International Journal of Pure and Applied Mathematics
Volume 107 No. 1 2016, 267-278
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v107i1.19

ijpam.eu

L-FUZZY QUASI-UNIFORM SPACES INDUCED BY L-FUZZY INTERIOR SPACES

Young Sun Kim[1], Yong Chan Kim[2 §]

[1]Department of Applied Mathematics
Pai Chai University
Dae Jeon, 302-735, KOREA
[2]Department of Mathematics
Gangneung-Wonju University
Gangneung, Gangwondo, 210-702, KOREA

Abstract: In this paper, we introduce the notions of L-fuzzy quasi-uniformities and L-fuzzy interior operators in complete residuated lattices. We investigate the L-fuzzy quasi-uniformities induced by L-fuzzy interior operators. We study the relationships between L-fuzzy interior operators and L-fuzzy quasi-uniformities. We give their examples.

AMS Subject Classification: 03E72, 06A15, 06F07, 54F05
Key Words: complete residuated lattices, L-fuzzy interior operators, L-fuzzy quasi-uniformities, LF- interior map, LF-uniformly continuous

1. Introduction

Many researcher introduced the notion of fuzzy uniformities in unit interval [0,1] [1], complete distributive lattices [3,8,9], commutative unital quantales [6] and complete quasi-monoidal lattices [15].

Kim [9] introduced the notion of fuzzy quasi-uniformities as an extension of Lowen in strictly two-sided, commutative quantales. Ramadan [12,13] investigated the relations among the families of L-fuzzy topology, L-neighborhood system, L-fuzzy topogenous structures and L-fuzzy quasi-uniformity.

Received: November 11, 2015
Published: March 31, 2016
© 2016 Academic Publications, Ltd.
url: www.acadpubl.eu

§Correspondence author

In this paper, we introduce the notions of L-fuzzy interior operators and L-fuzzy quasi-uniformities in complete residuated lattices [5]. We investigate the L-fuzzy quasi-uniformities induced by L-fuzzy interior operators. We study the relationships between L-fuzzy interior operators and L-fuzzy quasi-uniformities. We give their examples.

2. Preliminaries

Definition 1. [2,5,6,15] An algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a completely distributive with the greatest element 1 and the least element 0;

(C2) $(L, \odot, 1)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

In this paper, we assume that $(L, \leq, \odot, \rightarrow, \oplus, ^*)$ is a complete residuated lattice with an order reversing involution $x^* = x \rightarrow 0$ which is defined by $x \oplus y = (x^* \odot y^*)^*$ unless otherwise specified and we denote $L_0 = L - \{0\}$.

Lemma 2. [2,5,6,15] For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.

(1) $1 \rightarrow x = x$, $0 \odot x = 0$,

(2) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$,

(3) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$,

(4) $x \rightarrow (\bigwedge_i y_i) = \bigwedge_i (x \rightarrow y_i)$ and $(\bigvee_i x_i) \rightarrow y = \bigwedge_i (x_i \rightarrow y)$,

(5) $x \rightarrow (\bigvee_i y_i) \geq \bigvee_i (x \rightarrow y_i)$ and $(\bigwedge_i x_i) \rightarrow y \geq \bigvee_i (x_i \rightarrow y)$,

(6) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(7) $x \odot (x \rightarrow y) \leq y$ and $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,

(8) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \odot z) \rightarrow (y \odot w)$,

(9) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \oplus z) \rightarrow (y \oplus w)$,

(10) $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$ and $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$,

(11) $x \rightarrow y = y^* \rightarrow x^*$.

Definition 3. [6,13] A mapping $int : L^X \times L_0 \rightarrow L^X$ is called an L- fuzzy interior operator if it satisfies the following conditions;

(I1) $int(1_X, r) = 1_X$,

(I2) $int(\lambda, r) \leq \lambda$,

(I3) If $\lambda_1 \leq \lambda_2$, then $int(\lambda_1, r) \leq int(\lambda_2, r))$,

(I4) If $r_1 \leq r_2$, then $int(\lambda, r_1) \geq int(\lambda, r_2)$,
(I5) $int(\lambda_1 \odot \lambda_2, r \odot s) \geq int(\lambda_1, r) \odot int(\lambda_2, s)$.
The pair (X, int) is called an L-fuzzy interior space.

Let (X, int_X) and (Y, int_Y) be L-fuzzy interior spaces and $f : X \to Y$ be a map. Then f is called an LF- interior map if, for each $\rho \in L^Y, r \in L_0$,

$$int_X(f^{\leftarrow}(\rho), r) \geq f^{\leftarrow}(int_Y(\rho, r)).$$

Definition 4. [9,13] A mapping $\mathcal{U} : L^{X\times X} \to L$ is called an L-fuzzy quasi-uniformity on X iff it satisfies the conditions.
(QU1) There exists $u \in L^{X\times X}$ such that $\mathcal{U}(u) = 1$,
(QU2) If $v \leq u$, then $\mathcal{U}(v) \leq \mathcal{U}(u)$,
(QU3) For every $u, v \in L^{X\times X}$, $\mathcal{U}(u \odot v) \geq \mathcal{U}(u) \odot \mathcal{U}(v)$,
(QU4) If $\mathcal{U}(u) \neq 0$, then $1_\triangle \leq u$, where

$$1_\Delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y, \end{cases}$$

(QU5) $\bigvee\{\mathcal{U}(v) \mid v \circ v \leq u\} \geq \mathcal{U}(U)$,

$$v \circ w(x, z) = \bigvee_{z\in X} (v(y, z) \odot w(x, y)), \quad \forall\ x, y \in X.$$

The pair $(X,\mathcal{U})$ is called an L-fuzzy quasi-uniform space.

Let $\mathcal{U}_X$ and $\mathcal{U}_Y$ be L-fuzzy quasi-uniformities on X and Y, respectively and $f : X \to Y$ be a map. Then f is called LF-uniformly continuous if

$$\mathcal{U}_Y(v) \leq \mathcal{U}_X((f \times f)^{\leftarrow}(v)),\ \forall v \in L^{Y\times Y}.$$

Remark 5. Let $\mathcal{U}$ be an L-fuzzy quasi-uniformity on X. Then by (QU1) and (QU2), we have $\mathcal{U}(1_{X\times X}) = 1$ because $u \leq 1_{X\times X}$ for all $u \in L^{X\times X}$.

3. *L*-Fuzzy Quasi-Uniform Spaces Induced by *L*-Fuzzy Interior Spaces

Theorem 6. Let $(X,\mathcal{U})$ be an L-quasi-uniform space. Define mappings $int_{\mathcal{U}} : L^X \times L_0 \to L^X$ as

$$int_{\mathcal{U}}(\lambda, r) = \bigvee\{\ \overrightarrow{u}\{\lambda\} \mid \mathcal{U}(u) \geq r\ \},$$

where

$$\overrightarrow{u}\{\lambda\}(y) = \bigwedge_{x\in X} (u(y,x) \to \lambda(x)).$$

We have the following properties.

(1)

$$\begin{array}{l} \bigwedge_{x\in X} \big(int_{\mathcal{U}}(\lambda_1, r)(x) \to int_{\mathcal{U}}(\lambda_2, r)(x)\big) \\ \geq \bigwedge_{y\in X} \ \{\lambda_1(y) \to \lambda_2(y)\}. \end{array}$$

(2) $int_{\mathcal{U}}$ is an L-fuzzy interior operator.

(3) $int_{\mathcal{U}}(int_{\mathcal{U}}(\lambda, r_1), r_1) \geq int_{\mathcal{U}}(\lambda, r)$ for each $r_1 < r$.

Proof. (1)

$$\begin{array}{l} \bigwedge_{x\in X} \big(int_{\mathcal{U}}(\lambda_1, r)(x) \to int_{\mathcal{U}}(\lambda_2, r)(x)\big) \\ = \bigwedge_{x\in X} \bigvee\{(\bigwedge_{y\in X} u(x,y) \to \lambda_1(y)) \mid \mathcal{U}(u) \geq r\} \\ \to (\bigwedge_{y\in X}(u(x,y) \to \lambda_2(y)) \mid \mathcal{U}(u) \geq r\} \\ \geq \bigwedge_{x\in X} \bigvee \bigwedge_{y\in X} \ \{(u(x,y) \to \lambda_1(y)) \to (u(x,y) \to \lambda_2(y)) \mid \mathcal{U}(u) \geq r\} \\ \geq \bigwedge_{y\in X} \ \{\lambda_1(y) \to \lambda_2(y)\}. \end{array}$$

(2) (I1) Since $u(x,y) \to 1_X(x) = 1$, $int_{\mathcal{U}}((1_X, A), r) = (1_X, A)$.

(I2) Since

$$\overrightarrow{u}\{\lambda\}(y) = \bigwedge_{x\in X}(u(y,x) \to \lambda(x)) \leq u(y,y) \to \lambda(y) = \lambda(y),$$

then $\overrightarrow{u}\{\lambda\} \leq \lambda$. Hence $int_{\mathcal{U}}(\lambda, r) \leq \lambda$.

(I3) By (1), if $\lambda_1 \leq \lambda_2$, then $int_{\mathcal{U}}(\lambda_1, r) \leq int_{\mathcal{U}}(\lambda_2, r)$.

(I4) It is obvious from the definition of $int_{\mathcal{U}}$.

(I5) Since $(u(x,y) \to \lambda_1(x)) \odot (v(x,y) \to \lambda_2(x)) \leq u(x,y) \odot v(x,y) \to (\lambda_1 \odot \lambda_2(x)$,

$$\begin{array}{l} int_{\mathcal{U}}(\lambda_1, r) \odot int_{\mathcal{U}}(\lambda_2, s) \\ = \bigvee\{\bigwedge_{x\in X}(u(y,x) \to \lambda_1(x)) \mid \mathcal{U}(u) \geq r\} \\ \odot \bigvee\{\bigwedge_{x\in X}(v(y,x) \to \lambda_2(x)) \mid \mathcal{U}(v) \geq s\} \\ \leq \bigvee\{\bigwedge_{x\in X}((u(y,x) \odot v(y,x)) \to (\lambda_1 \odot \lambda_2)(x)) \mid \mathcal{U}(u \odot v) \geq r \odot s\} \\ \leq int_{\mathcal{U}}(\lambda_1 \odot \lambda_2, r \odot s), \end{array}$$

(3) Suppose that there exists $\lambda \in L^X$ and $r, s \in L_0$ with $r > s$ such that

$$int_{\mathcal{U}}(int_{\mathcal{U}}(\lambda, s), s) \not\geq int_{\mathcal{U}}(\lambda, r).$$

From the definition of $int_{\mathcal{U}}(\lambda, r)$, there exists $u \in S(X\times X, A)$ with $\mathcal{U}(u) \geq r$ such that

$$int_{\mathcal{U}}(int_{\mathcal{U}}(\lambda, s), s) \not\geq \bigwedge_{x\in X} u(-,x) \to \lambda(x).$$

On the other hand, since $\bigvee\{\mathcal{U}(v) \mid v \circ v \leq u\} \geq \mathcal{U}(u) \geq r$, for each $s < r$, there exists $v \in L^{X\times X}$ with $v \circ v \leq u$ and $\mathcal{U}(v) \geq s$.

$$\begin{aligned}
& int_{\mathcal{U}}(int_{\mathcal{U}}(\lambda, s), s)(y) \\
& = \bigvee\{\bigwedge_{x\in X}(w(y,x) \to int_{\mathcal{U}}(\lambda, s)(x)) \mid \mathcal{U}(w) \geq s\} \\
& \geq \bigwedge_{x\in X}\big(v(y,x) \to \bigwedge_{z\in X}(v(x,z) \to \lambda(z))\big) \\
& = \bigwedge_{x,z\in X}((v(y,x) \odot v(x,z)) \to \lambda(z)) \\
& = \bigwedge_{z\in X}(\bigvee_{x\in X}(v(y,x) \odot v(x,z)) \to \lambda(z)) \\
& \geq \bigwedge_{z\in X}(u(y,z) \to \lambda(z)).
\end{aligned}$$

It is a contradiction. Hence the result hold.

□

Lemma 7. For every $\lambda \in L^X$, we define $u_\lambda, u_\lambda^{-1} : L^{X\times X} \to L$ by

$$u_\lambda(x,y) = \lambda(x) \to \lambda(y)$$

$$u_\lambda^{-1}(x,y) = u_\lambda(y,x).$$

then we have the following statements
(1) $1_{X\times X} = u_{0_X} = u_{1_X}$ and $1_\triangle \leq u_\lambda$,
(2) For every $u_\rho \in L^{X\times X}$, we have $u_\rho \circ u_\rho \leq u_\rho$,
(3) $u_\lambda \odot u_\rho \leq u_{\lambda\odot\rho}$ and $u_\lambda \odot u_\rho \leq u_{\lambda\oplus\rho}$,
(4) $u_\rho^{-1} = u_{\rho^*}$ and $u_{\lambda\odot\rho}^{-1} = u_{\lambda^*\oplus\rho^*}$.

Proof. (1)

$$\begin{aligned}
& 1_{X\times X}(x,y) = 1 = u_{0_X}(x,y) = 0_X(x) \to 0_X(y) \\
& = 1_X(x) \to 1_X(y) = u_{1_X}(x,y).
\end{aligned}$$

Since $u_\lambda(x,x) = \lambda(x) \to \lambda(x) = 1,\ 1_\triangle \leq u_\lambda$.

(2)

$$\begin{aligned}
& (u_\rho \circ u_\rho)(x,z) = \bigvee_{y\in X}(u_\rho(x,y) \odot u_\rho(y,z)) \\
& = \bigvee_{y\in X}((\rho(x) \to \rho(y)) \odot (\rho(y) \to \rho(z)) \\
& = \rho(x) \to \rho(z) = u_\rho(x,z).
\end{aligned}$$

(3)

$$\begin{aligned}
& (u_\lambda \odot u_\rho)(x,y) = u_\lambda(x,y) \odot u_\rho(x,y) = (\lambda(x) \to \lambda(y)) \odot (\rho(x) \to \rho(y)) \\
& \leq \lambda(x) \odot \rho(x) \to \lambda(y) \odot \rho(y) = u_{\lambda\odot\rho}(x,y).
\end{aligned}$$

$$\begin{aligned}
& (u_\lambda \odot u_\rho)(x,y) = u_\lambda(x,y) \odot u_\rho(x,y) = (\lambda(x) \to \lambda(y)) \odot (\rho(x) \to \rho(y) \\
& \leq \lambda(x) \oplus \rho(x) \to \lambda(y) \oplus \rho(y) = u_{\lambda\oplus\rho}(x,y).
\end{aligned}$$

(4)

$$u_\lambda^{-1}(x,y) = u_\lambda(y,x) = \lambda(y) \to \lambda(x) = \lambda^*(x) \to \lambda^*(y) = u_{\lambda^*}(x,y).$$
$$u_{\lambda\odot\rho}^{-1} = u_{(\lambda\odot\rho)^*} = u_{\lambda^*\oplus\rho^*}.$$

In the following theorem, we obtain an L-fuzzy quasi-uniform structure from an L-fuzzy interior operator.

Theorem 8. Let (X, int) be an L-fuzzy interior space. Define $\mathcal{U}_{int} : L^{X\times X} \to L$ by

$$\mathcal{U}_{int}(u) = \bigvee\{\odot_{i=1}^n r_i \mid \lambda_i = int(\lambda_i, r_i),\ \odot_{i=1}^n u_{\lambda_i} \le u\},$$

where $\bigvee$ is taken over every finite family $\{u_{\lambda_i} \mid i = 1, 2, 3, ..., n\}$. Then we have the following properties.

(1) $\mathcal{U}_{int}$ is an L-fuzzy quasi-uniformity on X.

(2) If $int(int(\lambda, r), r) = int(\lambda, r)$, then $int_{\mathcal{U}_{int}} \ge int$.

(3) If $\mathcal{U}$ is an L-fuzzy quasi-uniformity on X such that $\mathcal{U}(\bigwedge_{i\in\Gamma} u_i) = \bigwedge_{i\in\Gamma} \mathcal{U}(u_i)$ for each u_i and $i \in \Gamma$, then $\mathcal{U}_{int_{\mathcal{U}}}(u) \le \mathcal{U}(u)$.

Proof. (1) (QU1) Since $int(1_X, r) = 1_X, cl(0_X, r) = 0_X$, there exists $1_{X\times X} = u_{0_X} = u_{1_X} \in L^{X\times X}$. It follows $\mathcal{U}_{int}(1_{X\times X}) = 1$.

(QU2) It is trivial from the definition of $\mathcal{U}_{int}$.

(QU3) Suppose there exist $u, v \in L^{X\times X}$ such that

$$\mathcal{U}_{int}(u \odot v) \not\ge \mathcal{U}_{int}(u) \odot \mathcal{U}_{int}(v).$$

There exist two finite families $\{\lambda_i \in L^X \mid int(\lambda_i, r_i) = \lambda_i,\ \odot_{i=1}^m u_{\lambda_i} \le u\}$ and $\{\rho_j \in L^X \mid int(\rho_j, s_j) = \rho_j,\ \odot_{j=1}^n u_{\rho_j} \le v\}$ such that

$$\mathcal{U}_{int}(u \odot v) \not\ge (\odot_{i=1}^m r_i) \odot (\odot_{j=1}^n s_j).$$

On the other hand, since $u \odot v \ge (\odot_{i=1}^m u_{\lambda_i}) \odot (\odot_{j=1}^n u_{\rho_j})$ and,

$$(\odot_{i=1}^m int(\lambda_i, r_i) \odot (\odot_{j=1}^n int(\rho_j, s_j)) = (\odot_{i=1}^m \lambda_i) \odot (\odot_{j=1}^n \rho_j).$$

we have

$$\mathcal{U}_{int}(u \odot v) \ge (\odot_{i=1}^m r_i) \odot (\odot_{j=1}^n s_j).$$

It is a contradiction.

(QU4) Let $\mathcal{U}_{int}(u) \neq 0$. Then there exists a finite family $\{\lambda_i \in L^X \mid int(\lambda_i, r_i) = \lambda_i,\ \odot_{i=1}^m u_{\lambda_i} \leq u\}$ such that

$$\mathcal{U}_{int}(u) = \odot_{i=1}^n r_i \neq 0.$$

Since $u_{\lambda_i} \geq 1_\triangle$ from Lemma 7(1),

$$1_\triangle \leq \odot_{i=1}^m u_{\lambda_i} \leq u.$$

(QU5) Suppose there exists $u \in L^{X\times X}$ such that

$$\bigvee\{\mathcal{U}_{int}(v) \mid v \circ v \leq u\} \not\geq \mathcal{U}_{int}(u).$$

There exists a finite family $\{\rho_i \in L^{X\times X} \mid int(\rho_i, r_i) = \rho_i,\ \odot_{i=1}^m u_{\rho_i} \leq u\}$ such that

$$\bigvee\{\mathcal{U}_{int}(v) \mid v \circ v \leq u\} \not\geq \odot_{i=1}^m r_i.$$

On the other hand, since $u_{\rho_i} \circ u_{\rho_i} = u_{\rho_i}$ for each $i \in \{1, ..., m\}$ from Lemma 7(2), we have $(\odot_{i=1}^m u_{\rho_i} \circ (\odot_{i=1}^m u_{\rho_i}) \leq \odot_{i=1}^m u_{\rho_i}$ from

$$\begin{aligned}
&\bigvee_{y\in X}((\odot_{i=1}^m u_{\rho_i}(x,y)) \odot (\odot_{i=1}^m u_{\rho_i}(y,z)))\\
&= \bigvee_{y\in X}((\odot_{i=1}^m (\rho_i(x) \to \rho_i(y)) \odot (\odot_{i=1}^m (\rho_i(y) \to \rho_i(z))))\\
&= \bigvee_{y\in X}((\odot_{i=1}^m (\rho_i(x) \to \rho_i(y)) \odot (\rho_i(y) \to \rho_i(z))))\\
&\leq \odot_{i=1}^m (\rho_i(x) \to \rho_i(z)).
\end{aligned}$$

Put $v = \odot_{i=1}^m u_{\rho_i}$. Since $int(\rho_i, r_i) = \rho_i,\ \odot_{i=1}^m u_{\rho_i} \leq v$, $v \circ v \leq u$ and

$$\bigvee\{\mathcal{U}_{int}(w) \mid w \circ w \leq u\} \geq \mathcal{U}_{int}(v) \geq \odot_{i=1}^m r_i.$$

It is a contradiction. Thus $\bigvee\{\mathcal{U}_{int}(v) \mid v \circ v \leq u\} \geq \mathcal{U}_{int}(u)$.

Hence $\mathcal{U}_{int}$ is an L-fuzzy quasi-uniformity on X.

(2) Let $int(int(\lambda, r), r) = int(\lambda, r)$. Then $\mathcal{U}_{int}(u_{int(\lambda,r)}) \geq r$.

$$\begin{aligned}
&int_{\mathcal{U}_{int}}(\lambda, r)(y) = \bigvee\{\bigwedge_{x\in X}(U(y,x) \to \lambda(x)) \mid \mathcal{U}_{int}(v) \geq r\}\\
&\geq \bigwedge_{x\in X}(u_{int(\lambda,r)}(y,x) \to \lambda(x))\\
&\geq \bigwedge_{x\in X}((int(\lambda,r)(y) \to int(\lambda,r)(x)) \to \lambda(x))\\
&\geq int(\lambda, r)(y)
\end{aligned}$$

because

$$\begin{aligned}
&int(\lambda,r)(y) \odot (int(\lambda,r)(y) \to int(\lambda,r)(x))\\
&\leq int(\lambda,r)(x) \leq \lambda(x).
\end{aligned}$$

Hence $int_{\mathcal{U}_{int}} \geq int$.

(3) Suppose there exists $u \in L^{X\times X}$ such that $\mathcal{U}_{int_{\mathcal{U}}}(u) \not\leq \mathcal{U}(u)$. By the definition of $\mathcal{U}_{int_{\mathcal{U}}}$, there exists $r_i \in L_0$ with $int_{\mathcal{U}}(\lambda_i, r_i) = \lambda_i$ and $\odot_{i=1}^{n} u_{\lambda_i} \leq u$ such that $\odot_{i=1}^{n} r_i \not\leq \mathcal{U}(u)$.

On the other hand, $int_{\mathcal{U}}(\lambda_i, r_i) = \bigvee_k \{\overrightarrow{w_{i_k}}\{\lambda_i\} \mid \mathcal{U}(w_{i_k}) \geq r_i\ \}$. Since L is a complete distributive lattice,

$$\bigvee_k \overrightarrow{w_{i_k}}\{\lambda_i\} = \overrightarrow{\bigwedge_k w_{i_k}}\{\lambda_i\}$$

Put $w_i = \bigwedge_k w_{i_k}$. Then $\mathcal{U}(w_i) = \mathcal{U}(\bigwedge_k w_{i_k}) = \bigwedge_k \mathcal{U}(w_{i_k}) \geq r_i$ and $int_{\mathcal{U}}(\lambda_i, r_i) = \overrightarrow{w_i}\{\lambda_i\} = \lambda_i$. Since $w_i \leq u_{\lambda_i}$,

$$\mathcal{U}(u) \geq \mathcal{U}(\odot_{i=1}^{n} u_{\lambda_i}) \geq \odot_{i=1}^{n} \mathcal{U}(u_{\lambda_i}) \geq \odot_{i=1}^{n} \mathcal{U}(w_i) \geq \odot_{i=1}^{n} r_i.$$

It is a contradiction.

Theorem 9. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be L-fuzzy quasi uniform spaces and $f : X \to Y$ be LF-uniformly continuous. Then $int_{\mathcal{U}}(f^{\leftarrow}(\rho), r) \geq f^{\leftarrow}(int_{\mathcal{V}}(\rho, r))$.

Proof. Since

$$\begin{aligned} &\overrightarrow{(f\times f)^{\leftarrow}(v)}\{f^{\leftarrow}(\rho)\}(y) = \textstyle\bigwedge_{x\in X}((f\times f)^{\leftarrow}(v)(x,y) \to f^{\leftarrow}(\rho)(x)) \\ &= \textstyle\bigwedge_{x\in X}(v(f(x), f(y)) \to \rho(f(x))) \geq \bigwedge_{z\in Y}(v(z, f(y)) \to \rho(z)) \\ &= f^{\leftarrow}(\overrightarrow{v}\{\rho\})(y). \end{aligned}$$

we have

$$\begin{aligned} &f^{\leftarrow}(int_{\mathcal{V}}(\rho, r)) = f^{\leftarrow}(\textstyle\bigvee\{\vec{v}\{\rho\} \mid \mathcal{V}(v) \geq r\}) \\ &= \textstyle\bigvee\{f^{\leftarrow}(\vec{v}\{\rho\}) \mid \mathcal{V}(v) \geq r\} \\ &\leq \textstyle\bigvee\{\overrightarrow{(f\times f)^{\leftarrow}(v)}\{f^{\leftarrow}(\rho)\} \mid \mathcal{U}((f\times f)^{\leftarrow}(v)) \geq r\} \\ &\leq int_{\mathcal{U}}(f^{\leftarrow}(\rho), r). \end{aligned}$$

Theorem 10. Let (X, int_X) and (Y, int_Y) be L-fuzzy interior spaces and $f : X \to Y$ be LF-interior map. Then the mapping $f : (X,\mathcal{U}_{int_X}) \to (Y,\mathcal{U}_{int_Y})$ is LF-uniformly continuous.

Proof. We have $(f\times f)^{\leftarrow}(u_\rho) = u_{f^{\leftarrow}(\rho)}$ from:

$$\begin{aligned} &(f\times f)^{\leftarrow}(u_\rho)(x,y) = u_\rho(f(x), f(y)) = \rho(f(x)) \to \rho(f(y)) \\ &= f^{\leftarrow}(\rho)(x) \to f^{\leftarrow}(\rho)(y) = u_{f^{\leftarrow}(\rho)}(x,y). \end{aligned}$$

Since $f : (X, int_X) \to (Y, int_Y)$ is an LF-interior map, then

$$
\begin{aligned}
&\mathcal{U}_{int_Y}(v) = \bigvee\{\odot_{i=1}^n r_i \mid int_Y(\rho_i, r_i) = \rho_i, \odot_{i=1}^n u_{\rho_i} \le v\}\\
&\le \bigvee\{\odot_{i=1}^n r_i \mid int_X(f^{\leftarrow}(\rho_i), r_i) = f^{\leftarrow}(\rho_i), \odot_{i=1}^n u_{f^{\leftarrow}(\rho_i)} \le (f\times f)^{\leftarrow}(v)\}\\
&\le \bigvee\{\odot_{i=1}^n r_i \mid int_X(\lambda_i, r_i) = \lambda_i, \odot_{i=1}^n u_{\lambda_i} \le (f\times f)^{\leftarrow}(v)\}\\
&= \mathcal{U}_{int_X}((f\times f)^{\leftarrow}(v)).
\end{aligned}
$$

Example 11. Let $X = \{x, y, z\}$ be a set and $(L = [0, 1], \odot, \rightarrow)$ be a residuated lattice defined as

$$a \odot b = (a + b - 1) \vee 0, \quad a \rightarrow b = (1 - a + b) \wedge 1.$$

(1) Put $w \in L^{X\times X}$ such that

$$w = \begin{pmatrix} 1 & 0.5 & 0.3 \\ 0.7 & 1 & 0.5 \\ 0.6 & 0.6 & 1 \end{pmatrix}, \quad w \odot w = \begin{pmatrix} 1 & 0 & 0 \\ 0.4 & 1 & 0 \\ 0.2 & 0.2 & 1 \end{pmatrix}.$$

Define $\mathcal{U} : L^{X\times X} \rightarrow L$ as follows

$$\mathcal{U}(u) = \begin{cases} 1, & \text{if } u = 1_{X\times X}, \\ 0.6, & \text{if } w \le u \ne 1_{X\times X}, \\ 0.3, & \text{if } \ w \odot w \le u \not\ge w, \\ 0, & \text{otherwise.} \end{cases}$$

Since $w \circ w = w$ and $(w \odot w) \circ (w \odot w) = w \odot w$, $\mathcal{U}$ is an L-fuzzy quasi-uniformity on X. By Theorem 6, we obtain L-fuzzy interior operator $int_{\mathcal{U}} : L^X \times L_0 \rightarrow L^X$ as follows

$$int_{\mathcal{U}}(\lambda, r) = \begin{cases} \bigwedge_{x\in X} \lambda(x), & \text{if } r > 0.6 \\ \overrightarrow{w}\{\lambda\}, & \text{if } 0.3 < r \le 0.6 \\ \overrightarrow{w \odot w}\{\lambda\}, & \text{if } 0 < r \le 0.3, \end{cases}$$

By Theorem 8, we obtain $\mathcal{U}_{int_{\mathcal{U}}} : L^{X\times X} \rightarrow L$ as follows

$$\mathcal{U}_{int_{\mathcal{U}}}(u) = \begin{cases} 1, & \text{if } u = 1_{X\times X}, \\ 0.6, & \text{if } u_{\overrightarrow{w}\{\lambda\}} \le u \ne 1_{X\times X}, \\ 0.3, & \text{if } \ u_{\overrightarrow{w\odot w}\{\lambda\}} \le u \not\ge u_{\overrightarrow{w}\{\lambda\}}, \\ 0.2, & \text{if } \ u_{\overrightarrow{w}\{\lambda\}} \odot u_{\overrightarrow{w}\{\lambda\}} \le u \not\ge u_{\overrightarrow{w\odot w}}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $w(x,y) \le \overrightarrow{w}\{\lambda\}(x) \to \overrightarrow{w}\{\lambda\}(y)$ from

$$\begin{aligned}&w(y,p)\odot w(x,y)\odot \overrightarrow{w}\{\lambda\}(x)\\ &\le w(x,p)\odot \textstyle\bigwedge_{z\in X}(w(x,z)\to\lambda(z))\\ &\le w(x,p)\odot (w(x,p)\to\lambda(p))\le\lambda(p)\\ &\text{iff } w(x,y)\odot \overrightarrow{w}\{\lambda\}(x)\le w(y,p)\to\lambda(p)\\ &\text{iff } w(x,y)\le \overrightarrow{w}\{\lambda\}(x)\to\overrightarrow{w}\{\lambda\}(y).\end{aligned}$$

Hence $\mathcal{U}_{int_{\mathcal{U}}}(u) \le \mathcal{U}(u)$ for all $u \in L^{X\times X}$.

(2) Define a $[0,1]$-fuzzy interior operator $int : [0,1]^X \times (0,1] \to [0,1]^X$ as follows:

$$int(\lambda,r)=\begin{cases}1_X, & \text{if } \lambda=1_X, r\in[0,1],\\ \rho, & \text{if } \rho\le\lambda,\ r\le 0.6,\\ \rho\odot\rho & \text{if } \rho\odot\rho\le\lambda\not\ge\rho,\ r\le 0.3\\ 0_X, & \text{otherwise.}\end{cases}$$

We have $int(int(\lambda,r),r) = int(\lambda,r)$.

By Theorem 8, we can obtain $\mathcal{U}_{int} : [0,1]^{X\times X} \to [0,1]$ as follows:

$$\mathcal{U}_{int}(u)=\begin{cases}1, & \text{if } u=u_{1_{X\times X}}\\ 0.6, & \text{if} u\ge u_\rho, u\not\ge u_{\rho\odot\rho},\\ 0.3, & \text{if} u\ge u_{\rho\odot\rho}, u\not\ge u_\rho,\\ 0.2, & \text{if} u\ge u_\rho\odot u_\rho, u\not\ge u_{\rho\odot\rho},\\ 0, & \text{otherwise.}\end{cases}$$

$$u_\rho=\begin{pmatrix}1 & 0.6 & 1\\ 1 & 1 & 1\\ 0.8 & 0.4 & 1\end{pmatrix},\ u_{\rho\odot\rho}=\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 0.6 & 0.6 & 1\end{pmatrix}.$$

$$u_\rho\odot u_\rho=\begin{pmatrix}1 & 0.2 & 1\\ 1 & 1 & 1\\ 0.6 & 0 & 1\end{pmatrix}.$$

From Theorem 6,

$$int_{\mathcal{U}_{int}}(\lambda,r)(y)=\bigvee\{\textstyle\bigwedge_{y\in X}(u(y,x)\to\lambda(x)) \mid \mathcal{U}_{int}(u)\ge r\}.$$

For $0.6 < r$, $int_{\mathcal{U}_{int}}(\lambda,r) = \bigwedge_{y\in X}(1_{X\times X}(y,x) \to \lambda(x)) =$

$$\begin{pmatrix}\lambda(x)\wedge\lambda(y)\wedge\lambda(z)\\ \lambda(x)\wedge\lambda(y)\wedge\lambda(z)\\ \lambda(x)\wedge\lambda(y)\wedge\lambda(z)\end{pmatrix}$$

For $0.3 < r \leq 0.6$, $int_{\mathcal{U}_{int}}(\lambda, r) = \bigwedge_{y\in X}(u_\rho(y,x) \to \lambda(x)) =$

$$\begin{pmatrix} \lambda(x) \wedge (0.4+\lambda(y)) \wedge \lambda(z) \\ \lambda(x) \wedge \lambda(y) \wedge \lambda(z) \\ (0.2+\lambda(x)) \wedge (0.6+\lambda(y)) \wedge \lambda(z) \end{pmatrix}$$

For $0.2 < r \leq 0.3$, $int_{\mathcal{U}_{int}}(\lambda, r) = (\bigwedge_{y\in X}(u_{\rho\odot\rho}(y,x) \to \lambda(x))) \vee (\bigwedge_{y\in X}(u_\rho(y,x) \to \lambda(x))) =$

$$\begin{pmatrix} \lambda(x) \wedge (0.4+\lambda(y)) \wedge \lambda(z) \\ \lambda(x) \wedge \lambda(y) \wedge \lambda(z) \\ (0.4+\lambda(x)) \wedge (0.6+\lambda(y)) \wedge \lambda(z) \end{pmatrix}$$

For $0 < r \leq 0.2$, $int_{\mathcal{U}_{int}}(\lambda, r) = \bigwedge_{y\in X}(u_\rho \odot u_\rho)(y,x) \to \lambda(x)) =$

$$\begin{pmatrix} \lambda(x) \wedge (0.8+\lambda(y)) \wedge \lambda(z) \\ \lambda(x) \wedge \lambda(y) \wedge \lambda(z) \\ (0.4+\lambda(x)) \wedge \lambda(z) \end{pmatrix}$$

Since $int_{\mathcal{U}_{int}}(\rho, r) = \rho$ if $0.3 < r \leq 0.6$ and $int_{\mathcal{U}_{int}}(\rho \odot \rho, r) = \rho \odot \rho$ if $0.2 < r \leq 0.3$, we have $int_{\mathcal{U}_{int}} \geq int$.

References

[1] R. Badard, A.A. Ramadan, A.S. Mashhour, Smooth preuniform and proximity spaces, *Fuzzy Sets and Systems*, **59** (1993), 95-107.

[2] R. Bělohlávek,*Fuzzy Relational Systems*, Kluwer Academic Publishers, New York , (2002), **doi:** 10.1007/978-1-4615-0633-1.

[3] D. Čimoka, A.P. Šostak, L-fuzzy syntopogenous structures, Part I: Fundamentals and application to L-fuzzy topologies, L-fuzzy proximities and L-fuzzy uniformities, *Fuzzy Sets and Systems*, **232** (2013), 74-97.

[4] Fang Jinming, *I*-fuzzy Alexandrov topologies and specialization orders, *Fuzzy Sets and Systems,* **158**(2007), 2359-2374, **doi:** 10.1016/j.fss.2007.05.001.

[5] P. Hájek, *Metamathematices of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht (1998), **doi:** 10.1007/978-94-011-5300-3.

[6] U. Höhle, S.E. Rodabaugh, *Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory*, The Handbooks of Fuzzy Sets Series 3, Kluwer Academic Publishers, Boston, 1999, **doi:** 10.1007/978-1-4615-5079-2.

[7] B. Hutton, Uniformities in fuzzy topological spaces, *J. Math. Anal. Appl.*, **58** (1977), 74-79. **doi:** 10.1016/0022-247x(77)90192-5.

[8] A.K. Katsaras, On fuzzy uniform spaces, *J. Math. Anal. Appl.*, **101**, 1984, 97-113. **doi:** 10.1016/0022-247x(84)90060-x.

[9] Y.C. Kim, Y.S. Kim, L-approximation spaces and L-fuzzy quasi-uniform Spaces, *Information Sciences*, **179** (2009), 2028-2048.

[10] R. Lowen, Fuzzy uniform spaces, *J. Math. Anal. Appl.*, **82** (1981), 370-385, **doi:** 10.1016/0022-247x(81)90202-x.

[11] H. Lai, D. Zhang, Fuzzy preorder and fuzzy topology, *Fuzzy Sets and Systems,* **157** (2006), 1865-1885, **doi:** 10.1016/j.fss.2006.02.013.

[12] A.A.Ramadan, E.H.Elkordy, Yong Chan Kim, Perfect L-fuzzy topogenous space, L-fuzzy quasi-proximities and L-fuzzy quasi-uniform spaces, *J. Intelligent and Fuzzy Systems* **28** (2015), 2591-2604, **doi:** 10.3233/IFS-151538.

[13] A.A.Ramadan, E.H.Elkordy, Yong Chan Kim, Relationships between L-fuzzy quasi-uniform structures and L-fuzzy topologies, *J. Intelligent and Fuzzy Systems* **28** (2015), 2319-2327, **doi:** 10.3233/IFS-141515.

[14] A.A. Ramadan,Y.C. Kim, M.K. El-Gayyar, On fuzzy uniform spaces, *The Journal of Fuzzy Mathematics*, **11** (2003), 279-299.

[15] S.E. Rodabaugh, E.P. Klement, *Topological and Algebraic Structures In Fuzzy Sets*, The Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Kluwer Academic Publishers, Boston, Dordrecht, London, 2003, **doi:** 10.1007/978-94-017-0231-7.