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A GENERAL INTEREST RATE MODEL INCORPORATING CURRENT YIELD CURVE

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Abstract: For an interest rate model to predict future rates, it is highly desirable that it has embedded in it the current term structure of interest rates known in the market. For the general four-parameter interest rates model incorporating the current yield into the model seems intractable. In the present paper we consider a first order approximation, valid for large time to maturity, to fit the current yield curve into the model. The resulting interest rate model is described which includes, as a particular case, the well-known interest rate model of Vasicek [11].

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1. Introduction

The first short rate model to be proposed was by Vasicek [11]:

$$dr = a\,(b - r)\,dt + \sigma dX. \tag{1.1}$$

This is a three constant-parameter model. Here, σ is volatility of interest rate and dX is a Wiener process drawn from a normal distribution with mean zero and variance dt. To overcome the possibility of interest rates becoming negative in the Vasicek model, Cox, Ingersoll and Ross [4] proposed the short rate model:

$$dr = a\,(b - r)\,dt + \sigma\sqrt{r}dX. \tag{1.2}$$

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It has a standard deviation of $\sigma\sqrt{r}$ which avoids possibility of negative or zero interest rates for $\sigma^2 < 2ab$.

For an interest rate model to provide future rates, it is highly desirable that it has embedded in it the current term structure of interest rates known in the market. To incorporate a pre-assigned term structure of interest rates, short rate models have been considered with time dependent parameters. The Vasicek model was extended to include time-dependent parameters by Hull and White [6]. A more general treatment is given by Maghsoodi [10] while a more tractable approach can be found in Brigo and Mercurio [2]. Hull and White [7] proposed another model of the form

$$dr = [\theta(t) - ar]\, dt + \sigma dX. \tag{1.3}$$

For more discussion of interest rate models, see Black, Derman and Toy [1], Duffie and Kan [5] and Klugman [8].

Klugman and Wilmott [9] considered a four-parameter random walk for the short rate described by

$$dr = u(r,t)\, dt + w(r,t)\, dX, \tag{1.4}$$

where

$$w(r,t) = \sqrt{\alpha r - \beta}, \quad u(r,t) = (\eta - \gamma r) + \lambda w(r,t).$$

See also Wilmott et al. [12] for more discussion of this short rate model. In the following we call an interest rate model based on this short rate as a *Four-Parameter Interest Rate Model* (FP-IRM). For constant parameters, Chawla [3] obtained complete explicit solutions of the bond pricing equation for FP-IRM.

For the general FP-IRM incorporating the current yield into the model seems intractable. In the present paper we consider a first order approximation, valid for large time to maturity, to fit the current term structure of interest rates into the model. The resulting interest rate model is described which includes, as a particular case, the well-known interest rate model of Vasicek [11].

2. Fitting Current Yield Into FP-IRM

Let $B(t,T)$ denote the value of a zero-coupon bond at time t with maturity value Z at time T, $t < T$, so that $V(T,T) = Z$. Let $\tau = T - t$ denote *time to maturity* and define

$$Y(t,T) = -\frac{1}{\tau}\ell n\left(\frac{B(t,T)}{B(T,T)}\right). \tag{2.1}$$

Yield curve is the plot of Y against τ, and dependence of yield curve on τ is often called the *term structure of interest rates.* For fixed t, the *interest rate implied by the yield curve* is given by

$$r(T) = r(t,T) = \frac{d}{dT}\left[Y(t,T)(T-t)\right]. \tag{2.2}$$

Based on the short rate model (1.4), the zero-coupon bond pricing equation, providing value $B(t,T)$ of a bond at time $t < T$, is

$$\frac{\partial B}{\partial t} + \frac{1}{2}(\alpha r - \beta)\frac{\partial^2 B}{\partial r^2} + (\eta - \gamma r)\frac{\partial B}{\partial r} - rB = 0. \tag{2.3}$$

Generally, an affine-yield solution of the bond pricing equation is sought in the form:

$$B\ (t,T) = Ze^{A(t,T)-rC(t,T)}. \tag{2.4}$$

The differential equations providing $A(t,T)$ and $C(t,T)$ are

$$\frac{\partial A}{\partial t} = \eta C + \frac{\beta}{2}C^2, \tag{2.5}$$

and

$$\frac{\partial C}{\partial t} = \frac{\alpha}{2}C^2 + \gamma C - 1, \tag{2.6}$$

with final conditions $A(T,T) = 0$ and $C(T,T) = 0$.

A general solution of (2.5) and (2.6) is obtained and discussed in Chawla [3]. We refer to the resulting interest rate model as FP-IRM. For the following we set

$$\psi = \sqrt{\gamma^2 + 2\alpha},\ a = \frac{-\gamma + \psi}{\alpha},\ b = \frac{\gamma + \psi}{\alpha}.$$

Since (2.6) involves only $C(t,T)$, it can be solved independently of (2.5). For $\alpha \neq 0$, the solution of (2.6) is given by

$$C(t,T) = \frac{2}{\alpha}\left(\frac{1 - e^{-\psi\tau}}{b + ae^{-\psi\tau}}\right). \tag{2.7}$$

Then, for FP-IRM, $A(t,T)$ is given by

$$A(t,T) = \left(\frac{\delta a - \beta}{\alpha}\right)\tau + \frac{\beta}{\alpha}C(\tau) + \frac{2\delta}{\alpha^2}\ln\left(\frac{b + ae^{-\psi\tau}}{b + a}\right), \tag{2.8}$$

where $\delta = \beta\gamma - \alpha\eta$, and the yield is given by

$$Y(t,T) = \left(\frac{\beta - \delta a}{\alpha}\right) - \frac{1}{\tau}\left[\left(\frac{\beta}{\alpha} - r\right)C(\tau) + \frac{2\delta}{\alpha^2}\ln\left(\frac{b + ae^{-\psi\tau}}{b + a}\right)\right], \tag{2.9}$$

with asymptotic value, for $\tau \to \infty$,

$$Y(t,T) \sim \frac{\beta - \delta a}{\alpha}.$$

For later use, we simplify this asymptotic value. Substituting for δ, we can write

$$\frac{\beta - \delta a}{\alpha} = a\eta + \frac{\beta}{\alpha}(1 - a\gamma) = a\eta + \frac{\beta}{\alpha^2}\left(\gamma^2 - \psi\gamma + \alpha\right).$$

Since $\gamma^2 - \psi\gamma + \alpha = \frac{1}{2}(a\alpha)^2$, asymptotic value for the yield can be written as

$$Y(t,T) \sim a\left(\eta + \frac{\beta a}{2}\right). \tag{2.10}$$

Note that FP-IRM includes for $\alpha = 0$ the Vasicek interest rate model and for $\beta = 0$ it includes the Cox-Ingersoll-Ross interest rate model.

In order to fit the initial yield into FP-IRM, we start with equation (2.5) treating η as a function of time:

$$\frac{\partial A}{\partial t} = \eta(t)\, C(t,T) + \frac{\beta}{2} C(t,T)^2. \tag{2.11}$$

Integrating (2.11) from $T \to t$ we get

$$A(t,T) = -\int_t^T \eta(s)\, C(s,T)\, ds - \frac{\beta}{2} I(t,T), \tag{2.12}$$

where we have set

$$I(t,T) = \int_t^T C(s,T)^2\, ds.$$

To fit the current yield curve at $t = 0$, setting $t = 0$ in (2.12) we have

$$\int_0^T \eta(s)\, C(s,T)\, ds = -A_0(T) - \frac{\beta}{2} I_0(T), \tag{2.13}$$

where, for any $\phi(t,T)$, for $t = 0$ we set $\phi(0,t) = \phi_0(T)$. From the yield curve (2.1) for the affine-yield solution (2.4),

$$Y_0(T) = \frac{1}{T}\left[-A_0(T) + r_0 C_0(T)\right],$$

we have

$$A_0(T) = r_0 C_0(T) - T Y_0(T).$$

Substituting in (2.13) we get

$$\int_0^T \eta(s)\, C(s,T)\, ds = F_0(T), \tag{2.14}$$

where we have set

$$F_0(T) = TY_0(T) - r_0 C_0(T) - \frac{\beta}{2} I_0(T). \tag{2.15}$$

With $C(t,T)$ given by (2.7), solution of the integral equation (2.14) seems intractable to provide a "simple" explicit solution. Instead, in the present paper we work with a first order approximation for $C(t,T)$, valid for large time to maturity, to obtain a version of FP-IRM incorporating into it the initial yield.

3. A First Order Approximation

We next find a first order approximation for $C(t,T)$. For the purpose, we go back to the solution of (2.6) or, with a change over from $t \to \tau$, equivalently to

$$\frac{dC}{(C-a)(C+b)} = -\frac{\alpha}{2} dt. \tag{3.1}$$

Partial fractioning gives

$$\left(\frac{1}{C-a} - \frac{1}{C+b}\right) dC = -\psi d\tau, \tag{3.2}$$

since $\frac{\alpha}{2}(a+b) = \psi$. Integrating we have

$$\frac{C-a}{C+b} = ke^{-\psi\tau},$$

for constant of integration k. Solving for C we get

$$C(\tau) = \frac{a + bke^{-\psi\tau}}{1 - ke^{-\psi\tau}}. \tag{3.3}$$

A first order approximation is given by

$$\begin{aligned} C(\tau) &\approx \left(a + bke^{-\psi\tau}\right)\left(1 + ke^{-\psi\tau}\right) \\ &\approx a + (a+b)\, ke^{-\psi\tau}. \end{aligned}$$

In order that this approximation satisfies the final condition we must have $k = -\frac{a}{a+b}$. Thus, the first order approximation for $C(t,T)$ satisfying the final condition, we call it $C^*(t,T)$, is

$$C^*(t,T) = a\left(1 - e^{-\psi\tau}\right). \tag{3.4}$$

This approximation can be interpreted as simply dropping the term $\frac{1}{C+b}$ in (3.2). Note that for $\alpha \to 0$, since $\psi \to \gamma$ and $a \to \frac{1}{\gamma}$, $C^*(t,T)$ in (3.4) becomes $C_V(t,T) = \frac{1}{\gamma}(1 - e^{-\gamma\tau})$ for the Vasicek case.

In the following we call the interest rate model resulting with $C^*(t,T)$ as IRM*. Before we fit the current yield to IRM*, we first describe this model IRM*. With $C^*(t,T)$ given by (3.4) we have

$$A^*(t,T) = -\eta \int_t^T C^*(s,T)\,ds - \frac{\beta}{2} I^*(t,T). \tag{3.5}$$

We calculate

$$\int_t^T C^*(s,T)\,ds = a\tau - \frac{1}{\psi} C^*(t,T),$$

and

$$\begin{aligned} I^*(t,T) = &\int_t^T C^*(s,T)^2\,ds \\ =&a^2\tau - \frac{a}{\psi} C^*(t,T) - \frac{1}{2\psi}\left(C^*(t,T)\right)^2. \end{aligned} \tag{3.6}$$

With these from (3.5) we obtain

$$A^*(t,T) = \left(\eta + \frac{\beta a}{2}\right)\left(\frac{1}{\psi} C^*(t,T) - a\tau\right) + \frac{\beta}{4\psi} C^*(t,T)^2. \tag{3.7}$$

The yield for IRM* is given by

$$\begin{aligned} Y^*(t,T) =&a\left(\eta + \frac{\beta a}{2}\right) + \left[r - \frac{1}{\psi}\left(\eta + \frac{\beta a}{2}\right)\right]\frac{1}{\tau} C^*(t,T) \\ &- \frac{\beta}{4\psi\tau} C^*(t,T)^2, \end{aligned} \tag{(3.8)}$$

with asymptotic value

$$Y^*(t,T) \sim a\left(\eta + \frac{\beta a}{2}\right). \tag{(3.9)}$$

Note that this asymptotic behavior of the yield for the interest rate model IRM* is the same as that for the general FP-IRM given in (2.10). In particular this includes asymptotic yields for the Vasicek and for the Cox-Ingersoll-Ross interest rate models.

4. Incorporating Current Yield Into IRM*

Now we proceed to fit current yield curve into the interest rate model IRM*. Treating η as a function of time, and with $C^*(t,T)$ and $A^*(t,T)$ for the model IRM*, from (2.12) we have

$$A^*(t,T) = -\int_t^T \eta(s)\, C^*(s,T)\, ds - \frac{1}{2}\beta I^*(t,T). \tag{4.1}$$

where $I^*(t,T)$ is given in (3.6). To fit the yield at $t=0$ from (2.14) we have, for the present case,

$$\int_0^T \eta(s)\, C^*(s,T)\, ds = F_0^*(T), \tag{4.2}$$

where now

$$F_0^*(T) = TY_0(T) - r_0 C_0^*(T) - \frac{\beta}{2} I_0^*(T). \tag{4.3}$$

We now solve integral equation (4.2) for $\eta(s)$. Substituting for $C^*(s,T)$ we write (4.2) as

$$\int_0^T \eta(s)\left(1 - e^{-\psi(T-s)}\right) ds = \frac{1}{a} F_0^*(T). \tag{4.4}$$

Differentiating (4.4) with respect to T and using the differentiation rule:

$$\frac{d}{dx}\int_a^x f(s,x)\, ds = \int_a^x \frac{\partial}{\partial x} f(s,x)\, ds + f(x,x), \tag{4.5}$$

we get

$$\int_0^T \eta(s)\, e^{-\psi(T-s)} ds = \frac{1}{\psi a} F_0^{*\prime}(T). \tag{4.6}$$

Adding (4.4) and (4.6) we have

$$\int_0^T \eta(s)\, ds = \frac{1}{a} F_0^*(T) + \frac{1}{\psi a} F_0^{*\prime}(T). \tag{4.7}$$

Again differentiating (4.7) with respect to T we get

$$\eta(T) = \frac{1}{\psi a}\left[\psi F_0^{*\prime}(T) + F_0^{*\prime\prime}(T)\right]. \tag{4.8}$$

To find the corresponding $A^*(t,T)$, substituting for $\eta(s)$ from (4.8) in (4.1) we have

$$A^*(t,T) = -J^*(t,T) - \frac{\beta}{2} I^*(t,T), \tag{4.9}$$

where we have set

$$J^*(t,T) = \int_t^T \left(1 - e^{-\psi(T-s)}\right) F_0^{*\prime} ds + \frac{1}{\psi} \int_t^T \left(1 - e^{-\psi(T-s)}\right) F_0^{*\prime\prime} ds.$$

and $I^*(t,T)$ is given by (3.6). Integrating the second integral by parts we obtain

$$J^*(t,T) = -\frac{1}{\psi}\left(1 - e^{-\psi\tau}\right) F_0^{*\prime}(t) + \int_t^T F_0^{*\prime}(s)\, ds,$$

and therefore

$$J^*(t,T) = F_0^*(T) - F_0^*(t) - \frac{1}{\psi a} C^*(t,T)\, F_0^{*\prime}(t). \tag{4.10}$$

We define *forward yield* at time $t = 0$ by

$$f(t,T) = \frac{1}{\tau}\left[Y_0(T)\, T - Y_0(t)\, t\right]. \tag{4.11}$$

With (4.3) we have

$$F_0^*(T) - F_0^*(t) = f(t,T)\,\tau - r_0\left[C_0^*(T) - C_0^*(t)\right] - \frac{\beta}{2}\left[I_0^*(T) - I_0^*(t)\right].$$

Now,

$$C_0^*(T) - C_0^*(t) = e^{-\psi t} C^*(t,T),$$
$$C_0^*(T) + C_0^*(t) = 2C_0^*(t) + e^{-\psi t} C^*(t,T),$$
$$C_0^*(T)^2 - C_0^*(t)^2 = 2e^{-\psi t} C^*(t)\, C^*(t,T) + e^{-2\psi t} C^*(t,T)^2,$$

and from (3.6) we get

$$\begin{aligned} I_0^*(T) - I_0^*(t) &= a^2\tau - \frac{a}{\psi}\left(C_0^*(T) - C_0^*(t)\right) - \frac{1}{2\psi}\left(C_0^*(T)^2 - C_0^*(t)^2\right) \\ &= a^2\tau - \frac{1}{\psi}\left(a + C_0^*(t)\right) e^{-\psi t} C^*(t,T) - \frac{1}{2\psi} e^{-2\psi t} C^*(t,T)^2. \end{aligned}$$

Thus we obtain

$$\begin{aligned} F_0^*(T) - F_0^*(t) = &\left(f(t,T) - \frac{\beta a^2}{2}\right)\tau - \left(r_0 - \frac{\beta}{2\psi}\left(a + C_0^*(t)\right)\right) e^{-\psi t} C^*(t,T) \\ &+ \frac{\beta}{4\psi} e^{-2\psi t} C^*(t,T)^2. \end{aligned} \tag{4.12}$$

Again, from (4.3) with (2.2) we have

$$\frac{d}{dT}F_0^*(T) = r(T) - r_0 C_0^{*\prime}(T) - \frac{\beta}{2} I_0^{*\prime}(T).$$

Since

$$C_0^{*\prime}(T) = a - C_0^*(T),$$

and from (3.6),

$$I_0^{*\prime}(T) = a^2 - \frac{1}{\psi}(a + C_0^*(T))\, C_0^{*\prime}(T),$$

changing $T \to t$ we obtain

$$F_0^{*\prime}(t) = r(t) - \frac{\beta a^2}{2} - (a - C_0^*(t))\, r_0 + \frac{\beta}{2\psi}\left(a^2 - C_0^*(t)^2\right). \tag{4.13}$$

Finally with (4.10) we obtain

$$A^*(t,T) = -\left[F_0^*(T) - F_0^*(t)\right] + \frac{1}{\psi a} C^*(t,T)\, F_0^{*\prime}(t) - \frac{\beta}{2} I^*(t,T), \tag{4.14}$$

where the required three terms are given, respectively, by (4.12), (4.13) and (3.6). This completes the determination of $A^*(t,T)$ for the case when the current yield curve has been incorporated into the interest rate model IRM*.

As a check, setting $t = T$ we find $A^*(T,T) = 0$. Again, for $t = 0$,

$$A_0^*(T) = -F_0^*(T) + F_0^*(0) + \frac{1}{\psi a} C_0^*(T)\, F_0^{*\prime}(0) - \frac{\beta}{2} I_0^*(T).$$

From (4.3) it is easy to see that $F_0^*(0) = 0$ and since $C_0^*(0) = 0$, therefore $A_0^*(T)$ reduces to

$$A_0^*(T) = -F_0^*(T) - \frac{1}{2}\beta I_0^*(T) = -TY_0(T) + r_0 C_0^*(T),$$

as it should so that our present interest rate model IRM* produces and is consistent with the initial term structure of interest rates observed in the market.

The yield curve for the interest rate model IMR*, after current yield has been embedded in it, is given by

$$Y^*(t,T) = -\frac{1}{\tau}\left[A^*(t,T) - rC^*(t,T)\right]. \tag{4.15}$$

With the help of (4.12) and (3.6) we find the coefficient of τ in $A^*(t,T)$ is

$$-\left[f(t,T) - \frac{\beta a^2}{2}\right] - \frac{\beta}{2}\left[a^2\right] = -f(t,T).$$

Since all other terms in (4.15) go to zero as $\tau \to \infty$, it follows that the asymptotic value of this yield:

$$Y^*(t,T) \sim f(t,T), \tag{4.16}$$

is identical with the forward rate at time $t = 0$.

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