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VAGUE PRIME LI-IDEALS OF LATTICE IMPLICATION ALGEBRAS

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Abstract: In this paper,we study the notion of vague prime LI - ideals of lattice implication algebras. We provide the equivalent conditions for vague prime LI - ideals. Extension property of a vague prime LI - ideal is built. We study the relations between vague prime LI - ideals and Vague ultra LI - ideals, VILI - ideals and Vague ultra LI - ideals.

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1. Introduction

In order to investigate a many - valued logical system whose proportional value is given in a lattice, in 1993 Y. Xu [12] first established the lattice implication algebra by combining lattice and implication algebra, and explored many useful structures. The ideal theory serves a vital function for the development of lattice implication algebras. Y. Xu, Y.B. Jun and E.H. Roh [6] introduced the notion of LI - ideals of a lattice implication algebras. In particular, Y.B. Jun

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[5] introduced the concept of prime LI - ideals of lattice implication algebras and discussed some of their properties. Later Y. Lin Liu, San Yang Liu, Y. Xu, Ke Yun Qin [13] introduced the notion of ILI - ideals of a lattice implication algebras and obtained relations between prime LI - ideals and LI -ideals, prime LI - ideals and ILI -ideals etc. Ke Yun Qin, Y. Xu and Y.B. Jun [9] introduced the notion of Ultra LI - ideals in lattice implication algebras.

The concept of fuzzy set was introduced by Zadeh [14]. With the development of fuzzy set, it is widely used in many fields. A fuzzy set is a single function, it cannot express the evidence of supporting and opposing. The concept of vague set introduced by W.L. Gau and D.J. Buehrer [4] in 1993. The idea of vague set is that the membership of every element can be divided into two aspects including supporting and opposing. Ranjit Biswas [3] initiated the study of vague algebra by studying vague groups. At first Ya Qin and Yi Liu [10] applied the concept of vague set theory to lattice implication algebras and introduced the notion of v- filter, and investigated their some properties.

In this paper, we introduce the notion of vague Prime LI - ideals of a lattice implication algebras and discuss some of their properties. Extension property of a vague prime LI - ideal is built. We study the relations between vague prime LI - ideals and Vague ultra LI - ideals, VILI - ideals and Vague ultra LI - ideals.

Throughout this article, L denote a lattice implication algebra.

2. Preliminaries

Definition 1. *(see [11]) Let $(L, \vee, \wedge, 0, I)$ be a complemented lattice with the universal bounds $0, I$. $\rightarrow$ is another binary operation of L. In the sequel the binary operation $\rightarrow$ will be denoted by juxtaposition. $\left(L, \vee, \wedge, \rightarrow, ', 0, I\right)$ is called a lattice implication algebra, if the following axioms hold, $\forall x, y, z \in L$,*
(I_1) $x\,(yz) = y\,(xz)$;
(I_2) $xx = I$;
(I_3) $xy = y'x'$;
(I_4) $xy = yx = I$ implies $x = y$;
(I_5) $(xy)\,y = (yx)\,x$;
(L_1) $(x \vee y)\,z = (xz) \wedge (yz)$;
(L_2) $(x \wedge y)\,z = (xz) \vee (yz)$.

A lattice implication algebra $\left(L, \vee, \wedge, \rightarrow, ' 0, I\right)$ is called a lattice H implication algebra, if $x \vee y \vee ((x \wedge y)z) = I, \forall x, y, z \in L$.

Theorem 2. (see [12]) *Let L be a lattice implication algebra, then for any*

$x, y, z \in L$, the following conclusions hold:

1. *If $Ix = I$ then x = I.*
2. $Ix = x$ and $x0 = x$.
3. $0x = I$ and $xI = I$.
4. $x \leq y$ *if and only if* $xy = I$.
5. $(xy) \vee (yx) = I$
6. $xy \geq x' \vee y$.

Definition 3. *(see [6]) Let I be a nonempty subset of L. I is said to be a LI - ideal of L if it satisfies the following conditions:*

1. $0 \in I$;
2. $\forall x, y \in L$, $(xy)' \in I$ *and* $y \in I$ *imply* $x \in I$.

Definition 4. *(see [5]) A proper LI - ideal P of L is said to be prime LI - ideal if whenever $x \wedge y \in P$ then $x \in P$ or $y \in P$ for all $x, y \in L$.*

Definition 5. *(see [9]) An LI - ideal A of L is said to be an ultra if for every $x \in L$, the equivalence holds: $x \in A \Leftrightarrow x' \notin A$.*

Definition 6. *(see [4]) A vague set A in the universal of discourse X is characterized by two membership functions given by:*

1. *A truth membership function $t_A : X \rightarrow [0, 1]$ and*
2. *A false membership function $f_A : X \rightarrow [0, 1]$.*

Here t_A is a lower bound of the grade of membership of x derived from the 'evidence for x', and f_A is a lower bound on the negation of x derived from the 'evidence against x' and $t_A + f_A \leq 1$.Thus the grade of membership of x in the vague set A is bounded by subinterval $[t_A, 1 - f_A]$ of $[0, 1]$.The vague set A is written as $A =\{\langle x, [t_A, f_A]\rangle / x \in X\}$.Where the interval $[t_A, 1 - f_A]$ is called the vague value of x in the vague set A and denoted by $V_A(x)$.

Definition 7. *(see [4]) The α - cut, A_α of the vague set A is the (α, α) - cut of A and hence given by $A_\alpha = \{x \in X / t_A(x) \geq \alpha\}$*

Definition 8. *(see [1]) Let A be a vague set of a lattice implication algebra L. A is said to be a vague LI - ideal $(brieflyVLI - ideal)$ of L if it satisfies the following conditions:*

1. $\forall x \in L, V_A(0) \geq V_A(x)$;
2. $\forall x, y \in L, V_A(x) \geq imin\left\{V_A\left((xy)'\right), V_A(y)\right\}$.

Definition 9. *(see [2]) Let A be a vague set set of a lattice implication algebra L. A is said to be a vague implicative LI - ideal* $(brieflyVILI - ideal)$ *of L if it satisfies the following conditions:*

1. $\forall x \in L, V_A(0) \geq V_A(x)$;
2. $\forall x, y, z \in L, V_A\left((xy)'\right) \geq imin\left\{V_A\left(\left(\left((xy)'y\right)'z\right)'\right), V_A(z)\right\}$.

3. Vague Prime Li - Ideals

In this section, we introduce the notion of Vague prime LI - ideals and investigate some of their properties. We study the relations between vague prime LI - ideals and Vague ultra LI - ideals, VILI - ideals and Vague ultra LI - ideals.

Definition 10. *A vague LI - ideal A of L is said to be a vague prime LI - ideal if it is a non- constant and* $V_A(x \wedge y) = V_A(x) \vee V_A(y)$ *for any* $x, y \in L$.

Example 11. *(see [12]) Let* $L = \{0, a, b, c, d, I\}$ *be a set with cayley table as follows:*

$\rightarrow$	0	a	b	c	d	I
0	I	I	I	I	I	I
a	c	I	b	c	b	I
b	d	a	I	b	a	I
c	a	a	I	I	a	I
d	b	I	I	b	I	I
I	0	a	b	c	d	I

Define $'$, $\vee, \wedge$ *operations on L as follows:*
$x' = x \rightarrow 0$, $x \vee y = (x \rightarrow y) \rightarrow y$, $x \vee y = ((x' \rightarrow y') \rightarrow y')'$ *for all* $x, y \in L$.

Then $(L, \vee, \wedge, \rightarrow, ', 0, I)$ *is a lattice implication algebra. Let A be a vague set of L defined by*

$$A = \{\langle 0, [0.6, 0.3] \rangle, \langle a, [0.6, 0.3] \rangle, \langle b, [0.5, 0.4] \rangle, \langle c, [0.5, 0.4] \rangle, \\ \langle d, [0.6, 0.3] \rangle, \langle I, [0.5, 0.4] \rangle\}.$$

Then A is a vague prime LI - ideal of L.

Lemma 12. *Let A be a vague LI - ideal of L. Then A is a constant vague set if and only if $V_A(0) = V_A(I)$*

Proof. Necessariness is obvious and we need to prove the sufficiency:
Assume that A satisfies $V_A(0) = V_A(I)$. Since A is a vague LI - ideal, so A is order reversing. For any $x \in L$, $0 \le x \le I$, it follows that $V_A(0) \ge V_A(x) \ge V_A(I)$. Hence $V_A(0) = V_A(I) = V_A(x)$ for any $x \in L$. Then A is constant. □

Theorem 13. *Let A be a non - constant vague LI - ideal of L. Then the following are equivalent:*

1. *A is a vague prime LI - ideal.*
2. *$\forall x, y \in L$, if $V_A(x \wedge y) = V_A(0)$, then $V_A(x) = V_A(0)$ or $V_A(y) = V_A(0)$.*
3. *$\forall x, y \in L$, $V_A\left((xy)'\right) = V_A(0)$ or $V_A\left((yx)'\right) = V_A(0)$.*

Proof. (1) $\Rightarrow$ (2)

Assume that A is vague prime LI - ideal of L.

$$\begin{array}{llll} Let & x, y \in L & such that & V_A(x \wedge y) = V_A(0) \\ & & \Rightarrow & V_A(x) \vee V_A(y) = V_A(0) \\ & & \Rightarrow & V_A(x) = V_A(0)\, or V_A(y) = V_A(0). \end{array}$$

(2) $\Rightarrow$ (3)

$$\begin{array}{llll} Let & x, y \in L & then & (xy)' \wedge (yx)' = 0 \\ & & \Rightarrow & V_A\left((xy)' \wedge (yx)'\right) = V_A(0) \\ & & \Rightarrow & V_A\left((xy)'\right) = V_A(0)\, or V_A\left((yx)'\right) = V_A(0). \end{array}$$

(3) $\Rightarrow$ (1)

Let $x, y \in L$. Suppose that, $V_A\left((xy)^{'}\right) = V_A(0)$. Since A is Vague LI - ideal of L, it follows that

$$\begin{aligned} V_A(x) &\geq imin\left\{V_A\left(\left(x(xy)^{'}\right)^{'}\right), V_A\left((xy)^{'}\right)\right\} \\ &= imin\left\{V_A\left(\left((xy)x^{'}\right)^{'}\right), V_A(0)\right\} \\ &= imin\left\{V_A(x \wedge y), V_A(0)\right\} \\ &\geq V_A(x \wedge y). \end{aligned}$$

Since $x \geq x \wedge y$, $V_A(x) \leq V_A(x \wedge y)$. Therefore $V_A(x \wedge y) = V_A(x)$.
Similarly, $V_A(x \wedge y) = V_A(y)$. Thus $V_A(x \wedge y) = V_A(x) \vee V_A(y)$.
Simillarly we can prove that,
$V_A\left((yx)^{'}\right) = V_A(0)$ implies $V_A(x \wedge y) = V_A(x) \vee V_A(y)$.
Therefore, A is a vague prime LI - ideal of L. □

Corollary 14. *Let P be a vague prime LI - ideal of L then every non - constant VLI - ideal containing P is also vague prime LI - ideal of L.*

Corollary 15. *Let A be a vague prime LI - ideal of L then the α -cut, A_α is also vague prime LI - ideal of L.*

Theorem 16. *Let A be a vague LI - ideal of L. Then A is a vague prime LI - ideal if and only if the set $I = \{x \in L / V_A(x) = V_A(0)\}$ is a prime LI - ideal.*

Proof. Suppose that A is a vague prime LI - ideal L.
Since A is non - constant, I is proper.

$$\begin{aligned} Let \quad x, y \in L \quad suchthat \quad & x \wedge y \in I \\ \Rightarrow \quad & V_A(0) = V_A(x \wedge y) = V_A(x) \vee V_A(y) \\ \Rightarrow \quad & V_A(x) = V_A(0) \quad or \quad V_A(y) = V_A(0) \\ \Rightarrow \quad & x \in I \quad or \quad y \in I. \end{aligned}$$

Therefore, I is a prime LI - ideal of L.
Conversely, assume that I is a prime LI - ideal of L.
Since I is proper, A is non - constant.

$$\begin{aligned} Let \quad x, y \in L \quad then \quad & (xy)^{'} \wedge (yx)^{'} = 0 \\ \Rightarrow \quad & (xy)^{'} \in I \quad or \quad (yx)^{'} \in I \end{aligned}$$

$$\Rightarrow \quad V_A\left(xy\right)^{'} = V_A\left(0\right) \quad or \quad V_A\left(yx\right)^{'} = V_A\left(0\right).$$

So A is a vague prime LI - ideal L. □

Example 17. *In example 11, the vague set A is a vague prime LI - ideal of L. Then $I = \{0, a, d\}$ is a prime LI - ideal of L.*

Let I be a subset of of L and $\alpha \in [0, 1]$. Now we define vague set A_I by

$$V_{A_I}\left(x\right) = \begin{cases} [\alpha, \alpha] & if \quad x \in I \\ [0, 0] & otherwise \end{cases}$$

Theorem 18. *Let I be a VLI - ideal of L. Then I is a prime LI - ideal of L if and only if A_I is a vague prime LI - ideal of L.*

Proof. Assume that I is a prime LI - ideal of L.
Since I is proper, A_I is non - constant.

$$\begin{aligned} Let x, y \in L \quad then \quad & \left(xy\right)^{'} \wedge \left(yx\right)^{'} = 0 \\ \Rightarrow \quad & \left(xy\right)^{'} \in I \quad or \quad \left(yx\right)^{'} \in I \\ \Rightarrow \quad & V_{A_I}\left(\left(xy\right)^{'}\right) = [\alpha, \alpha] = V_{A_I}\left(0\right) \end{aligned}$$

or

$$V_{A_I}\left(\left(yx\right)^{'}\right) = [\alpha, \alpha] = V_{A_I}\left(0\right).$$

Therefore A_I is a vague prime LI - ideal of L.
Conversely, assume that A_I is a vague prime LI - ideal of L.

$$\begin{aligned} Then \quad & V_{A_I}\left(\left(xy\right)^{'}\right) = V_{A_I}\left(0\right) \quad or \quad V_{A_I}\left(\left(yx\right)^{'}\right) = V_{A_I}\left(0\right) for all x, y \in L \\ \Rightarrow \quad & V_{A_I}\left(\left(xy\right)^{'}\right) = [\alpha, \alpha] \quad or \quad V_{A_I}\left(\left(yx\right)^{'}\right) = [\alpha, \alpha] \\ \Rightarrow \quad & \left(xy\right)^{'} \in I \quad or \quad \left(yx\right)^{'} \in I. \end{aligned}$$

So I is a prime LI - ideal of L. □

Theorem 19. *Let A be a proper VLI - ideal of L. Then L is a chain if and only if A is prime LI - ideal of L.*

Proof. Assume that L is a chain and A be any proper VLI - ideal of L.

$$\begin{aligned} Then \quad & x \leq y \quad or \quad y \leq x for all x, y \in L \\ \Rightarrow \quad & xy = I \quad or \quad yx = I \end{aligned}$$

$$\Rightarrow \quad V_A\left((xy)^{'}\right) = V_A(0) \quad or \quad V_A\left((yx)^{'}\right) = V_A(0).$$

Therefore A is vague prime LI - ideal of L.

Conversely, suppose that A is vague prime LI - ideal of L.

$$\begin{aligned} Then \quad & V_A\left((xy)^{'}\right) = V_A(0) \quad or \quad V_A\left((yx)^{'}\right) = V_A(0)\, forall x, y \in L \\ \Rightarrow \quad & xy = I \quad or \quad yx = I \\ \Rightarrow \quad & x \leq y \quad or \quad y \leq x. \end{aligned}$$

So, L is a chain. □

Theorem 20. *Let A be a non - constant VLI - ideal of L such that for any VLI - ideals A_1, A_2 of L, $A_1 \wedge A_2 \leq A$ implies $A_1 \leq A$ or $A_2 \leq A$. Then A is a vague prime LI - ideal of L.*

Proof. On the contradictory assume that A is not a vague prime LI - ideal of L. Then $I = \{x \in L / V_A(x) = V_A(0)\}$ is not a prime LI - ideal of L by theorem 16. Let I_1, I_2 be two LI - ideals such that $I = I_1 \cap I_2$. Implies $I \subset I_1$ and $I \subset I_2$ (otherwise $I = I_1$ or $I = I_2$, then I is irreducible, that is I is prime). Then there exist $x_1, x_2 \in L$ such that $x_1 \in I_1$ but $x_1 \notin I$, $x_2 \in I_2$ but $x_2 \notin I$. Therefore $V_A(x_1) < V_A(0)$ and $V_A(x_2) < V_A(0)$. We define two vague sets A_1, A_2 as follows:

$$V_{A_1}(x) = \begin{cases} V_A(0) & if \quad x \in I_1, \\ 0 \quad if \quad x \notin I_1, \end{cases}$$

$$V_{A_2}(x) = \begin{cases} V_A(0) & if \quad x \in I_2, \\ 0 \quad if \quad x \notin I_2. \end{cases}$$

Now, we need to prove A_1, A_2 are VLI - ideals of L, but we only to prove A_1.

Obviously, $V_{A_1}(0) \geq V_{A_1}(x)$ for any $x \in L$. Let $x, y \in L$

(a) If $y \in I_1$and $(xy)^{'} \in I_1$, then $x \in I_1$. By definition of A_1, it follows that $V_{A_1}(y) = V_A(0) = V_{A_1}\left((xy)^{'}\right)$ and $V_{A_1}(x) = V_A(0)$.

Therefore $V_{A_1}(x) \geq imin\left\{V_{A_1}\left((xy)^{'}\right), V_{A_1}(y)\right\}$.

(b) If $y \notin I_1$ or $(xy)^{'} \notin I_1$, it is easy to verify

$$V_{A_1}(x) \geq imin\left\{V_{A_1}\left((xy)^{'}\right), V_{A_1}(y)\right\}.$$

Therefore, A_1 is a VLI - ideal of L by (a) and (b). Similarly we can prove A_2 is a VLI - ideal of L.

For any $x \in L$, it is to verify that $(A_1 \wedge A_2) \leq A$. By hypothesis, $A_1 \leq A$ and $A_2 \leq A$. But, since $V_{A_1}(x_1) = V_A(0) > V_A(x)$ and $V_{A_2}(x_2) = V_A(0) > V_A(x)$, it follows that $A_1 > A$ and $A_2 > A$. Contradiction. Thus, A is a vague prime LI - ideal of L. □

Theorem 21. *Let L_1 and L_2 are two lattice implication algebras. The mapping $f : L_1 \to L_2$ is an onto lattice implication homomorphism. Then:*

(a) If B is a vague prime LI - ideal of L_2 then $f^{-1}(B)$ is a vague prime LI - ideal of L_1.

(b) If A is a vague prime LI - ideal of L_1 then $f(A)$ is a vague prime LI - ideal of L_2.

Proof. (a) Let B is a vague prime LI - ideal of L_2. By theorem 3.19 in [1], $f^{-1}(B)$ is a vague LI - ideal of L_1. Let $x, y \in L_1$.

$$\begin{aligned} Then f^{-1}(V_B(x)) \vee f^{-1}(V_B(y)) &= V_B(f(x)) \vee V_B(f(x)) \\ &= V_B(f(x) \wedge f(y)) \\ &= V_B(f(x \wedge y)) \\ &= f^{-1}(V_B(x \wedge y)). \end{aligned}$$

So $f^{-1}(B)$ is a vague prime LI - ideal of L_1.

(b) Let A is a vague prime LI - ideal of L_1. By theorem 3.19 in [1], $f(A)$ is a vague LI - ideal of L_2.Since $f : L_1 \to L_2$ is an onto then for any $x, y \in L_2$ there exist $u, v \in L_1$ such that $f(u) = x$, $f(v) = y$.

$$\begin{aligned} Then f(V_A(x)) \vee f(V_A(y)) &= V_A\left(f^{-1}(x)\right) \vee V_A\left(f^{-1}(y)\right) \\ &= V_A(u) \vee V_A(v) \\ &= V_A(u \wedge v) \\ &= V_A\left(f^{-1}(x) \wedge f^{-1}(y)\right) \\ &= V_A\left(f^{-1}(x \wedge y)\right) \\ &= f(V_A(x \wedge y)). \end{aligned}$$

So $f(A)$ is a vague prime LI - ideal of L_2. □

Let A be a vague set of a lattice implication algebra L such that $t_A(x) + f_A(x) \leq t_A(0) + f_A(0)$ for all $x \in L$. Define A^* such that $V_{A^*}(x) = V_A(x) + 1 - V_A(0)$, for all $x \in L$, then A^* is a vague set of L.

Theorem 22. *Let A be a vague set of L. Then A is a vague prime LI - ideal of L if and only if the vague set A^* is vague prime LI - ideal of L.*

Proof. Proof: Let A be a vague prime LI - ideal of a lattice implication algebra L, then $V_A(0) \geq V_A(x)$ for all$x \in L$. For any $x, y \in L$,
$V_{A^*}(0) = V_A(0) + 1 - V_A(0) = 1 \geq V_{A^*}(x)$
and

$$\begin{aligned} imin\left\{V_{A^*}(y), V_{A^*}((xy)^{'})\right\} &= imin\left\{V_A(y) + 1 - V_A(0), V_A((xy)^{'}) + 1 - V_A(0)\right\} \\ &= imin\left\{V_A(y), V_A((xy)^{'})\right\} + 1 - V_A(0) \\ &\leq V_A(x) + 1 - V_A(0) \\ &= V_{A^*}(x). \end{aligned}$$

Therefore A^* is a vague LI - ideal of L. Now, we prove A^* is prime.
Since A is vague prime LI - ideal, for any $x, y \in L$,

$$\begin{aligned} V_A\left(x \wedge y\right) &= V_A\left(x\right) \vee V_A\left(y\right) \\ V_A\left(x \wedge y\right) + 1 - V_A(0) &= \left(V_A\left(x\right) \vee V_A\left(y\right)\right) + 1 - V_A(0) \\ V_A\left(x \wedge y\right) + 1 - V_A(0) &= \left(V_A\left(x\right) + 1 - V_A(0)\right) \vee \left(V_A\left(y\right) + 1 - V_A(0)\right) \\ V_{A^*}\left(x \wedge y\right) &= V_{A^*}\left(x\right) \vee V_{A^*}\left(y\right), \end{aligned}$$

so A^* is a vague prime LI - ideal of L.

Conversely, suppose that A^* is a vague prime LI - ideal, then $V_{A^*}\left(x\right) \leq V_{A^*}\left(0\right)$, that is $V_A(x) + 1 - V_A(0) \leq V_A(0) + 1 - V_A(0)$, it follows that $V_A(x) \leq V_A(0)$.

Since min $\left\{V_{A^*}\left(y\right), V_{A^*}\left(\left(xy\right)^{'}\right)\right\} \leq V_{A^*}\left(x\right)$, so min $\left\{V_A\left(y\right), V_A\left(\left(xy\right)^{'}\right)\right\} \leq V_A\left(x\right)$.

As A^* is prime, it follows that $V_{A^*}\left(x \wedge y\right) = V_{A^*}\left(x\right) \vee V_{A^*}\left(y\right)$, we have $V_A\left(x \wedge y\right) = V_A\left(x\right) \vee V_A\left(y\right)$. Therefore, A is a vague prime LI - ideal of L. □

Definition 23. *A vague LI - ideal of L is said to be a vague ultra LI - ideal of L if for any $x \in L$, either $V_A\left(x\right) = V_A\left(0\right)$ or $V_A\left(x^{'}\right) = V_A\left(0\right)$.*

Example 24. *Clearly the vague set A in the example 11 of L is a vague ultra LI - ideal if L.*

Theorem 25. *Every non constant vague ultra LI - ideal of L is a vague prime LI - ideal of L.*

Proof. Suppose that A is a vague ultra LI - ideal of L.
Then for any $x \in L$, either $V_A(x) = V_A(0)$ or $V_A\left(x^{'}\right) = V_A(0)$.
Let us consider for any $x, y \in L$, $V_A(x \wedge y) = V_A(0)$.
If $V_A(x) = V_A(0)$ then obviously, A is a vague prime LI - ideal.
If $V_A\left(x^{'}\right) = V_A(0)$. For any $x, y \in L$.
As $\left(yx^{'}\right)^{'} \leq \left(y^{'} \vee x^{'}\right)^{'} = y \wedge x$ and A is a VLI - ideal, it follows that

$$\begin{aligned} V_A(y) &\geq imin\left\{V_A\left(\left(yx^{'}\right)^{'}\right), V_A\left(x^{'}\right)\right\} \\ &\geq imin\left\{V_A(y \wedge x), V_A\left(x^{'}\right)\right\} \\ &\geq imin\{V_A(0), V_A(0)\} \\ &= V_A(0). \end{aligned}$$

and $V_A(y) \leq 0$. Thus $V_A(y) = V_A(0)$.By theorem 13 A is a vague prime LI - ideal. □

Theorem 26. *Every vague ultra LI - ideal of L is a VILI -ideal of L.*

Proof. Suppose that A is a vague ultra LI - ideal of L. Since A is a vague prime LI - ideal, $V_A\left((xy)^{'}\right) = V_A(0)$ or $V_A\left((yx)^{'}\right) = V_A(0)$ for all $x, y \in L$.

For instance $V_A\left((xy)^{'}\right) = V_A(0)$, since A is a VLI - ideal of L,$V_A(0) \geq V_A(x)$ for all $x \in L$.

$$\begin{aligned} It follows that V_A\left((xy)^{'}\right) &= V_A(0) \\ &\geq V_A\left(\left((xy)^{'} y\right)^{'}\right) \\ &\geq imin\left\{V_A\left(\left(\left((xy)^{'} y\right)^{'} z\right)^{'}\right), V_A(z)\right\}. \end{aligned}$$

Similarly, If $V_A\left((yx)^{'}\right) = V_A(0)$, then we obtain

$$V_A\left((yx)^{'}\right) \geq imin\left\{V_A\left(\left(\left((yx)^{'} x\right)^{'} z\right)^{'}\right), V_A(z)\right\}.$$

Hence A is a VILI - ideal of L. □

Theorem 27. *In a Lattice H implication algebra, every vague prime LI - ideal is a vague ultra LI - ideal.*

Proof. Let A be a vague prime LI - ideal of a Lattice H implication algebra L. Then for any $x \in L$, we have $x \vee x^{'} = I$. So, $V_A\left(x^{'} \wedge x\right) = V_A(0)$. By theorem 13, we have $V_A(x) = V_A(0)$ or $V_A\left(x^{'}\right) = V_A(0)$. Hence A is a vague ultra LI - ideal of L. □

Remark 28. *In a Lattice H implication algebra, the concepts of vague prime LI - ideals and vague ultra LI - ideals coincide.*

Theorem 29. *Let A be a proper vague LI - ideal of L. Then A is vague ultra LI - ideal of L if and only if $V_A(x \otimes y) = V_A(0)$ implies $V_A(x) = V_A(0)$ or $V_A(y) = V_A(0)$ for any $x, y \in L$.*

Proof. Suppose that A is a vague ultra LI - ideal of L and $V_A(x \otimes y) = V_A(0)$. So $V_A\left(x^{'} \oplus y^{'}\right) = V_A\left(xy^{'}\right) = V_A\left((x \otimes y)^{'}\right) \neq V_A(0)$.

Since A is a vague LI - ideal, $V_A\left(x^{'} \oplus y^{'}\right) \geq imin\left\{V_A\left(x^{'}\right), V_A\left(y^{'}\right)\right\}$.

It follows that $V_A\left(x^{'}\right) \neq V_A(0)$ or $V_A\left(y^{'}\right) \neq V_A(0)$.

That is $V_A(x) = V_A(0)$ or $V_A(y) = V_A(0)$.

Conversely, suppose that is a for any $x, y \in L$, $V_A(x \otimes y) = V_A(0)$ implies $V_A(x) = V_A(0)$ or $V_A(y) = V_A(0)$.

For any $x \in L$, we have $V_A\left(x \otimes x^{'}\right) = V_A\left((xx)^{'}\right) = V_A(0)$, which implies $V_A(x) = V_A(0)$ or $V_A\left(x^{'}\right) = V_A(0)$.

So A is a vague ultra LI - ideal of L. □

Theorem 30. *Let L_1 and L_2 are two lattice implication algebras. The mapping $f : L_1 \to L_2$ is an onto lattice implication homomorphism and $f(0) = 0$. Then:*

(a) If B is a vague ultra LI - ideal of L_2 then $f^{-1}(B)$ is a vague ultra LI - ideal of L_1.

(b) If A is a vague ultra LI - ideal of L_1 and f is bijective then $f(A)$ is a vague ultra LI - ideal of L_2.

Proof. (a) Let B is a vague ultra LI - ideal of L_2.

By theorem 3.19 in [1], $f^{-1}(B)$ is a vague LI - ideal of L_1.

Let $x, y \in L_1$, then $V_B(f(x)) = V_B(f(0)) = V_B(0)$ or $V_B\left(f\left(x^{'}\right)\right) = V_B(f(0)) = V_B(0)$.

That is $f^{-1}(V_B(x)) = f^{-1}(V_B(0))$ or $f^{-1}\left(V_B\left(x^{'}\right)\right) = f^{-1}(V_B(0))$.
So $f^{-1}(B)$ is a vague ultra LI - ideal of L_1.

(b) Let A is a vague ultra LI - ideal of L_1.
By theorem 3.19 in [1], $f(A)$ is a vague LI - ideal of L_2.
Let $f : L_1 \to L_2$ is an bijective.
Let us consider $u, v \in L_2$ there exist $x, x^{'} \in L_1$ such that $f(x) = u$, $f\left(x^{'}\right) = v$.
Since A is a vague ultra LI - ideal of L_1, we have

$$V_A(x) = V_A(0) \text{ or } V_A\left(x^{'}\right) = V_A(0).$$

If $V_A(x) = V_A(0)$, then $f(V_A(u)) = V_A f^{-1}((u)) = V_A(x) = V_A(0)$.
If $V_A\left(x^{'}\right) = V_A(0)$,then $f(V_A(v)) = V_A f^{-1}((v)) = V_A\left(x^{'}\right) = V_A(0)$.
So $f(A)$ is a vague ultra LI - ideal of L_2. □

4. Conclusion

Since W.L.Gau, D.J.Buehrer proposed the notion of vague sets, these ideas have been applied to various fields. In this paper, we applied these ideas to Lattice implication algebras and introduced the notion of vague prime LI - ideal. We obtained some properties of vague Prime LI - ideals and established the extension theorem for vague prime LI - ideals. we derive the relations between the vague Prime LI - ideals and vague ultra LI - ideals, VILI - ideals and Vague ultra LI - ideals. it is our sincere with that our work helps in supporting and augmenting various researches in this field.

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