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# NUMERICAL SOLUTIONS FOR LINEAR AND NON-LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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**Abstract:** In this paper, a new numerical technique to solve linear and nonlinear fractional differential equations of order $0 < \alpha < 1$ in sense of Caputo's definition is proposed. The efficiency of this technique will be illustrated by solving several examples of linear and nonlinear fractional differential equations of order $0 < \alpha < 1$.

**AMS Subject Classification:** 65D25
**Key Words:** finite difference formulas, composite trapezoidal rule, numerical solutions, linear fractional differential equation, nonlinear fractional differential equation

## 1. Introduction

The concept of Fractional Calculus is known by L'Hopital's question to Leibniz in the year 1695, his question was about the differentiation of order 1/2. In his reply, dated 30September 1695, Leibniz wrote to L'Hopital as follows "This is an apparent Paradox from which, one day, useful consequences will be drawn" [11].

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Fractional Calculus is a calculus of integrals and derivatives of any arbitrary real or complex order which is investigated during the past three decades in numerous areas of science and engineering, namely in signal processing, control engineering [1] and [14], electromagnetism [12], bioscience [17], fluid mechanics [3], electrochemistry[7], diffusion processes [20], continuum and statistical mechanics [3] and propagation of spherical flames [2]. In general, most of the fractional differential equations do not have exact solutions. Particularly, there is no known method for finding exact solutions of fractional differential equations. Therefore, several methods for approximate solutions to classical differential equations are extended to solve differential equations of fractional order numerically. Some of these methods include: Adomian Decomposition Method [19], Perturbation Method [13], Homotopy Analysis Method [5], Variational Iteration Method [4], Extrapolation Method [8] and Generalized Differential Transform Method [21].

The fractional derivative of order $\alpha > 0$ has several definitions. Riemann-Liouville and Caputo's definitions are the most commonly used for the derivative of this order. For the fractional derivative, the Caputo's definition is used, which is a modification of the Riemann-Liouville definition; because it has an advantage of dealing properly with the initial value problem since the initial condition is given in terms of the field variables and their integer order. This case is widely used in physical applications [9].

Usually, Taylor Series is used to obtain numerical differentiation methods. Methods using Taylor Series are: Backward Difference Method, Forward Difference Method and Central Difference Method to evaluate the derivative [6] and [15]. Finite difference formulas are the most common formulas which are used to solve the ordinary and partial differential equations numerically [6]. The derivatives in these equations can be replaced with suitable finite difference approximations on a discretized domain. The accuracy of the solution depends on the number of mesh points such that if the number of mesh points is increased, then the solution will be more accurate.

In this paper, numerical solutions are presented for linear and nonlinear fractional differential equations using FFDM. The suggested FFDM is adopted because of using the finite difference formulas to approximate the $1^{st}$ and $2^{nd}$ derivatives more than once. Caputo's definition of fractional derivative is used in this method and the Composite Trapezoidal Rule is utilized to approximate the integral term in Caputo's definition, then the finite difference formulas in three types (Forward, Central and Backward) are used to present approximations to the $1^{st}$ and $2^{nd}$ derivatives. FFDM is useful to solve the linear and nonlinear fractional differential equations of order $0 < \alpha < 1$ numerically. This method is a technique by which the obtained derivatives ($1^{st}$ and $2^{nd}$ derivatives), from some operations worked on the Caputo's definition, are approximated by finding the differences in the values of $y(t)$ by applying the independent variable t and small increment $(t + h)$ [16]. To solve linear and nonlinear fractional differential equations, the derivatives in these equations are replaced with finite difference approximations.

Further, applications are given to show the applicability and effectiveness of the FFDM for finding accurate approximate solutions to the general fractional differential equation of the form

$$\begin{aligned} D_*^\alpha y(t) + a_m y^{(m)}(t) + a_{m-1} y^{(m-1)}(t) + ... \\ + a_0 y(t) + N(y(t), y'(t)) = f(t), \\ t \geqslant 0, \quad m - 1 < \alpha \leqslant m, \end{aligned} \tag{1}$$

Subject to

$$y^{(i)}(0) = y_i, \quad i = 0, 1, ..., m - 1.$$

Here $D_*^\alpha$ is the derivative of y of order $\alpha$ in the sense of Caputo, fractional differential operator $y(t)$ is an unknown function of the independent variable $t$, and $N$ is a nonlinear differential operator.

FFDM's efficiency is demonstrated by comparing the obtained approximate solutions for some linear and nonlinear fractional differential equations by that method with the exact solutions and with other approximate solutions that obtained by other solvers and methods. For the linear fractional differential equations, the mentioned method is utilized to find approximate solutions for these equations. Moreover, Mathematica (Ver. 9.0) is used to solve the system of linear equations that is obtained from some operations. Like the way of approximating solutions for the linear fractional differential equations, the same method is utilized to find approximate solutions for the nonlinear fractional differential equations. Also, the same program is used to solve the system of nonlinear equations that is obtained from some substitutions.

## 2. Fractional Derivatives

We start this section by illustrating some notations that will be used frequently. The notation $D_a^\alpha f(t)$ will be introduced as a notation which is denoted to the fractional derivative of a function $f(t)$ along *t-axis* of an arbitrary order $\alpha > 0$, where the subscript a is denoted to the lower limit of the integration. However, to get a simplified notation, we may drop the subscript a. Further, the fractional derivative can be defined using the definition of the fractional integral. The following definitions illustrates that:

**Definition 1.** Let $\alpha \in R^+$. For a positive integer $m$ such that $m-1 < \alpha \leq m$, then the Riemann-Liouville Fractional Derivative of a function $f(t)$ of order $\alpha$ is defined by

$$D_a^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \left( \int_a^t (t-x)^{m-\alpha-1} f(x)dx \right) \tag{2}$$

In addition, one observation that should be appointed, that's the most used version of $D_\alpha^\alpha$ is when $a = 0$, so

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \left( \int_0^t (t-x)^{m-\alpha-1} f(x)dx \right) \tag{3}$$

In 1967, M. Caputo presented a new definition of a fractional derivative called Caputo's Fractional Derivative which is a modification of the Riemann-Liouville Fractional Derivative [9]. However, the definition of the Caputo's Fractional Derivative uses almost the same notations of the definition of the Riemann-Liouville Fractional Derivative.

**Definition 2.** Let $\alpha \in R^+$ and $n \in N$ such that $n-1 < \alpha < n$, then the Caputo's Fractional Derivative of order $\alpha$ is defined by

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx \tag{4}$$

In addition, if $a = 0$ in Eq.(4), then the most used version of the Caputo's Fractional Derivative is obtained, i.e.

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx \tag{5}$$

## 3. Approximating the Fractional Derivative of Order $0 < \alpha < 1$ using FFDM

For the fractional derivative, the Caputo's definition is chosen. Now, according to Eq.(5), if $0 < \alpha < 1$, then the Caputo's Fractional Derivative is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(x)}{(t-x)^\alpha} dx \tag{6}$$

For $t \geq 0$ and $\alpha \in R^+$.

By applying the method of integration by parts on the right hand side of Eq.(6), we get

$$D^\alpha f(t) = \frac{1}{(1-\alpha)\Gamma(1-\alpha)}(f'(0)t^{1-\alpha} + \int_0^t (t-x)^{1-\alpha} f''(x)\, dx \tag{7}$$

Last integral in Eq.(7) can be approximated using the Composite Trapezoidal Rule, as follows
$\int_0^t (t-x)^{1-\alpha} f''(x) dx \approx$

$$\frac{h}{2}[t^{1-\alpha} f''(0) + 2\sum_{j=1}^{n-1}(t-x_j)^{1-\alpha} f''(x_j) + (t-b)^{1-\alpha} f''(b)] \tag{8}$$

with $h = (b-a)/n$ and $x_j = a + jh$ for each $j = 0, 1, \ldots, n-1$.

Now, by substituting Eq. (8) in Eq.(14), we obtain:
$D^\alpha f(t) =$

$$\frac{1}{(1-\alpha)\Gamma(1-\alpha)} \times \left\{ \begin{array}{c} \frac{h}{2}[t^{1-\alpha} f''(0) + (t-b)^{1-\alpha} f''(b) + \\ 2\sum\limits_{j=1}^{n-1}(t-x_j)^{1-\alpha} f''(x_j)] + f'(0)t^{1-\alpha} \end{array} \right\} \tag{9}$$

Here, $f'$ and $f''$ can be approximated at specific points. Forward finite difference formula for the $1^{st}$ order derivative, is used to approximate $f'(0)$ such that

$$f'(0) \approx \frac{-3f(0) + 4f(h) - f(2h)}{2h} \tag{10}$$

for small values of $h$.

To approximate $f''(0)$, also forward finite difference formula for the $2^{nd}$ derivative, is used as follows

$$f^{''}(0) \approx \frac{f(0) - 2f(h) + f(2h)}{h^2} \tag{11}$$

for small values of $h$.

While the central finite difference formula for the $2^{nd}$ derivative, is used to approximate$f''(x_j)$ in the sum term of Eq.(9) as

$$f^{''}(x_j) \approx \frac{f(x_j - h) - 2f(x_j) - f(x_j + h)}{h^2} \tag{12}$$

for small values of $h$ and for $j = 0, 1, \ldots, n-1$.

Finally, $f''(b)$ can be approximated using backward finite difference formula for the $2^{nd}$ derivative, as follows

$$f^{''}(b) \approx \frac{f(b) - 2f(b-h) - f(b-2h)}{h^2} \tag{13}$$

for small values of $h$.

By substituting Eq.'s (10), (11), (12) and (13) in Eq.(9), we get

$$D^{\alpha} f(t) \approx \frac{1}{(1-\alpha)\Gamma(1-\alpha)} \times \left( A_1 + \frac{h}{2} [A_2 + A_3 + A_4] \right) \tag{14}$$

Where

$$A_1 = \frac{-3f(0) + 4f(h) - f(2h)}{2h} t^{1-\alpha},$$

$$A_2 = \frac{f(0) - 2f(h) + f(2h)}{h^2} t^{1-\alpha},$$

$$A_3 = 2 \sum_{j=1}^{n-1} \frac{f(x_j - h) - 2f(x_j) + f(x_j + h)}{h^2} (t - x_j)^{1-\alpha},$$

and

$$A_4 = \frac{f(b) - 2f(b-h) + f(b-2h)}{h^2} (t-b)^{1-\alpha}.$$

We observe that, according to Eq.(14), the approximation of $D^{\alpha}\ f(t)$ depends on the value of $h$ such that if the value of h is decreased, then the approximate result of $D^{\alpha}\ f(t)$ will be more accurate. Moreover, Eq.(14) is the purposed equation that needs an algorithm to approximate the fractional derivative of order $0 < \alpha < 1$ of a given function $f(t)$. Also, the value of $h$ according to the Composite Trapezoidal Rule, is $(b-a)/n$ .

### 3.1. Error Analysis

In the Composite Trapezoidal Rule the error term equal to

$$-\frac{b-a}{12} h^2 (\frac{d^2}{dx^2} [(t-x)^{1-\alpha} f''(x)]|_{x=\mu} \tag{15}$$

for some $\mu$ between $a$ and $b$, and as $h$ approaches zero the error term approaches zero.

| | | $h = 0.01$ | | $h = 0.0001$ | |
|---|---|---|---|---|---|
| t | Exact | App. | Abs. Error | App. | Abs. Error |
| 0.1 | 1.06639 | 1.06571 | $6.82 \times 10^{-4}$ | 1.06638 | $4.30 \times 10^{-6}$ |
| 0.2 | 1.39931 | 1.39779 | $1.51 \times 10^{-3}$ | 1.39930 | $9.55 \times 10^{-5}$ |
| 0.3 | 1.73265 | 1.73022 | $2.43 \times 10^{-3}$ | 1.73264 | $1.54 \times 10^{-5}$ |
| 0.4 | 2.07345 | 2.07006 | $3.40 \times 10^{-3}$ | 2.07343 | $2.14 \times 10^{-5}$ |
| 0.5 | 2.41926 | 2.41495 | $4.31 \times 10^{-3}$ | 2.41923 | $2.72 \times 10^{-5}$ |
| 0.6 | 2.76233 | 2.75730 | $5.03 \times 10^{-3}$ | 2.76230 | $3.17 \times 10^{-5}$ |
| 0.7 | 3.08998 | 3.08462 | $5.36 \times 10^{-3}$ | 3.08994 | $3.37 \times 10^{-5}$ |
| 0.8 | 3.38461 | 3.37958 | $5.03 \times 10^{-3}$ | 3.38458 | $3.14 \times 10^{-5}$ |
| 0.9 | 3.62453 | 3.62077 | $3.76 \times 10^{-3}$ | 3.62451 | $2.32 \times 10^{-5}$ |
| 1.0 | 3.78586 | 3.78452 | $1.33 \times 10^{-3}$ | 3.78585 | $7.57 \times 10^{-6}$ |

Table 1: Numerical fractional derivative of $f(t)$ for Example 1, where $\alpha = 0.9$.

On the other hand, in the finite difference formula we used, the error term also approaches zero as $h$ approaches to zero.

## 4. Applications

The following example shows the efficiency of the method of (FFDM) to approximate the fractional derivative.

We can find $D^{\alpha}\ f(t)$ approximately by taking increasingly $n\prime s$, that's we may take $h = \{0.01, 0, 001, 0.0001\}$.

**Example1.** For the function $f(t) = \sin(t^2) + e^t + 1,\ t \in (0, 1)$. We will take $\alpha = 0.9$ to approximate $D^{0.9}f(t)$ on its interval using FFDM and compare the obtained approximate solutions with the exact solutions. Thus, to approximate $D^{0.9}f(t)$, we may take $h = 0.010$ and $.0001$. Hence, the following tables illustrates the concluded results using FFDM by computing the fractional derivatives of order $\alpha = 0.9$ for $f(t)$ at specific points of $t$.

Thus, our method has been built to introduce numerical solutions for the fractional derivatives of order $0 < \alpha < 1$ of various functions at specific points of $t$ using FFDM. We note that, this method is a reliable and very useful to obtain approximate solutions for fractional derivatives of order $0 < \alpha < 1$. Moreover, the occurred errors between the approximate solutions which are obtained by

| t | App. | Exact | Abs. Error |
|---|---|---|---|
| 0.1 | $-0.089851$ | $-0.09$ | $1.49 \times 10^{-4}$ |
| 0.2 | $-0.159781$ | $-0.16$ | $2.19 \times 10^{-4}$ |
| 0.3 | $-0.209728$ | $-0.21$ | $2.72 \times 10^{-4}$ |
| 0.4 | $-0.239683$ | $-0.24$ | $3.17 \times 10^{-4}$ |
| 0.5 | $-0.249643$ | $-0.25$ | $3.57 \times 10^{-4}$ |
| 0.6 | $-0.239607$ | $-0.24$ | $3.93 \times 10^{-4}$ |
| 0.7 | $-0.209574$ | $-0.21$ | $4.26 \times 10^{-4}$ |
| 0.8 | $-0.159544$ | $-0.16$ | $4.56 \times 10^{-4}$ |
| 0.9 | $-0.089515$ | $-0.09$ | $4.85 \times 10^{-4}$ |
| 1.0 | 0.000512 | 0.00 | $5.12 \times 10^{-4}$ |

Table 2: Numerical solutions for Example 2 using FFDM when $h = 0.01$ and $\alpha = 0.5$.

FFDM and the exact solutions are steady and stable according to the values of $h$ which can be taken decreasingly.

To show the efficiency of the FFDM, we will approximate the solutions for some linear and nonlinear fractional differential equations of order $0 < \alpha < 1$ and comparing them with the exact solutions and with other approximate solutions that obtained by other solvers and methods. As we mentioned before, Eq.(14) is considered as an approximation for $D^\alpha y(t)$; so whenever we find $D^\alpha y(t)$ in any fractional differential equation of order $0 < \alpha < 1$ of the form of Eq.(1), we may replace it by that approximation.

**Example 2.** For the linear fractional differential equation:

$$D^\alpha y(t) = \frac{8}{3}\sqrt{\frac{t^3}{\pi}} - 2\sqrt{\frac{t}{\pi}}, \quad y(0) = 0.$$

The exact solution is $y(t) = t^2 - t$.

Here, we will take $h = 0.01$. Thus, the following table illustrates the concluded results using FFDM by computing the approximate solutions for this problem at various values of $t$.

**Example 3.** (see [18]) For the linear fractional differential equation:

$$D^\alpha y(t) = t^2 + \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - y(t), \quad y(0) = 0.$$

| t | App. | Exact | Abs. Error |
|---|---|---|---|
| 0.1 | 0.010116 | 0.01 | $1.16 \times 10^{-4}$ |
| 0.2 | 0.040156 | 0.04 | $1.56 \times 10^{-4}$ |
| 0.3 | 0.090181 | 0.09 | $1.81 \times 10^{-4}$ |
| 0.4 | 0.160200 | 0.16 | $2.00 \times 10^{-4}$ |
| 0.5 | 0.250215 | 0.25 | $2.15 \times 10^{-4}$ |
| 0.6 | 0.360227 | 0.36 | $2.27 \times 10^{-4}$ |
| 0.7 | 0.490237 | 0.49 | $2.37 \times 10^{-4}$ |
| 0.8 | 0.640246 | 0.64 | $2.46 \times 10^{-4}$ |
| 0.9 | 0.810254 | 0.81 | $2.54 \times 10^{-4}$ |
| 1.0 | 1.000261 | 1.00 | $2.61 \times 10^{-4}$ |

Table 3: Numerical solutions for example 3 using FFDM and the corresponding errors when $h = 0.01$ and $\alpha = 1/2$.

The exact solution to this problem is $y(t) = t^2$.

We may take $h = 0.01$. So; the following table illustrates the concluded results using FFDM by computing the approximate solutions for this problem.

**Example 4.** (see [18]) For the nonlinear fractional differential equation:

$$D^{\alpha}y(t) = \frac{40320}{\Gamma(9-\alpha)}t^{8-\alpha} - 3\frac{\Gamma(5+\frac{\alpha}{2})}{\Gamma(5-\frac{\alpha}{2})}t^{4-\frac{\alpha}{2}} + \frac{9}{4}\Gamma(\alpha+1) + (\frac{3}{2}t^{\frac{\alpha}{2}} - t^4)^3 - y^{\frac{3}{2}}(t),$$
$$y(0) = 0.$$

The exact solution is $y(t) = t^8 - 3t^{4+\frac{\alpha}{2}} + \frac{9}{4}t^{\alpha}$

We may take $h = 0.01$. However, the following table illustrates the concluded results using FFDM by computing the approximate solutions for this problem.

**Example 5.** (see [10]): For the nonlinear fractional differential equation:

$$D^{\alpha}y(t) = 2y(t) - y^2(t) + 1, \quad y(0) = 0.$$

Before we present the approximate solutions using FFDM, we will introduce a previous study that deals this problem. As we mentioned before, the

| | $\alpha = 0.2$ | | | $\alpha = 0.5$ | | |
|---|---|---|---|---|---|---|
| t | App. | Exact | Abs. Error | App. | Exact | Abs. Error |
| 0.1 | 1.414387 | 1.419416 | $5.029 \times 10^{-3}$ | 0.700025 | 0.711344 | 0.011319 |
| 0.2 | 1.624381 | 1.626670 | $2.289 \times 10^{-3}$ | 0.997030 | 1.003023 | $5.993 \times 10^{-3}$ |
| 0.3 | 1.745556 | 1.747029 | $1.473 \times 10^{-3}$ | 1.210491 | 1.214457 | $3.966 \times 10^{-3}$ |
| 0.4 | 1.802753 | 1.803824 | $1.071 \times 10^{-3}$ | 1.359608 | 1.362604 | $2.996 \times 10^{-3}$ |
| 0.5 | 1.786804 | 1.787701 | $8.970 \times 10^{-4}$ | 1.434719 | 1.437228 | $2.509 \times 10^{-3}$ |
| 0.6 | 1.678080 | 1.678839 | $7.590 \times 10^{-4}$ | 1.415158 | 1.417451 | $2.293 \times 10^{-3}$ |
| 0.7 | 1.457012 | 1.457674 | $6.620 \times 10^{-4}$ | 1.279068 | 1.281281 | $2.213 \times 10^{-3}$ |
| 0.8 | 1.117227 | 1.117881 | $6.540 \times 10^{-4}$ | 1.015963 | 1.018106 | $2.143 \times 10^{-3}$ |
| 0.9 | 0.685292 | 0.685880 | $5.880 \times 10^{-4}$ | 0.645971 | 0.647831 | $1.860 \times 10^{-3}$ |
| 1.0 | 0.249488 | 0.250000 | $5.120 \times 10^{-4}$ | 0.248838 | 0.250000 | $1.162 \times 10^{-3}$ |

Table 4: Numerical solutions for Example 4 using FFDM and the corresponding errors, where $h = 0.01$.

exact solution is not available for most nonlinear fractional differential equations. However, for such equations, solutions are obtained approximately by some of approximate analytic solvers such as Homotopy Perturbation Method, Chebyshev Wavelets (CW) and others [10]. Muhammad Asif Zahoor Raja, Junaid Ali Khan and Ijaz Mansoor Qureshi develop a new stochastic Technique for a solution of this problem. They use a Particle Swarm Optimization (PSO) as a tool for the Rapid Global Search Method and simulating annealing for efficient local search [10]. However, our FFDM is also applied to solve this problem and compared with some other solvers and methods according to their approximate solutions. Now, by following the same procedure and by taking $h = 0.01$, we get the following results ("(figure1 and figure 2) which illustrate the approximate solutions for this problem using FFDM in comparison with other mentioned solvers and methods that are used to solve the same problem.

Regarding to [10], the authors concluded that their proposed computing approach is reliable, effective and easily applicable to nonlinear differential equation of arbitrary order. So; according to their conclusions, we find that the FFDM gives approximate solutions which are very close to the approximate solutions that obtained using PSO and CW algorithms according to their approximate solutions for that problem.

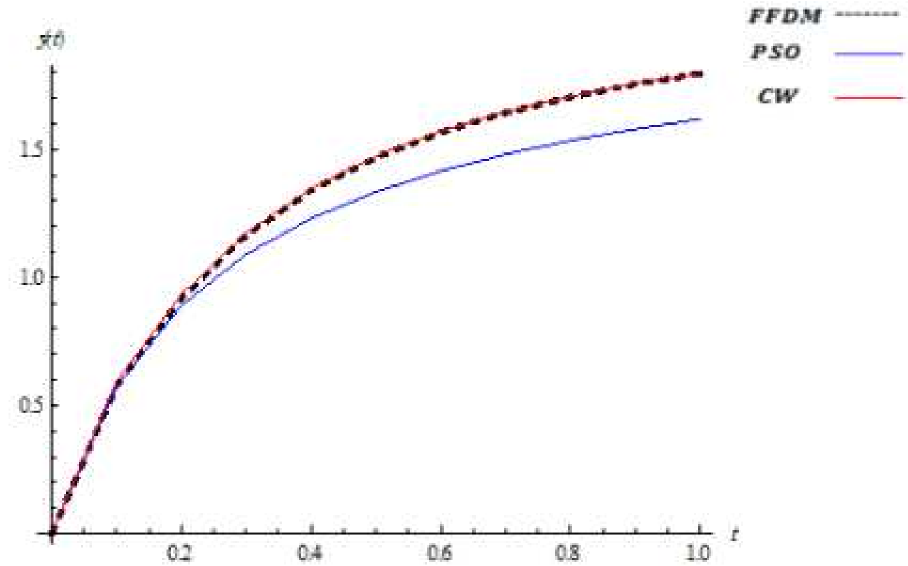


Figure 1: The comparison between the FFDM solutions (where $h = 0.01$ and $\alpha = 0.5$) and the approximate solutions which are obtained by some methods, according to Example 5.

## 5. Conclusions

The major goal of this paper is to find accurate approximate solutions for linear and nonlinear fractional differential equations. Hence, we carry out this goal by preparing a new method called Fractional Finite Difference Method “FFDM”. In this paper, we discussed and illustrated the numerical solutions of linear and nonlinear fractional differential equations using FFDM. Also, the efficiency of FFDM was illustrated by solving several examples of linear and nonlinear fractional differential equations.

Successfully, FFDM was applied to solve the linear and nonlinear fractional differential equations of order $0 < \alpha < 1$. All ideas were illustrated to be efficient in applying the proposed technique to several linear and nonlinear fractional differential equations of that order. We found that our method is powerful and efficient in finding numerical solutions for those equations. Moreover, the FFDM solutions demonstrate excellent approximations in comparison with the exact solutions and with other methods and solvers through the applicable domain. Also, we observed that if the value of his decreased, then the approximate results for the linear and nonlinear fractional differential equa-

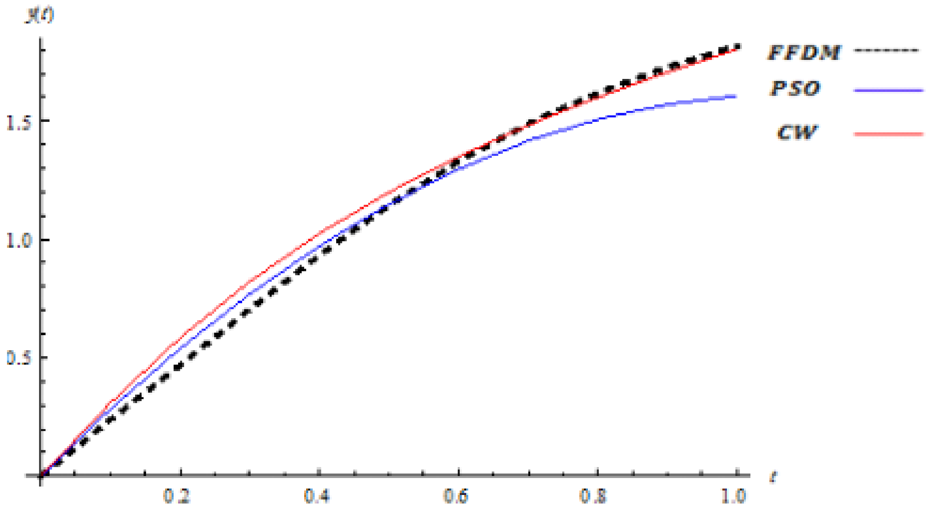


Figure 2: The comparison between the FFDM solutions (where h=0.01 and $\alpha = 0.75$) and the approximate solutions which are obtained by some methods, according to Example 5.

tions of order $0 < \alpha < 1$ will be more accurate. In addition, the occurred errors between the FFDM solutions for those equations of that orders and the exact solutions are steady and stable.

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