria Susai Manuel[1 §], G. Dominic Babu[2],
D.S. Dilip[3], G. Britto Antony Xavier[4]

[1]Department of Mathematics, R.M.D. Engineering College,
Kavaraipettai - 601 206, Tamil Nadu, S. INDIA
[2]Department of Mathematics
Annai Velankanni College
Tholaiyavattam, Kanyakumari District, INDIA
[3]Department of Mathematics
St. John's College, Anchal, Kollam Dt., INDIA
[4]Department of Mathematics
Sacred Heart College
Tirupattur, 635 601, Vellore District,Tamil Nadu, INDIA

Abstract: In this paper, the authors discuss the oscillatory and nonoscillatory behaviour of solutions of some generalized mixed difference equations of the form

$$\Delta_{\ell}^{2}\Big(\Delta_{\alpha(\ell)}u(k)\Big) + \delta p(k)u(k) = 0, k \in [a, \infty), \tag{1}$$

$$\Delta_{\ell}^{3}\Big(\Delta_{\alpha(\ell)}u(k)\Big) + \delta p(k)u(k) = 0, k \in [a, \infty), \tag{2}$$

where $\delta = \pm 1$ and the function p is real with $p(k) \geq c$ and α, ℓ are positive real.

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§Correspondence author

1. Introduction

The basic theory of difference equations is based on the operator Δ defined as $\Delta u(k) = u(k+1) - u(k),\ k \in \mathbb{N} = \{0, 1, 2, 3, \cdots\}$. Eventhough many authors (see [1], [19]-[23]) have suggested the definition of Δ as

$$\Delta u(k) = u(k+\ell) - u(k),\ k \in \mathbb{R},\ \ell \in \mathbb{R} - \{0\}, \tag{3}$$

no significant progress took place on this line. But recently, E. Thandapani, M.M.S. Manuel, G.B.A.Xavier [7] considered the definition of Δ as given in (3) and developed the theory of difference equations in a different direction. For convenience, the operator Δ defined by (3) is labelled as Δ_ℓ and by defining its inverse Δ_ℓ^{-1}, many interesting results and applications in number theory were obtained. By extending the study related to the sequences of complex numbers and ℓ to be real, some new qualitative properties of the solutions like rotatory, expanding, shrinking, spiral and weblike were obtained for difference equation involving Δ_ℓ. The results obtained using Δ_ℓ are found in (see [7]-[14],[17],[18]).

Jerzy Popenda and B. Szmanda (see [5],[6]) defined Δ as

$$\Delta_\alpha u(k) = u(k+1) - \alpha u(k) \tag{4}$$

and based on this definition they studied the qualitative properties of a particular difference equation and no one else has handled this operator.

In [15] the authors extended the definition of Δ_α to $\Delta_{\alpha(\ell)}$ defined on $u(k)$ as $\Delta_{\alpha(\ell)} v(k) = v(k+\ell) - \alpha v(k)$, where $\alpha \neq 0$, $\ell > 0$ are fixed and $k \in [0, \infty)$ is variable. By defining the inverse $\Delta_{\alpha(\ell)}^{-1}$, several interesting results on number theory were obtained (see [12],[14],[15],[16]).

An equation involving both Δ and Δ_α is called mixed difference equation. Oscillatory behaviour of solutions of certain types of mixed difference equations have been dicussed in [3, 4, 21, 22]. An equation involving Δ_ℓ and $\Delta_{\alpha(\ell)}$ is called as generalized mixed difference equation.

B. Smith and W.E. Taylor (see [21]) investigated the oscillatory behavior of solutions of certain mixed difference equations.

In this paper the theory is extended from Δ to Δ_ℓ and Δ_α to $\Delta_{\alpha(\ell)}$ for all real $k \in [a, \infty)$ and we discuss the oscillatory and nonoscillatory behavior of solutions of the generalized mixed difference equations (1) and (2).

Throughout this paper, we make use the following assumptions:

(i) $\mathbb{N}_\ell(j) = \{j, j+\ell, j+2\ell, \ldots\}$.

(ii) $\lceil x \rceil$ and $[x]$ denote upper integer and integer part of x respectively.

(iii) $j = k - k_i - \left[\frac{k-k_i}{\ell}\right]\ell, k_i \in [0, \infty)$.

2. Preliminaries

In this section, we present some preliminaries of generalized difference operator and its inverse which will be useful for future discussion.

Definition 2.1. [7] Let $u(k), k \in [0, \infty)$, be a real or complex valued function and $\ell > 0$ be fixed. Then, the inverse of Δ_ℓ denoted by Δ_ℓ^{-1} is defined as follows;

$$\text{If } \Delta_\ell v(k) \;=\; u(k), \; \textit{then } v(k) \;=\; \Delta_\ell^{-1} u(k) + c_j, \tag{5}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j),\ j = k - \left[\frac{k}{\ell}\right]\ell$.

In general $\Delta_\ell^{-n} u(k) = \Delta_\ell^{-1}(\Delta_\ell^{-(n-1)} u(k))$ for the integers $n \geq 2$.

Definition 2.2. [11] The inverse of the Generalized α-difference operator, denoted by $\Delta_{\alpha(\ell)}^{-1}$, on $u(k)$ is defined as follows. If $\Delta_{\alpha(\ell)} v(k) \;=\; u(k)$, then

$$\Delta_{\alpha(\ell)}^{-1} u(k) \;=\; v(k) - \alpha^{\left[\frac{k}{\ell}\right]} v(j), \tag{6}$$

where $k \in \mathbb{N}_\ell(j),\ j = k - \left[\frac{k}{\ell}\right]\ell$.

Lemma 2.3. *[7](Finite Summation formula) If the real valued function $u(k)$ is defined for all $k \in [0, \infty)$, then*

$$\Delta_\ell^{-1} u(k)\Big|_j^k = \sum_{r=1}^{\left[\frac{k}{\ell}\right]} u(k - r\ell) + c_j, \tag{7}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$, $j = k - \left[\frac{k}{\ell}\right]\ell$.

Definition 2.4. [11] The solution $u(k)$ of a generalized difference equation is called oscillatory if for any $k_1 \in [a, \infty)$ there exists a $k_2 \in \mathbb{N}_\ell(k_1)$ such that $u(k_2)u(k_2 + \ell) \leq 0$. The difference equation itself is called oscillatory if all its solutions are oscillatory. If the solution $u(k)$ is not oscillatory, then it is said to be nonoscillatory (i.e. $u(k)u(k + \ell) > 0$ for all $k \in [k_1, \infty)$).

3. Main Results

The higher order (nth order) generalized α_i- difference equation of the form $\Delta_{\alpha_1(\ell_1)}(\Delta_{\alpha_2(\ell_2)}(\cdots\Delta_{\alpha_n(\ell_n)}(v(k))\cdots)) = u(k),\ k \in [0,\infty),\ \ell_i > 0\ \alpha_i \neq 0$ becomes generalized mixed difference equation if $\alpha_i = 1$ for some i and $n \geq 2$. In this section we study the asymptotic behavior of the non-oscillatory solutions of the generalized mixed difference equation (1) and (2).

Theorem 3.1. *Suppose $u(k)$ is a nonoscillatory solution of equation (1) if*

$$sgn\ u(k) = sgn\ \Delta_\ell^2 u(k) \neq sgn\ \Delta_\ell u(k) = sgn\ \Delta_\ell^3 u(k) \tag{8}$$

and

$$\lim_{n\to\infty} u(k) = 0. \tag{9}$$

Proof. A nonoscillatory solution of (1) may not exist if $0 < \alpha < 1$, but if it does exist, we show that it must satisfy (8) and (9). As the negative solution of equation (1) is also a solution of the same equation, it suffices to prove that a positive solution of (1) satisfies (8). Let $u(k) > 0$ be a non-oscillatory solution of (1) for $\delta = 1$.

Setting $r(k) = \Delta_{\alpha(\ell)} u(k) = u(k+\ell) - \alpha u(k)$, we get

$$\Delta_\ell^2 r(k) = -p(k)u(k) < 0, \tag{10}$$

and so $\Delta_\ell r(k)$ is (eventually) strictly decreasing. From (10) it follows that if $\Delta_\ell r(k)$ is eventually negative we must have $r(k) \to -\infty$. However this is contradictory, since $r(k) = u(k+\ell) - \alpha u(k) = \Delta_\ell u(k) + (1-\alpha)u(k) \to -\infty$ implies $\Delta_\ell u(k)$, forces $u(k)$ to be eventually negative. Hence we must have

$$\Delta_\ell r(k) > 0 \tag{11}$$

for all large k. Indeed we will show that $\lim\limits_{k\to\infty} u(k) = 0$.

Writing (1) as $\Delta_\ell^2\Big(\Delta_{\alpha(\ell)} u(k)\Big) = -p(k)u(k)$, and by Lemma 2.3, when k_0 is chosen large enough so that $\Delta_\ell r(k) > 0$ for all $k \geq k_0$, we get

$$\Delta_\ell r(k) - \Delta_\ell r(k_0) = -\sum_{r=0}^{\left[\frac{k-\ell}{\ell}\right]} p(k_0 + r\ell)u(k_0 + r\ell).$$

The lim inf condition on $p(k)$ yields

$$0 < c\sum_{r=0}^{\left[\frac{k-\ell}{\ell}\right]} u(k_0 + r\ell) \leq \sum_{r=0}^{\left[\frac{k-\ell}{\ell}\right]} p(k_0 + r\ell)u(k_0 + r\ell) < \Delta_\ell r(k).$$

Letting $k \to \infty$, we see that $\sum_{r=0}^{\infty} u(k_0 + r\ell) < \infty$ and therefore $\lim_{k\to\infty} u(k) = 0$. Since $u(k) \to 0$ as $k \to \infty$ it follows that $r(k) \to 0$ as $k \to \infty$. From (11) we get $r(k)$ is increasing and hence $r(k) < 0$ eventually. It then follows from the inequality $r(k) = \Delta_\ell u(k) + (1-\alpha)u(k) < 0$ that $\Delta_\ell u(k) < 0$ and from (11) we obtain the relation $\Delta_\ell r(k) = \Delta_\ell^2 u(k) + (1-\alpha)\Delta_\ell u(k) > 0$ and $\Delta_\ell^2 u(k) > 0$. From (10) we get $\Delta_\ell^2 r(k) = \Delta_\ell^3 u(k) + (1-\alpha)\Delta_\ell^2 u(k) < 0$, $\Delta_\ell^3 u(k) < 0$ and the proof is complete. □

Example 3.2. *The solution of the third order generalized mixed difference equation*

$$\Delta_\ell^2\Big(\Delta_{\alpha(\ell)} u(k)\Big) + \frac{(k+2\ell)_\ell^{(3)} - (2+\alpha)(k+3\ell)(k+\ell)_\ell^{(2)} + (1+2\alpha)(k+3\ell)_\ell^{(2)} k - \alpha(k+3\ell)_\ell^{(3)}}{(k+3\ell)_\ell^{(3)}} \times u(k) = 0,$$

satisfies all the conditions of Theorem 3.1 and $\lim_{n\to\infty} u(k) = 0$. *Infact* $u(k) = \frac{1}{k}$ *is one such solution.*

Theorem 3.3. *If $u(k)$ is a nonoscillatory solution of (1) for $\delta = -1$ with $\alpha > 1$ then for all k sufficiently large*

$$sgn\ u(k) = sgn\ \Delta_\ell u(k) = sgn\ \Delta_\ell^2 u(k), \tag{12}$$

and

$$\lim_{k\to\infty} |u(k)| = \lim_{k\to\infty} |\Delta_\ell u(k)| = \lim_{k\to\infty} |\Delta_\ell^2 u(k)| = \infty. \tag{13}$$

Proof. Assume that $u(k) > 0$ for all k sufficiently large.
Taking $r(k) = \Delta_{\alpha(\ell)} u(k) = u(k+\ell) - \alpha u(k)$, we get

$$\Delta_\ell^2 r(k) = p(k)u(k) > 0 \tag{14}$$

and $\Delta_\ell r(k)$ is increasing. If $\Delta_\ell r(k)$ is eventually positive, then as $k \to \infty, r(k) \to \infty$ and since $r(k) = \Delta_\ell u(k) + (1-\alpha)u(k)$ and $\alpha > 1$ it follows that $\Delta_\ell u(k) \to \infty$, which in turn implies $u(k) \to \infty$. To see $\Delta_\ell^2 u(k) \to \infty$, note that $u(k) \to \infty$ implies $\Delta_\ell^2 r(k) \to \infty$ and $\Delta_\ell r(k) \to \infty$ because of (14). Hence the result follows from $\Delta_\ell r(k) = \Delta_\ell^2 u(k) + (1-\alpha)\Delta_\ell u(k)$. Now, if $\Delta_\ell r(k)$ is eventually negative and increasing, then $\Delta_\ell r(k)$ has a limit as $k \to \infty$. However $\Delta_\ell r(k)$ having a limit implies that $\sum_{k=0}^{\infty} u(k_0 + r\ell) < \infty$ and this implies $u(k) \to 0$. But $u(k) \to 0$ implies $r(k) \to 0$ and since $r(k)$ is decreasing to zero we get $r(k) > 0$. But the relation $r(k) = \Delta_\ell u(k) + (1-\alpha)u(k) > 0$ implies $\Delta_\ell u(k) > 0$, a contradiction since $u(k) > 0$ and $\Delta_\ell u(k) > 0$ is inconsistent with $u(k) \to 0$. Hence (13) cannot have a nonoscillatory solution with $\Delta_\ell r(k)\Delta_\ell^2 r(k) < 0$ for all k sufficiently large. □

Example 3.4. *The Theorem 3.3 holds for the generalized mixed difference equation*

$$\Delta_\ell^2\Big(\Delta_{\alpha(\ell)} u(k)\Big) + \frac{2\ell^2(1-\alpha)}{k^2} u(k) = 0.$$

Infact $u(k) = k^2$ is one solution.

Theorem 3.5. *Consider the equation (2) for $\delta = -1$ and $\alpha \geq 1$. If $u(k)$ is a nonoscillatory solution, then for all k sufficiently large either*

$$sgn\ u(k) = sgn\ \Delta_\ell u(k) = sgn\ \Delta_\ell^2 u(k) = sgn\ \Delta_\ell^3 u(k) \tag{15}$$

or

$$sgn\ u(k) = sgn\ \Delta_\ell\big(\Delta_{\alpha(\ell)} u(k)\big) \neq sgn\ \Delta_{\alpha(\ell)} u(k) = sgn\ \Delta_\ell^2\big(\Delta_{\alpha(\ell)} u(k)\big). \tag{16}$$

Proof. We prove the case for $\alpha > 1$. Assume $u(k)$ is eventually positive. Taking $r(k) = \Delta_{\alpha(\ell)} u(k) = u(k+\ell) - \alpha u(k)$, we get

$$\Delta_\ell^3 r(k) = p(k)u(k) > 0. \tag{17}$$

Clearly $\Delta_\ell^2 r(k)$ is increasing. In case $\Delta_\ell^2 r(k)$ is eventually positive, we will have $\lim_{k\to\infty} \Delta_\ell r(k) = \lim_{k\to\infty} r(k) = \infty$ and since $r(k) < u(k+\ell)$ it follows that $u(k) \to \infty$. Since $p(k) > c$ for large k, we have

$$\lim_{k\to\infty} \Delta_\ell^3 r(k) = \lim_{k\to\infty} \Delta_\ell^2 r(k) = \infty.$$

Since $r(k) = \Delta_\ell u(k) + (1-\alpha)u(k) \to \infty$ and $\alpha > 1$ it follows that $\Delta_\ell u(k) \to \infty$. Examining $\Delta_\ell r(k) = \Delta_\ell^2 u(k) + (1-\alpha)\Delta_\ell u(k)$ we see that $\Delta_\ell^2 u(k) \to \infty$ as $k \to \infty$. Continuing in this manner we see that (15) holds eventually.

Next, we consider the case where $\Delta_\ell^3 r(k) > 0$ and $\Delta_\ell^2 r(k) < 0$. Then, existence of $\lim_{k\to\infty} \Delta_\ell^2 r(k)$ and Lemma 2.3 yield

$$-\Delta^2 r(k_1) > \Delta^2 r(m) - \Delta^2 r(k_1) = \sum_{t=0}^{\frac{k-\ell-k_1-j}{\ell}} p(k_1+j+t\ell)u(k_1+j+t\ell)$$

$$\geq c \sum_{t=0}^{\frac{k-\ell-k_1-j}{\ell}} u(k_1+j+t\ell).$$

Letting $m \to \infty$, it then follows that $\sum_{t=0}^{\infty} u(k_1+j+t\ell) < \infty$ and hence $\lim_{k\to\infty} u(k) = 0$ which implies $r(k) \to 0$. Thus, if $\Delta_\ell^3 r(k) > 0$ and $\Delta_\ell^2 r(k) < 0$ then we have

$$\Delta_\ell^3 r(k) > 0, \Delta_\ell^2 r(k) < 0, \Delta_\ell r(k) > 0, \tag{18}$$

because $\Delta_\ell^2 r(k) < 0$ and $\Delta_\ell r(k) < 0$ is inconsistent with $r(k) \to 0$, it then follows that either (i) $r(k) > 0$ or (ii) $r(k) < 0$ eventually. We will show that (i) is impossible. If (i) holds then since $r(k) = \Delta_\ell u(k) + (1-\alpha)u(k) > 0$, it follows that $\Delta_\ell u(k) > 0$, in fact we have that $\Delta_\ell u(k) > c + (\alpha-1)u(k) > c$ for some positive constant c and so, $u(k) \to \infty$ as $k \to \infty$. But this implies $\Delta_\ell^3 r(k) \to \infty$, so we must have $\Delta_\ell^2 r(k) > 0$ eventually, contradicting (18). So (i) cannot hold, resulting (ii) holding eventually. □

Example 3.6. *The Theorem 3.5 holds for the generalized mixed difference equation*

$$\Delta_\ell^3\Big(\Delta_{\alpha(\ell)} u(k)\Big) + \frac{6\ell^3(1-\alpha)}{k^3} u(k) = 0.$$

Infact $u(k) = k^3$ is one such solution.

Theorem 3.7. *Every nontrivial bounded solution of (2) for $\delta = 1$, where $\alpha > 1$, is oscillatory.*

Proof. Suppose $u(k) > 0$ is bounded nonoscillatory for large k.

Letting $r(k) = \Delta_{\alpha(\ell)} u(k) = u(k+\ell) - \alpha u(k)$, we see that $r(k) \geq -\alpha u(k)$ and then

$$\Delta_\ell^3 r(k) = -p(k)u(k) < 0.$$

Obviouly $\Delta^2 r(k)$ is decreasing and if $\Delta^2 r(k)$ is eventually negative, we see that $r(k) \to \infty$. This clearly contradicts the boundedness of $u(k)$. Thus, we consider the case where $\Delta^2 r(k) > 0$. In this case, $\lim_{k\to\infty} \Delta^2 r(k) = t \geq 0$. Using the fact $p(k)$ is bounded away from zero for large k, it follows that $\Delta_\ell r(k) < 0$ and $r(k) > 0$ for large k. Furthermore $\lim_{k\to\infty} u(k) = 0$, since $\Delta_\ell^2 r(k) \to t$ implies

$\sum_{r=0}^{\infty} u(r\ell) < \infty$. Since $\alpha > 1$ and $r(k) = \Delta_\ell u(k) + (1-\alpha)u(k) > 0$, $\Delta_\ell u(k) > 0$ for all k sufficiently large. But this is a contradiction, since $u(k)\Delta_\ell u(k) > 0$ is congruent with $u(k) \to 0$. □

Example 3.8. *The Fourth generalized mixed difference equation*

$$\Delta_\ell^3\Big(\Delta_{\alpha(\ell)} u(k)\Big) + \frac{(1+\alpha^2)(1+\alpha)^3}{\alpha^4} u(k) = 0,$$

satisfies the conditions of Theorem 3.7 and hence the solution is oscillatory. Infact $u(k) = \frac{1}{(-\alpha)^{\lceil \frac{k}{\ell} \rceil}}$ *is one such solution.*

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