

EVEN VERTEX COLORING OF A GRAPH

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Abstract: As a generalization of Harary's notion of consistency in marked graphs, we define an even vertex coloring of a graph G as an assignment of colors to the vertices of G such that in every cycle of G there is a nonzero even number of vertices of at least one color. The even vertex coloring number $\varepsilon_v(G)$ of even-vertex colorable graph G is defined as the minimum number of colors in an even vertex coloring of G and a minimum even vertex coloring of G is one which uses exactly $n = \varepsilon_v(G)$ colors. A characterization of minimally edge-colored graphs is obtained and a result linking the notion to bipartite Eulerian multigraphs is established.

AMS Subject Classification: 05C22

Key Words: coloring, marked graphs, consistency, even vertex coloring

1. Introduction

For all terminology and notation in graph theory we refer the reader to consult any one of the standard text-books by Chartrand and Zhang [4], Harary [7] and West [8].

In this paper, unless mentioned otherwise, we shall consider only undirected simple graphs (i.e., any two vertices are joined by at most one edge) which are not necessarily finite and do not contain isolated vertices.

Received: August 12, 2015

Published: February 26, 2016

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url: www.acadpubl.eu

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As a generalization of Harary's notion of cognitive balance in signed graphs [5], the notion of an even edge coloring of a graph G was introduced by B. D. Acharya [2] as an assignment g of colors to the edges of G such that in every cycle of G there is a nonzero even number of edges of at least one color. The minimum number of colors in an even edge coloring of G will be denoted $\varepsilon(G)$.

We now define vertex analogous of even edge coloring which is generalization of consistent marked graphs (See [3]) defined as follows: An even vertex coloring is an assignment of colors to the vertices of G such that in every cycle of G , there is a nonzero even number of vertices of at least one color. Clearly a graph need not have an even vertex coloring. For example, complete graph on 5 vertices. A graph G is said to be even vertex-colorable if G has an even vertex coloring.

A graph G is said to be even vertex colorable, if G has an even vertex coloring. The following result characterizes complete graphs which are even vertex colorable.

Theorem 1. *The complete graph on n vertices K_n is even vertex colorable if, and only if, $n \leq 4$.*

Proof. (Necessary) Necessary part is immediate, since, complete graph K_n , $n \leq 4$ is even vertex colorable.

(Sufficiency) Suppose that $K_n, n \geq 5$ is even vertex colorable and let g be an even vertex coloring. Clearly K_n contains a cycle C of length 5. By our assumption, there exists at least one color c_i which occurs an even number of times on the cycle C . Suppose there are 3 vertices with color c_i , then since G is complete, these three vertices form a cycle of length 3 in G in which all vertices are colored c_i , a contradiction. On the other hand if there are exactly two vertices with color c_i , then C contains 3 vertices which are colored by the color other than c_i . Now suppose that all the vertices does not colored same, then the cycle C receives 3 colors. But a vertex from each color class forms a cycle of length 3 receiving 3 colors, a contradiction. This completes the proof. \square

The characterization of graphs which are even vertex colorable is not easy and so we pose the following problem:

Problem 2. Characterize even vertex colorable graphs?

Clearly, if G does not contain any cycle (such a graph is often said to be 'acyclic' or called a 'forest'), then by the vacuous case of the definition, it follows that every vertex coloring of G is an even vertex-coloring of G . More generally, if a vertex v of G not belonging to any cycle of G then its color does not figure in the definition of an even vertex coloring of G and hence it can be arbitrary. Hence, unless specified otherwise, by a 'graph' G , we mean a graph in which

every vertex belongs to a cycle of G . Let $|g|$ denote the number of colors in an even vertex coloring g of G if G has an even vertex coloring. The minimum number of colors in an even vertex coloring of G will be denoted $\varepsilon^v(G)$ and a minimum even vertex-coloring of G is one with exactly $\varepsilon^v(G)$ colors. If a G is not even vertex colorable we assume $\varepsilon^v(G) = 0$.

2. Minimum Even Vertex Colorings

Clearly, for any graph G , $\varepsilon^v(G) = 1 \iff G$ is bipartite, and the class of bipartite graphs is well studied in literature and a large number of their properties and uses are available in standard text-books of graph theory (e.g., see [4, 7, 8]). Hence, we consider graphs with $n = \varepsilon^v(G) > 1$.

Further, if in an even vertex coloring g of G , each color occurs a nonzero even number of vertices of at least one cycle in G , let us say that the coloring is complete and if for each vertex v that is in a cycle of G there is a cycle containing a nonzero even number of vertices of the color of v then the coloring is said to be optimal. For each color c_i in a minimum coloring g of G , let the subgraph spanned by the set of vertices receiving color c_i in g be denoted V_i . Clearly, if g is complete, V_i contains at least one cycle C of G such that $0 < |V(C) - V_i| \equiv 0 \pmod{2}$ and $P = \{V_1, V_2, \dots, V_n\}, n = |g|$, is a partition of $V(G)$.

Theorem 3. *For any graph G with $\varepsilon^v(G) \geq 3$, any minimum even vertex coloring g of G is both optimal and complete.*

Proof. Suppose g is not optimal. Then, there is a cycle C in G containing an vertex v such that there is no cycle in G that contains a nonzero even number of edges having the color of v , say $g(v) = c_i$. Therefore, every cycle in G that contains v has only an odd number of vertices with color $g(v)$. Let us recolor all the edges of G having the color $g(v)$ with one of the existing colors other than $g(x)$, say c_j , and let g_0 be the resulting vertex coloring of G . Since g is an even vertex coloring of G with more than three colors, every cycle containing v must contain an even number of edges of some color c_k other than c_i . Hence, suppose amongst the other cycles (that is, those which do not contain v) in G there is a cycle C_0 having an vertex u with color $g(v)$. Then, by our assumption, it follows that C_0 contains a nonzero even number of vertices of a color other than $g(v)$. Thus, every cycle of (G, g_0) has a nonzero even number of vertices of at least one color, implying thereby that g_0 is indeed an even vertex coloring of G . This contradicts the minimality of g . Hence, g must be optimal.

Next, suppose g is not complete. By definition, we then must have a color, say c_i , in g that does not occur on a nonzero even number of vertices in any cycle of G . That is, if there is any vertex having color c_i in a cycle of G it must occur on an odd number of its vertices. However, since g is an even vertex-coloring of G each such cycle must have a nonzero even number of vertices of some color other than c_i . Hence, repainting all the vertices in G having color c_i with one of the existing colors, say c_j , other than c_i , resulting in a new vertex-coloring g_0 of G , does not alter their requirement to be still in consonance with the definition of an even vertex-coloring of G , since $\varepsilon^v(G) \geq 3$. But then, we have a contradiction to the minimality of g again. Thus, it follows that g must be complete and the theorem is proved. \square

Remark: The converse of the above theorem need not be true. Since any even vertex coloring of even cycle of order > 4 satisfies both the conditions that is both optimal and complete but it is not minimal since $\varepsilon^v(G) = 1$.

A marked graph $M = (G, \mu)$ is consistent if every cycle of G is contains an even number of vertices marked $-$ (See [3]). Clearly a marked graph in which every vertex is marked $+$ is consistent and a marked graph in which every vertex is marked $-$ is consistent if and only if the underlying graph is bipartite.

Theorem 4. [L. W. Beineke and F Harary[3]] *If M is consistent and has some negative points, then the subgraph induced by the negative points must be bipartite.*

Theorem 5. [L. W. Beineke and F Harary[3]] *If M is consistent and point u is positive while point v is negative, there cannot be three disjoint $u : v$ paths.*

Given an unmarked graph, it can always be marked in a consistent way by giving each vertex by a positive sign. However, this cannot always be done if some negative sign must be used. Consider for example the complete graph K_4 . A graph G markable if it can be consistently marked using at least one $-$ sign. Beineke and Harary [3] introduced the problem of characterizing markable graphs.

A graph M is said to be markable if it is possible to mark the vertices of G such that the resulting marked graph is consistent and there exists at least one vertex is marked $-$.

If $\varepsilon^v(G) = 2$ and the colors are interpreted as being positive or negative then we have a marked graph in which every cycle contains a nonzero even number of positive vertices or a nonzero even number of negative vertices. We say that a marked graph G_μ is said to be quasiconsistant if every cycle of G contains a non-zero even number of vertices marked positive or a even number

of vertices marked negative. Clearly, a marked graph G is quasiconsistent if and only if its negation $\eta(G)$ (i.e., the marked graph obtained from G by reversing the sign of every vertex) is also so. There are social systems whose marked graph structures are quasi consistent and hence it would be of interest to find a structural characterization of such marked graphs. As there is no structural characterization of consistent marked graph, it is difficult to find such characterizations.

Problem 6. Characterize the quasi consistent marked graphs.

3. Multigraph Associated with Even Vertex Coloring

For any finite graph G together with a minimum vertex coloring g , one can define a bipartite multigraph $B_g(G)$ as follows: The vertex set of $B_g(G)$ has $V_1 = \{c_1, c_2, \dots, c_n\}$, $n = |g|$ as one partite set, where c_i s are the colors by g to the vertices of G , and $V_2 = \{C^1, C^2, \dots, C^\mu\}$ is a cycle-basis of G , that is, a set of μ linearly independent fundamental cycles C_j , $\mu = \mu(G)$ being the cycle-rank of G . Further, the multiset $E(B_g(G))$ of the edges of $B_g(G)$ is defined by the presence of exactly e_{ij} unordered pairs $c_i C_j$, whenever e_{ij} is the nonzero even number of vertices in C^j having color c_i . The following rather far-fetched fact points at a natural connection between even vertex colorings and Eulerian graphs.

Theorem 7. *Let (G, g) be any finite graph in which every vertex belongs to a cycle together with a minimum even vertex coloring g . Then, the multigraph $B_g(G)$ is Eulerian.*

Proof. Let $V_1 = c_1, c_2, \dots, c_n$ be the set of colors assigned by g to the vertices of G and let $V_2 = C^1, C^2, \dots, C^\mu$ is an arbitrarily fixed cycle-basis of G . Since every vertex of G belongs to a cycle, whence $\mu(G) > 0$ and V_1 and V_2 are nonempty sets. Since g is an even vertex coloring of G , the edge $c_i C^j$ appears in $B_g(G)$ exactly as many times as the nonzero even number of edges in C_j have the color c_i , the degree of the vertex $c_i \in V_1$ being a finite sum of at most $\mu(G)$ such even numbers must be even for each $i, 1 \leq i \leq n$. The same also holds for each vertex of V_2 in $B_g(G)$. Further, since $|g| = \varepsilon^v(G)$, by completeness and optimality of g , it follows that $B_g(G)$ is connected. Thus, by a well known characterization of Eulerian graphs, we see that $B_g(G)$ is indeed an Eulerian graph. \square

Acknowledgement

The authors are thankful to the anonymous referee for valuable suggestions and comments for the improvement of the paper. Also, the first author is grateful to Dr. M. N. Channabasappa, Director and Dr. Shivakumaraiah, Principal, Siddaganga Institute of Technology, Tumkur, for their constant support and encouragement.

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